Global existence and uniqueness of solutions of integral equations with delay: progressive contractions

Theodore A. Burton[⊠1] and **Ioannis K. Purnaras**[2]

[1]Northwest Research Institute, Port Angeles, WA, U.S.A.
[2]University of Ioannina, P. O. Box 1186, 451 10, Ioannina, Greece

Abstract. In the theory of progressive contractions an equation such as

$$x(t) = L(t) + \int_0^t A(t-s)[f(s,x(s)) + g(s,x(s-r(s)))]ds,$$

with initial function ω with $\omega(0) = L(0)$ defined by $t \leq 0 \implies x(t) = \omega(t)$ is studied on an interval $[0,E]$ with $r(t) \geq \alpha > 0$. The interval $[0,E]$ is divided into parts by $0 = T_0 < T_1 < \cdots < T_n = E$ with $T_i - T_{i-1} < \alpha$. It is assumed that f satisfies a Lipschitz condition, but there is no growth condition on g. When we try for a contraction on $[0,T_1]$ the terms with g add to zero and we get a unique solution ξ_1 on $[0,T_1]$. Then we get a complete metric space on $[0,T_2]$ with all functions equal to ξ_1 on $[0,T_1]$ enabling us to get a contraction. In n steps we have obtained a solution on $[0,E]$. When $r(t) > 0$ on $[0,\infty)$ we obtain a unique solution on that interval as follows. As we let $E = 1, 2, \ldots$ we obtain a sequence of solutions on $[0,n]$ which we extend to $[0,\infty)$ by a horizontal line, thereby obtaining functions converging uniformly on compact sets to a solution on $[0,\infty)$. Lemma 2.1 extends progressive contractions to delay equations.

Keywords: progressive contractions, integral and differential equations with delay, global existence, fixed points.

2010 Mathematics Subject Classification: 34A08, 34A12, 45D05, 45G05, 47H09, 47H10.

1 Introduction

In earlier papers [2–5] we studied variants of integral equations of the form

$$x(t) = g(t,x(t)) + \int_0^t A(t-s)f(s,x(s))ds$$

and introduced a technique which we called *progressive contractions* which allowed us to show:

1. Lipschitz maps become contractions;

2. continuous maps become compact maps;

⊠Corresponding author. Email: taburton@olypen.com

3. Krasnoselskii's theorem on the sum of two operators can collapse into Schauder's theorem;

4. the sum of two contractions can be a contraction even when the sum of the contraction constants exceed one;

5. solve a classical conjecture [9, p. 39] that certain maps of the unit ball in a Banach space have a fixed point.

   The purpose of this note is to broaden the application of progressive contractions to obtain global unique solutions of delay integral equations of the form

$$x(t) = L(t) + \int_0^t A(t-s)[f(s,x(s)) + g(s,x(s-r(s)))]ds \tag{1.1}$$

with a continuous initial function ω having several important properties given in (2.4) and (2.5) while satisfying $x(t-r(t)) = \omega(t-r(t))$ when $t-r(t) \leq 0$ and $t \geq 0$. It is assumed that f satisfies a global Lipschitz condition, g and r are continuous, and $r(t) > 0$. The Lipschitz constant can grow with t. Our Lemma 2.1 is the key to the extension to delay equations.

   The method is simple and it avoids classical arguments concerning problems which must be overcome if $r(t)$ tends to zero very quickly, while $g(t,x)$ increases at an arbitrarily large rate. Not only do progressive contractions get us past both difficulties but the questions do not even arise in the process. This is in sharp contrast to methods seen in the literature in which these problems force us to invoke, explicitly or implicitly, Zorn's lemma ([7, p. 42], [8, pp. 87–98]) to obtain a maximal solution.

   We point out that (1.1) is general. It includes, for example standard delay differential equations

$$x'(t) = f(t,x(t)) + g(t,x(t-r(t))$$

and more delay terms are added without difficulty so long as the added terms are either Lipschitz or the delay does not vanish.

