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Null controllability for a singular coupled system of degenerate parabolic equations in nondivergence form

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Abstract. We deal with a control problem for a coupled system of two degenerate singular parabolic equations in non-divergence form with degeneracy and singularity appearing at an interior point of the space domain. In particular, we consider the well-posedness of the problem and then we prove the null controllability property via an observability inequality for the adjoint system. The key ingredient is the derivation of a suitable Carleman-type estimate.

Keywords: Carleman estimates, degenerate parabolic systems, singular coefficients, observability inequalities, controllability.

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1 Introduction and main results

The control of coupled parabolic systems is an important subject which has been recently investigated in a large number of articles. The main issue is often to reduce the number of control functions acting on the system.

In this article, we are concerned with a class of control systems governed by degenerate singular parabolic equations in nondivergence form, in presence of singular coupling coefficients. More precisely, we study the null-controllability by one control force of systems of the form

$$u_t - a(x)u_{xx} - \frac{\lambda_1}{b_1(x)}u - \frac{\mu}{d(x)}v = h1_\omega, \quad (t,x) \in Q, \tag{1.1}$$

$$v_t - a(x)v_{xx} - \frac{\lambda_2}{b_2(x)}v - \frac{\mu}{d(x)}u = 0, \quad (t,x) \in Q, \tag{1.2}$$

$$u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \quad t \in (0,T), \tag{1.3}$$

$$u(0,x) = u_0(x), \ v(0,x) = v_0(x), \quad x \in (0,1), \tag{1.4}$$

where ω is an open subset of $(0,1)$, $T > 0$ fixed, $Q := (0,T) \times (0,1)$, 1_ω denotes the characteristic function of the set ω, $u_0, v_0 \in L^2_{1/a}(0,1)$ are the initial conditions, and $h \in L^2_{1/a}(Q) :=$

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$L^2(0,T;L^2_{1/a}(0,1))$ is the control input. Here $L^2_{1/a}(0,1)$ is the Hilbert space

$$L^2_{1/a}(0,1) := \left\{ u \in L^2(0,1) \ \mid \int_0^1 \frac{u^2}{a} dx < \infty \right\},$$

endowed with the associated norm $\|u\|^2_{L^2_{1/a}(0,1)} := \int_0^1 \frac{u^2}{a} dx$, $\forall\, u \in L^2_{1/a}(0,1)$.

Moreover, we assume that the constants λ_i, μ, $i = 1,2$, satisfy suitable assumptions described below, and the functions a, b_i, d, $i = 1,2$, degenerate at the same interior point $x_0 \in (0,1)$. In particular, we make the following assumptions.

Hypothesis 1.1. *Double weakly degenerate case (WWD) There exists $x_0 \in (0,1)$ such that $a(x_0) = b_i(x_0) = 0$, $a, b_i > 0$ in $[0,1] \setminus \{x_0\}$, $a, b_i \in C^1([0,1] \setminus \{x_0\})$ and there exists $K, L_i \in (0,1)$ such that $(x - x_0)a' \le Ka$ and $(x - x_0)b_i' \le L_i b_i$ a.e. in $[0,1]$.*

Hypothesis 1.2. *Weakly strongly degenerate case (WSD) There exists $x_0 \in (0,1)$ such that $a(x_0) = b_i(x_0) = 0$, $a, b_i > 0$ in $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\})$, $b_i \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, $\exists\, K \in (0,1), L_i \in [1,2)$ such that $(x - x_0)a' \le Ka$ and $(x - x_0)b_i' \le L_i b_i$ a.e. in $[0,1]$.*

Hypothesis 1.3. *Strongly weakly degenerate case (SWD) There exists $x_0 \in (0,1)$ such that $a(x_0) = b_i(x_0) = 0$, $a, b_i > 0$ in $[0,1] \setminus \{x_0\}$, $a \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, $b_i \in C^1([0,1] \setminus \{x_0\})$, $\exists K \in [1,2)$, $L_i \in (0,1)$ such that $(x - x_0)a' \le Ka$ and $(x - x_0)b_i' \le L_i b_i$ a.e. in $[0,1]$.*

Hypothesis 1.4. *Double strongly degenerate case (SSD). There exists $x_0 \in (0,1)$ such that $a(x_0) = b_i(x_0) = 0$, $a, b_i > 0$ in $[0,1] \setminus \{x_0\}$, $a, b_i \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$, there exists $K, L_i \in [1,2)$ such that $(x - x_0)a' \le Ka$ and $(x - x_0)b_i' \le L_i b_i$ a.e. in $[0,1]$.*

For our further results we shall admit two types of degeneracy for the coupling term d, namely weak and strong degeneracy. More precisely, we shall handle the two following cases.