   The kernel $A(t-s)$ plays a main role in progressive contractions as seen in (2.3) and (2.11). The kernel enters into the study of a typical problem

$$x'(t) = f(t,x(t)) + g(t,x(t-r(t))$$

by converting it to a form of (1.1). Write the equation as

$$[x'(t) + x(t)]e^t = e^t[x(t) + f(t,x(t)) + g(t,x(t-r(t)))]$$

or

$$(x(t)e^t)' = e^t[x(t) + f(t,x(t)) + g(t,x(t-r(t)))]$$

so that

$$x(t) = x(0)e^{-t} + \int_0^t e^{-(t-s)}[x(s) + f(s,x(s)) + g(s,x(s-r(s)))]ds.$$

2 The setting

In (1.1) it is assumed that $r : [0,\infty) \to (0,\infty)$, $f,g : \Re \times \Re \to \Re$ are continuous and for each $E > 0$ there is a $K > 0$ and $\alpha > 0$ such that $0 \leq t \leq E$ and $x,y \in \Re$ imply that

$$|f(t,x) - f(t,y)| \leq K|x-y|, \qquad r(t) \geq \alpha. \tag{2.1}$$

Note that K can grow with E and, in fact, be unbounded. Also, $A : (0, \infty) \to \Re$ is continuous and

$$\lim_{t \downarrow 0} \int_0^t |A(s)| ds = 0. \tag{2.2}$$

Select $\beta < 1$ and find T with

$$0 < T < \alpha \quad \& \quad K \int_0^T |A(s)| ds < \beta. \tag{2.3}$$

For the $E > 0$ there exists $H > 0$ such that $0 \leq t \leq E$ implies that

$$-H \leq t - r(t) \tag{2.4}$$

and there is a continuous initial function

$$\omega(t) : [-H, 0] \to \Re, \omega(0) = L(0),$$

with

$$x(t - r(t)) = \omega(t - r(t)), \qquad -H \leq t - r(t) \leq 0. \tag{2.5}$$

Progressive contractions allow both K and H to grow with E and be unbounded, while α can approach zero.

Now we are going to divide up the interval $[0, E]$ into n equal segments of length S on which our mapping derived from (1.1) will be a contraction yielding a unique segment of the solution of (1.1) and each of these segments will allow us to ignore $g(t, x(t - r(t))$ in the future contraction arguments.

For the T of (2.3) choose S with $0 < S < T$ so that $nS = E$ and label points on $[0, E]$ by

$$0 = T_0 < T_1 < \cdots < T_n = E, \qquad T_i - T_{i-1} = S. \tag{2.6}$$

The following simple result is a main theorem for delay equations to be treated by progressive contractions.

Lemma 2.1. *If $T_{i-1} \leq t \leq T_i$ and if $\phi(t) = \psi(t)$ for $-H \leq t \leq T_{i-1}$ then*

$$g(t, \phi(t - r(t)) - g(t, \psi(t - r(t)) \equiv 0. \tag{2.7}$$

Proof. Now for $T_{i-1} \leq t \leq T_i$ we have

$$t - r(t) \leq t - \alpha < T_i - T < T_i - S = T_{i-1}.$$

Hence, the arguments in (2.7) are equal. □

We turn now to our existence theorem and we name the type of proof *a progressive contraction*. The complete metric space used here is found in El'sgol'ts [6, p. 16] and repeated in Burton [1, p. 177].

Theorem 2.2. *Let (2.1)–(2.6) hold for (1.1). For every $E > 0$ there is a unique solution of (1.1) on $[0, E]$.*

Proof. We have divided the interval $[0, E]$ into n equal parts, each of length $S < T$, denoting the end points by

$$T_0 = 0, T_1, T_2, \ldots, T_n = E.$$

Step 1. Let $(\mathcal{M}_1, \| \cdot \|_1)$ be the complete metric space of continuous functions $\phi : [-H, T_1] \to \Re$ with the supremum metric and with $\phi(t) = \omega(t)$ for $-H \leq t \leq 0$. Define a mapping $P_1 : \mathcal{M}_1 \to \mathcal{M}_1$ by $\phi \in \mathcal{M}_1$ and $-H \leq t \leq 0$ implies that $(P_1\phi)(t) = \omega(t)$, while $0 < t \leq T_1$ implies that

$$(P_1\phi)(t) = L(t) + \int_0^t A(t - s)[f(s, \phi(s)) + g(s, \phi(s - r(s)))]ds. \tag{2.8}$$

Since $\omega(0) = L(0)$ in (2.5), $(P_1\phi)$ is continuous.

For $\phi, \psi \in \mathcal{M}_1$ and $-H \leq t \leq T_1$ we have

$$|(P_1\phi)(t) - (P_1\psi)(t)| \leq \int_0^t |A(t - s)|[|f(s, \phi(s)) - f(s, \psi(s))|$$
$$+ |g(s, \phi(s - r(s))) - g(s, \psi(s - r(s)))|]ds$$

and by Lemma 2.1

$$\leq K \int_0^t |A(t - s)||\phi(s) - \psi(s)|ds$$
$$\leq K|\phi - \psi|_1 \int_0^{T_1} |A(s)|ds$$
$$\leq \beta|\phi - \psi|_1,$$

a contraction with a unique fixed point ξ_1 on $[-H, T_1]$ and for $0 \leq t \leq T_1$ satisfying

$$(P_1\xi_1)(t) = \xi_1(t) = L(t) + \int_0^t A(t - s)[f(s, \xi_1(s)) + g(s, \xi_1(s - r(s)))]ds. \tag{2.9}$$

Note that $\xi_1(0) = L(0)$ and $\xi_1(t) = \omega(t)$ for $-H \leq t \leq 0$.

Step 2. Let $(\mathcal{M}_2, \| \cdot \|_2)$ be the complete metric space of continuous functions $\phi : [-H, T_2] \to \Re$ with the supremum metric and

$$\phi(t) = \xi_1(t) \quad \text{on } [-H, T_1].$$

Define $P_2 : \mathcal{M}_2 \to \mathcal{M}_2$ by $\phi \in \mathcal{M}_2$ and $-H \leq t \leq T_1$ implies $(P_2\phi)(t) = \xi_1(t)$, while $T_1 < t \leq T_2$ implies

$$(P_2\phi)(t) = L(t) + \int_0^t A(t - s)[f(s, \phi(s)) + g(s, \phi(s - r(s)))]ds. \tag{2.10}$$

We now prove that $P_2\phi$ is continuous on $[-H, T_2)$. Since $P_2\phi = \xi_1(t)$ on $[-H, T_1]$ then P_2 is continuous on $[-H, T_1)$ yet continuous from the left at the endpoint T_1. Also $P_2\phi$ is continuous on $(T_1, T_2]$ for L is continuous and the integrand in (2.10) consists of continuous functions, yet it is continuous from the right at T_1. It remains to prove that $P_2\phi$ is continuous at T_1. Indeed, we have

$$(P_2\phi)(T_1) = \xi_1(T_1)$$
$$= L(T_1) + \int_0^{T_1} A(T_1 - s)[f(s, \phi(s)) + g(s, \phi(s - r(s)))]ds$$
$$= \lim_{t \downarrow T_1}(P_2\phi)(t).$$

So $P_2\phi$ agrees with ξ_1 on $[-H, T_1]$ (by definition) and it is continuous on the whole interval $[-H, T_2]$, and this means that $P_2 : \mathcal{M}_2 \to \mathcal{M}_2$.

We will need a change of variable to see that by (2.3) and $T_1 \le t \le T_2$ we have

$$K \int_{T_1}^t |A(t-s)| ds < \beta. \tag{2.11}$$

For $\phi, \psi \in \mathcal{M}_2$ then

$$|(P_2\phi)(t) - (P_2\psi)(t)| \le \int_0^t |A(t-s)|[|f(s,\phi(s)) - f(s,\psi(s))|$$
$$+ |g(s,\phi(s-r(s))) - g(s,\psi(s-r(s))]ds$$

(by Lemma 2.1)

$$\le \int_0^t |A(t-s)|K|\phi(s) - \psi(s)|ds$$

(since $\phi(t) = \psi(t) = \xi_1(t)$ on $[-H, T_1]$, now take $t > T_1$)

$$\le \int_{T_1}^t |A(t-s)|K|\phi(s) - \psi(s)|ds$$

$$\le \left[\int_{T_1}^t K|A(t-s)|ds\right]|\phi - \psi|_2$$

$$\le \beta|\phi - \psi|_2$$

a contraction on $[-H, T_2]$ with unique fixed point ξ_2 on that entire interval. It is a unique continuous solution of (1.1) on $[0, T_2]$ and it agrees with ξ_1 on $[-H, T_1]$ by construction.

Step 3. The next step is essentially the inductive hypothesis. Here is a sketch of what we are doing. We define the complete metric space $(\mathcal{M}_3, \|\cdot\|_3)$ of continuous functions $\phi : [-H, T_3] \to \Re$ with $\phi(t) = \xi_2$ on $[-H, T_2]$. But ξ_2 is a fixed point and so P_3 would be defined as in Step 2 and map $\mathcal{M}_3$ into $\mathcal{M}_3$. Exactly as in Step 2 we obtain a continuous solution ξ_3 on $[0, T_3]$. By induction we then would obtain a unique continuous solution on $[0, E]$. While we feel this is sufficient for a complete understanding, here are the induction details.