Hypothesis 1.5. *The function d is weakly degenerate, that is, there exists $x_0 \in (0,1)$ such that $d(x_0) = 0$, $d > 0$ on $[0,1] \setminus \{x_0\}$, $d \in C^1([0,1] \setminus \{x_0\})$ and there exists $M \in (0,1)$ such that $(x - x_0)d' \le Md$ a.e. in $[0,1]$.*

Hypothesis 1.6. *The function d is strongly degenerate, that is, there exists $x_0 \in (0,1)$ such that $d(x_0) = 0$, $d > 0$ on $[0,1] \setminus \{x_0\}$, $d \in C^1([0,1] \setminus \{x_0\}) \cap W^{1,\infty}(0,1)$ and there exists $M \in [1,2)$ such that $(x - x_0)d' \le Md$ a.e. in $[0,1]$.*

The main controllability result of this paper can be stated as follows.

Theorem 1.7. *Under Hypotheses 3.1 and 3.6, for any time $T > 0$ and any initial datum $(u_0, v_0) \in (L^2_{1/a}(0,1))^2$, there exists a control function $h \in L^2_{1/a}(Q)$ such that the solution of (1.1)–(1.4) satisfies*

$$u(T,x) = v(T,x) = 0, \quad \text{for all } x \in (0,1). \tag{1.5}$$

By a classical duality argument (e.g., see [21]), null controllability will be studied through an observability estimate for the homogeneous backward system associated to (1.1)–(1.4). To get the observability inequality, we prove first a particular Carleman estimate, which is by now a classical technique in control theory. Then, via cut off functions, we prove that there exists a positive constant C_T such that every solution (U,V) of

$$\begin{aligned}
&U_t + a(x)U_{xx} + \frac{\lambda_1}{b_1(x)}U + \frac{\mu}{d(x)}V = 0, \quad (t,x) \in Q,\\
&V_t + a(x)V_{xx} + \frac{\lambda_2}{b_2(x)}V + \frac{\mu}{d(x)}U = 0, \quad (t,x) \in Q,\\
&U(t,0) = U(t,1) = \ V(t,0) = V(t,1) = 0, \quad t \in (0,T),\\
&U(T,x) = U_T(x), \ V(T,x) = V_T(x),
\end{aligned}$$

satisfies, under suitable assumptions, the following estimate:

$$\|(U,V)(0,.)\|^2_{L^2_{1/a}(0,1)^2} \le C_T \iint_{\omega\times(0,T)} \frac{U^2(t,x)}{a}\,dx\,dt. \tag{1.6}$$

Let us observe that in (1.6) we are estimating the $L^2_{1/a}$-norm of $(U,V)(0,.)$ by means of the $L^2_{1/a}$-norm of the first component of (U,V) localized in $\omega\times(0,T)$. One calls this property indirect observability since by observing only one component of the solution on ω, one can control all components of the state at the final time. Roughly, the method is the following: we will start by deriving an intermediate Carleman estimate with two observations which could be used to show the null controllability of the system with two controls. Then, thanks to an interpolation inequality, see Lemma 3.8, we deduce a Carleman estimate with one observation which yields the observability inequality (1.6). As a consequence, using the Hilbert uniqueness method, we then deduce an indirect null controllability result for the system (1.1)–(1.4), that is the state-vector vanishes identically at the final time by applying only one localized control force.

Before dealing with problem (1.1)–(1.4), let us first review some previous results. The general framework addressing the controllability problems of nondegenerate parabolic equations and nondegenerate coupled parabolic systems has been established in earlier papers, and there is nowadays an extended literature on this topic (see for instance, [2,3,20,23,31,32,34]). For more details, on actual methods concerning null or approximate controllability of linear parabolic systems, we refer to the survey [4].

Next results concern control issues for degenerate parabolic equations. In particular, new Carleman estimates (and consequently null controllability properties) were established for operators with degeneracy appearing at the boundary of the domain (see, for instance, [5, 14–16] and the references therein). To the best of our knowledge, [10,26,28,29] are the first papers dealing with Carleman estimates (and, consequently, null controllability) for operators (in divergence and in nondivergence form with Dirichlet or Neumann boundary conditions) with mere degeneracy at the interior of the space domain. For related systems of degenerate equations we refer to [1,11].

Also the question of whether it is possible to control heat equations involving singular inverse-square potentials has already been addressed both in the one-dimensional and in the multi-dimensional case, see [22,36] for the case of internal singularity, and [17] for the case of boundary singularity.

Another interesting situation that has received a lot of attention in recent years is the case of parabolic operators that couple a degenerate diffusion coefficient with a singular potential. Among the pioneering related works we mainly refer to the papers [24,35] in which the authors have studied the control of singular parabolic equations degenerating at origin. These results are complemented in [27], in which it is considered well posedness and null controllability for operators with Dirichlet boundary conditions in divergence form with a degeneracy and a singularity both occurring in the interior of the domain. We refer to the recent paper [25] for the analogous results for operators in nondivergence form under Dirichlet or Neumann boundary conditions.

More recently, in [33] the authors treat well posedness and null controllability for coupled degenerate/singular parabolic systems in divergence form.

However, as it is by now well-known (see, e.g., [9,30]), the equation in non-divergence form cannot be recast, in general, from the equation in divergence form. Indeed, the necessary condition that ensures the well posedness of the problem (1.1)–(1.4) makes it not null controllable. Thus, we cannot derive the null controllabilility for (1.1)–(1.4) by the one of the problem in divergence form. For this reason, in this paper as in [25], [26] or [28], we prove null controllability for (1.1)–(1.4) without deducing it by the previous results for the problem in divergence form.

The object of this paper is twofold: first we analyze the well-posedness of (1.1)–(1.4); second, under suitable conditions on all the parameters of (1.1)–(1.4), we prove related global Carleman estimates. To the best of our knowledge, this is a problem that has never been treated in precedence, although it is a natural extension of the results of the work [25] to the case of coupled 2-component degenerate system involving a singular coupling matrix. To be more precise, observe that the problem (1.1)–(1.4) takes the equivalent form

$$\begin{cases} \partial_t Y - \mathcal{K}Y - \mathcal{C}Y = e_1 h 1_\omega, & \text{in} \quad Q, \\ Y(t,0) = Y(t,1) = 0, & t \in (0,T), \\ Y(0,x) = Y_0(x), & x \in (0,1), \end{cases} \tag{1.7}$$

where $Y = (u,v)^\star$, $Y_0 = (u_0, v_0)^\star$, $\mathcal{K}$ is the matrix operator given by

$$\mathcal{K} = \operatorname{diag}(K, K),$$

and the differential operator K is defined by

$$Kw := a(x) w_{xx}.$$

Further, $\mathcal{C}$ is the singular coupling matrix given by

$$\mathcal{C} = \begin{pmatrix} \frac{\lambda_1}{b_1} & \frac{\mu}{d} \\ \frac{\mu}{d} & \frac{\lambda_2}{b_2} \end{pmatrix}, \tag{1.8}$$

and finally $e_1 = (1,0)^\star$ is the first element of the canonical basis of $\mathbb{R}^2$.

It is worth pointing out that analyzing the controllability properties of system (1.1)–(1.4) (and thus, (1.7)) is more intricate than the null controllability problem for a scalar degenerate singular parabolic equation ([25]) since we want a coupled parabolic system to be controlled by a unique distributed control and additional technical difficulties arise owing to the coupling of the equations.

The paper is organized as follows. In Section 2, we study the well-posedness of the problem via Hardy inequality, applying classical semi-group theory. The Carleman estimate is proved in Section 3. As a consequence, in Section 4, we prove observability inequality, and hence null controllability. Finally, we conclude our article with an appendix in which we prove a Caccioppoli type inequality that is fundamental in our analysis.

All along the article, we use generic constants for the estimates, whose values may change from line to line.