For $2 < i \le n$ let ξ_{i-1} be the unique solution of (1.1) on $[0, T_{i-1}]$. Let $(\mathcal{M}_i, |\cdot|_i)$ be the complete metric space of continuous functions $\phi : [-H, T_i] \to \Re$ with the supremum metric and $\phi = \xi_{i-1}$ on $[-H, T_{i-1}]$. Define $P_i : \mathcal{M}_i \to \mathcal{M}_i$ by $\phi \in \mathcal{M}_i$ implies that $(P_i\phi(t)) = \xi_{i-1}$ on $[-H, T_{i-1}]$ and for $0 \le t \le T_i$ let

$$(P_i\phi)(t) = L(t) + \int_0^t A(t-s)[f(s,\phi(s)) + g(s,\phi(s-r(s)))]ds.$$

Continuity of the function $P_i\phi$ is justified as in Step 2.

To see that this is a contraction, let $\phi, \psi \in \mathcal{M}_i$ and $-H \le t \le T_i$ and use Lemma 2.1 to see that

$$|(P_i\phi)(t) - (P_i\psi)(t)| \le \int_0^t |A(t-s)||f(s,\phi(s)) - f(s,\psi(s))|ds$$

$$\le \int_0^t |A(t-s)|K|\phi(s) - \psi(s)|ds.$$

Now, use $\phi = \psi$ on $[0, T_{i-1}]$ to see that T_{i-1} is the lower limit. Next, take $T_{i-1} < t$ to see that the last quantity is

$$\le \left[\int_{T_{i-1}}^t K|A(t-s)|ds\right]|\phi - \psi|_i.$$

Next use a change of variable to see that this quantity is

$$\leq \left[\int_0^{T_1} K|A(s)|ds \right] |\phi - \psi|_i$$
$$\leq \beta |\phi - \psi|_i$$

a contraction with unique fixed point ξ_i on $[-H, T_i]$. This completes the proof. $\qquad\square$

It is to be noted that as $E \to \infty$, the constant K may also tend to infinity. Still, we determine T from the same relation; as K increases, T decreases. The process works for any $E > 0$. This is important for our next result in that we need to see that we can let $E \to \infty$ and always get a solution on $[0, E]$.

We will now show that we can select a well-defined function on $[0, \infty)$ which is a unique solution of (1.1) and it involves no translations or unfinished steps on the road to a solution on $[0, \infty)$.

Theorem 2.3. *Under the conditions of Theorem 2.2 with $r(t) > 0$ on $[0, \infty)$ there is a unique solution of (1.1) on $[0, \infty)$.*

Proof. Using Theorem 2.2 we will obtain a sequence of uniformly continuous functions on $[0, \infty)$ which converge uniformly on compact sets to a continuous function which is the unique solution of (1.1). Here are the details.

For each positive integer n use Theorem 2.2 to obtain a solution of (1.1) on $[0, n]$. Then denote by $x_n(t)$ the solution on $[0, n]$ extended to a function on $[0, \infty)$ by $x_n(t) = x_n(n)$ for $t \geq n$. This sequence converges uniformly on compact sets to a continuous function, $x(t)$, a solution of (1.1) because at every t the function $x(t)$ agrees with a solution $x_n(t)$ where $n > t$. This completes the proof. $\qquad\square$

References

[1] T. A. Burton, *Stability and periodic solutions of ordinary and functional differential equations*, Academic Press, Orlando, 1985. MR837654

[2] T. A. Burton, An existence theorem for a fractional differential equation using progressive contractions, *J. Fractional Calculus Appl.* **8**(2017), No. 1, 168–172. MR3582599

[3] T. A. Burton, Existence and uniqueness results by progressive contractions for integro-differential equations, *Nonlinear Dyn. Syst. Theory* **16**(2016), No. 4, 366–371.

[4] T. A. Burton, A note on existence and uniqueness for integral equations with sum of two operators: progressive contractions, *Fixed Point Theory*, accepted for publication.

[5] T. A. Burton, The shrinking fixed point map and integral equations, preprint.

[6] L. E. El'sgol'ts, *Introduction to the theory of differential equations with deviating arguments*, Holden-Day, San Francisco, 1966. MR0192154

[7] J. K. Hale, *Theory of functional differential equations*, Springer, New York, 1977. MR0508721

[8] R. K. Miller, *Nonlinear Volterra integral equations*, Benjamin, Menlo Park, CA, 1971. MR0511193

[9] D. R. Smart, *Fixed point theorems*, Cambridge, 1980. MR0467717