2 Function spaces and well-posedness

It is commonly accepted that Hardy-type inequalities are the starting point to prove well-posedness of singular parabolic equations (see, for instance, [8], [13] and [37]). In the present context, such inequalities turn out to be fundamental for the proof of Proposition 2.10. In order to deal with these inequalities we consider different classes of weighted Hilbert spaces, which are suitable to study the four different situations given above, namely the (WWD), (WSD), (SWD) and (SSD) cases. Thus, as in [25] or [28, Chapter 2], we introduce

$$K_a^1(0,1) := L^2_{1/a}(0,1) \cap H^1_0(0,1)$$

and

$$K^1_{a,b_i}(0,1) := \left\{ u \in K_a \ : \ \frac{u}{\sqrt{ab_i}} \in L^2(0,1) \right\}$$

endowed with the inner products

$$\langle u, v\rangle_{K_a^1} := \int_0^1 \frac{uv}{a}\, dx + \int_0^1 u'v'\, dx,$$

and

$$\langle u, v\rangle_{K^1_{a,b_i}} := \int_0^1 \frac{uv}{a}\, dx + \int_0^1 u'v'\, dx + \int_0^1 \frac{uv}{ab_i}\, dx,$$

respectively.

Using the weighted spaces introduced before we can prove the next Hardy–Poincaré inequality. First, we make the following assumption (we refer to [25] for some comments).

Hypothesis 2.1.

1. *Hypothesis 1.1 holds with* $K + L_i < 1$, *or*
2. *Hypothesis 1.1 holds with* $1 \le K + L_i \le 2$ *and*

$$\exists c_1, c_{i2} > 0 \text{ such that } |x - x_0|^K \ge c_1 a \text{ and } |x - x_0|^{L_i} \ge c_{i2} b_i \ \forall x \in [0,1], \tag{2.1}$$

   or

3. *Hypothesis 1.2 or 1.3 with* $K + L_i \le 2$ *and* (2.1), *or*
4. *Hypothesis 1.4 holds with* $K = L_i = 1$.

Proposition 2.2 ([25, Lemma 2.4 and 2.5])**.** *Assume Hypothesis 2.1 holds. Then there exists a constant* $C_i > 0$ *such that for all* $w \in K^1_{a,b_i}(0,1)$ *we have*

$$\int_0^1 \frac{w^2}{ab_i}\, dx \le C_i \int_0^1 (w')^2\, dx. \tag{2.2}$$

Observe that the above Hardy–Poincaré inequality allows us to consider for the (SSD) case only the situation when K and L_i are both 1.

For the well-posedness of the problem (1.1)–(1.4), due to the presence of singular coupling terms, a natural functional setting involves the weighted space

$$K^1_{a,b_i,d}(0,1) := \left\{ u \in K^1_{a,b_i} \ : \ \frac{u}{\sqrt{ad}} \in L^2(0,1) \right\}$$

which is a Hilbert space for the scalar product

$$\langle u, v\rangle_{K^1_{a,b_i,d}} := \int_0^1 \frac{uv}{a}\, dx + \int_0^1 u'v'\, dx + \int_0^1 \frac{uv}{ab_i}\, dx + \int_0^1 \frac{uv}{ad}\, dx.$$

In the following we make the following assumptions on d.

Hypothesis 2.3.

1. *Hypothesis 1.5 holds with* $K + M < 1$, *or*
2. *Hypothesis 1.5 holds with* $1 \leq K + M \leq 2$ *and*
$$\exists c_3 > 0 \text{ such that } |x - x_0|^M \geq c_3 d \ \forall x \in [0,1], \tag{2.3}$$
or
3. *Hypothesis 1.6 with* $K + M \leq 2$ *and* (2.3), *or*
4. *Hypothesis 1.6 with* $K = M = 1$.

We will proceed with a Hardy-type estimate involving the coupling term under consideration. Such an estimate is valid in the following suitable Hilbert space $\mathcal{K}_i^1 := K^1_{a,b_i,d}(0,1)$, under hypothesis 2.3, and it states the existence of $C_d > 0$ such that for all $w \in \mathcal{K}_i^1$, we have

$$\int_0^1 \frac{w^2}{ad}\,dx \leq C_d \int_0^1 (w')^2\,dx. \tag{2.4}$$

Remark 2.4. If the assumptions 2.1 and 2.3 are satisfied, then the standard norm $\|.\|_{\mathcal{K}_i^1}$ is equivalent to $\|w\|^2_\sim := \int_0^1 (w')^2 dx$ for all $w \in \mathcal{K}_i^1$, $i = 1,2$.

From now on, we make the following assumptions on a, b_i, d, λ_i and μ.

Hypothesis 2.5. *Throughout this section, we assume the following hypotheses.*

1. *Hypothesis 2.1 holds.*
2. *We shall also admit Hypothesis 2.3.*
3. *Setting* $C_i^\star$ *and* $C_d^\star$ *the best constant in* $\mathcal{K}_i^1$ *of* (2.2) *and* (2.4) *respectively, we assume that* $\lambda_i, \mu \neq 0$ *and*
$$\lambda_i < \frac{1}{C_i^\star}, \tag{2.5}$$
$$\mu \in \left(0, \frac{\sqrt{\Lambda_1 \Lambda_2}}{C_d^\star}\right), \tag{2.6}$$
where Λ_i, $i = 1,2$ *is given in* (2.7).

We also need the following result which is a crucial tool to prove well-posedness and observability properties.

Proposition 2.6 ([25, Proposition 3.1])**.** *Assume Hypothesis 2.5. Then there exists* $\Lambda_i \in (0,1]$ *such that for all* $w \in \mathcal{K}_i^1$,
$$\int_0^1 (w'(x))^2\,dx - \lambda_i \int_0^1 \frac{w^2(x)}{a(x)b_i(x)}\,dx \geq \Lambda_i \|w\|^2_{\mathcal{K}_i^1}. \tag{2.7}$$

Finally, we introduce the Hilbert space

$$\begin{aligned}\mathcal{K}_i^2 :=& H^2_{a,b_i}(0,1)\\ :=& \{w \in K_a^1(0,1) : w' \in H^1(0,1) \text{ and } A_i w \in L^2_{1/a}(0,1)\},\end{aligned}$$

where $A_i w := a w_{xx} + \frac{\lambda_i}{b_i} w$, $i = 1, 2$.

In the Hilbert space $\mathbb{H}_{1/a} = L^2_{1/a}(0,1) \times L^2_{1/a}(0,1)$, the system (1.1)–(1.4) can be transformed into the following inhomogeneous Cauchy problem

$$X'(t) - \mathbb{A}X(t) = f(t), \qquad X(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \tag{2.8}$$

where $X = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$,

$$\mathbb{A} = \mathcal{A} + \mathcal{B}, \tag{2.9}$$

with

$$\mathcal{D}(\mathbb{A}) := \{X \in \mathcal{K}^2_1 \times \mathcal{K}^2_2 : \mathbb{A}X \in \mathbb{H}_{1/a}\}, \tag{2.10}$$

where

$$\mathcal{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 0 & \frac{\mu}{d} \\ \frac{\mu}{d} & 0 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} h(t,\cdot)1_\omega \\ 0 \end{pmatrix}.$$

Remark 2.7. Observe that if $X \in \mathcal{D}(\mathbb{A})$, then $(\frac{u}{d}, \frac{v}{d})$ and $(\frac{u}{\sqrt{d}}, \frac{v}{\sqrt{d}}) \in \mathbb{H}_{1/a}$ so that $X \in \mathcal{K}^1_1 \times \mathcal{K}^1_2$. Thus inequalities (2.2) and (2.4) hold true if Hypotheses 2.1 and 2.3 are satisfied.

We recall the following formula of integration by parts which will be used in the rest of the paper.

Lemma 2.8 ([28, Lemma 2.2]). *For all $(u, v) \in K^2_a \times K^1_a$ one has*

$$\int_0^1 u'' v dx = -\int_0^1 u'v' dx, \tag{2.11}$$

where

$$K^2_a := \{u \in K^1_a : u' \in H^1(0,1)\}.$$

Let us now show that the operator $(\mathbb{A}, \mathcal{D}(\mathbb{A}))$ defined by (2.9)–(2.10) generates an analytic semi-group in the pivot space $\mathbb{H}_{1/a}$ for the equation (2.8). This aim relies on this fact.

Lemma 2.9. *Assume that Hypothesis 2.5 is satisfied. Then, the operator $\mathbb{A}$ with domain $\mathcal{D}(\mathbb{A})$ is nonpositive and self-adjoint on $\mathbb{H}_{1/a}$.*

Proof. Observe that $\mathcal{D}(\mathbb{A})$ is dense in $\mathbb{H}_{1/a}$.

(i) $\mathbb{A}$ is nonpositive. By Proposition 2.6 and Lemma 2.8, it follows that, for any $X = \binom{w_1}{w_2} \in \mathcal{D}(\mathbb{A})$ we have

$$\begin{aligned}
-\langle \mathbb{A}X, X\rangle_{\mathbb{H}_{1/a}} &= -\langle \mathcal{A}X + \mathcal{B}X, X\rangle_{\mathbb{H}_{1/a}}, \\
&= -\left\langle \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_{\mathbb{H}_{1/a}} - \left\langle \begin{pmatrix} 0 & \frac{\mu}{d} \\ \frac{\mu}{d} & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_{\mathbb{H}_{1/a}}, \\
&= -\int_0^1 (a w_1'' + \frac{\lambda_1}{b_1} w_1)\frac{w_1}{a}\, dx - \int_0^1 (a w_2'' + \frac{\lambda_2}{b_2} w_2)\frac{w_2}{a}\, dx \\
&\quad - 2\mu \int_0^1 \frac{w_1 w_2}{ad}\, dx, \\
&= \int_0^1 (w_1')^2\, dx - \lambda_1 \int_0^1 \frac{w_1^2}{ab_1}\, dx + \int_0^1 (w_2')^2\, dx - \lambda_2 \int_0^1 \frac{w_2^2}{ab_2} dx \\
&\quad - 2\mu \int_0^1 \frac{w_1 w_2}{ad}\, dx, \\
&\geq \Lambda_1 \int_0^1 (w_1')^2\, dx + \Lambda_2 \int_0^1 (w_2')^2\, dx - 2\mu \int_0^1 \frac{w_1 w_2}{ad}\, dx.
\end{aligned}$$

Using Young's inequality, the last term in the above right-hand side is estimated as

$$\left|\int_0^1 \frac{w_1 w_2}{ad}\,dx\right| \le \int_0^1 \frac{|w_1|}{\sqrt{ad}}\frac{|w_2|}{\sqrt{ad}}\,dx,$$
$$\le \delta \int_0^1 \frac{w_1^2}{ad}\,dx + \frac{1}{4\delta}\int_0^1 \frac{w_2^2}{ad}\,dx,$$

where $\delta > 0$ is a constant that will be chosen later on. Then, we can apply the Hardy–Poincaré inequality (2.4) obtaining

$$\left|\int_0^1 \frac{w_1 w_2}{ad}\,dx\right| \le \delta C_d^\star \int_0^1 (w_1')^2\,dx + \frac{C_d^\star}{4\delta}\int_0^1 (w_2')^2\,dx.$$

Hence,

$$-\langle \mathbb{A}X, X\rangle_{\mathbb{H}_{1/a}} \ge (\Lambda_1 - 2\mu\delta C_d^\star)\int_0^1 (w_1')^2\,dx + \left(\Lambda_2 - 2\mu\frac{C_d^\star}{4\delta}\right)\int_0^1 (w_2')^2\,dx.$$

Now, by (2.6) one can find δ such that

$$\frac{\mu C_d^\star}{2\Lambda_2} < \delta < \frac{\Lambda_1}{2\mu C_d^\star}. \tag{2.12}$$

For this choice, we deduce that there exists $\Sigma > 0$ such that

$$-\langle \mathbb{A}X, X\rangle_{\mathbb{H}_{1/a}} \ge \Sigma \|X\|^2_{\mathcal{K}_1^1\times\mathcal{K}_2^1} \ge 0.$$

(ii) $\mathbb{A}$ is self-adjoint. Let $T : \mathbb{H}_{1/a} \to \mathbb{H}_{1/a}$ be the mapping defined in the following usual way: to each $f \in \mathbb{H}_{1/a}$ associate the weak solution $X = T(f) \in \mathcal{K}_1^1 \times \mathcal{K}_2^1$ of

$$-\langle \mathbb{A}X, Y\rangle_{\mathbb{H}_{1/a}} = \langle f, Y\rangle_{\mathbb{H}_{1/a}},$$

for every $Y \in \mathcal{K}_1^1 \times \mathcal{K}_2^1$. Note that T is well defined by Lax–Milgram lemma via the part (i), which also implies that T is continuous. Now, it is easy to see that T is injective and symmetric. Thus it is self adjoint. As a consequence, $\mathbb{A} = T^{-1} : \mathcal{D}(\mathbb{A}) \to \mathbb{H}_{1/a}$ is self-adjoint (for example, see [19, Proposition X.2.4]). □

As a consequence of the previous lemma we immediately have the following well-posedness result in the sense of evolution operator theory.

Proposition 2.10. *Assume Hypothesis 2.5. Then, the operator $\mathbb{A} : \mathcal{D}(\mathbb{A}) \to \mathbb{H}_{1/a}$ generates an analytic contraction semigroup of angle $\pi/2$ on $\mathbb{H}_{1/a}$. Moreover, for all $h \in L^2_{1/a}(Q)$ and $u_0, v_0 \in L^2_{1/a}(0,1)$, there exists a unique weak solution $(u,v) \in C([0,T];\mathbb{H}_{1/a}) \cap L^2(0,T;\mathcal{K}_1^1\times\mathcal{K}_2^1)$ of (1.1)–(1.4). In addition, if $(u_0,v_0) \in \mathcal{D}(\mathbb{A})$ and $h \in W^{1,1}(0,T,L^2_{1/a}(0,1))$, then*

$$(u,v) \in C^1(0,T;\mathbb{H}_{1/a}) \cap C([0,T];\mathcal{D}(\mathbb{A})). \tag{2.13}$$

Proof. Since $\mathbb{A}$ is a nonpositive, self-adjoint operator on a Hilbert space, it is well known that $(\mathbb{A},\mathcal{D}(\mathbb{A}))$ generates a cosine family and an analytic contractive semigroup of angle $\pi/2$ on $\mathbb{H}_{1/a}$ (see [6, Example 3.14.16 and 3.7.5]). Being $\mathbb{A}$ the generator of a strongly continuous semigroup on $\mathbb{H}_{1/a}$, the assertion concerning the assumption $u_0, v_0 \in L^2_{1/a}(0,1)$ and the regularity of the solution (u,v) when $(u_0,v_0) \in \mathcal{D}(\mathbb{A})$ is a consequence of the results in [7] and [18, Lemma 4.1.5 and Proposition 4.1.6]. □

3 Carleman estimates

3.1 Carleman estimate for the inhomogeneous adjoint system

In this subsection we prove crucial estimates of Carleman type for the solutions (U, V) of the following nonhomogeneous adjoint problem:

$$U_t + a(x)U_{xx} + \frac{\lambda_1}{b_1}U + \frac{\mu}{d}V = h_1, \quad (t,x) \in Q, \tag{3.1}$$

$$V_t + a(x)V_{xx} + \frac{\lambda_2}{b_2}V + \frac{\mu}{d}U = h_2, \quad (t,x) \in Q, \tag{3.2}$$

$$U(t,1) = U(t,0) = V(t,1) = V(t,0) = 0, \quad t \in (0,T), \tag{3.3}$$

$$U(T,x) = U_T(x), V(T,x) = V_T(x), \quad x \in (0,1), \tag{3.4}$$

which is derived taking inspiration from the work [25]. Here $h_1, h_2 \in L^2_{\frac{1}{a}}(Q)$, while on a, b_i and d we make the following assumptions.

Hypothesis 3.1.

1. *Hypothesis 2.5 is satisfied;*
2. $\frac{(x-x_0)a'(x)}{a(x)} \in W^{1,\infty}(0,1)$;
3. *if $K \geq \frac{1}{2}$, then there exists a constant $\vartheta \in (0, K]$ such that the function $x \mapsto \frac{a(x)}{|x-x_0|^\vartheta}$ is nonincreasing on the left and nondecreasing on the right of $x = x_0$;*
4. *if $\lambda_i < 0$, then $(x - x_0)b_i'(x) \geq 0$ in $[0,1]$.*

To prove an estimate of Carleman type, as in [25] or in [28, Chapter 4], we introduce the function

$$\varphi(t,x) := \theta(t)\psi(x), \quad \forall (t,x) \in (0,T) \times (-1,1),$$

where

$$\theta(t) := \frac{1}{[t(T-t)]^4} \quad \text{and} \quad \psi(x) := d_1 \left[\int_{x_0}^x \frac{y - x_0}{\tilde{a}(y)} e^{R(y-x_0)^2} dy - d_2 \right]. \tag{3.5}$$

Here $d_2 > \tilde{d}_2^\star := \max_{x \in [-1,1]} \int_{x_0}^x \frac{y - x_0}{a(y)} e^{R(y-x_0)^2} dy$, R and d_1 are general strictly positive constants, while the function $\tilde{a}$ is defined as follows:

$$\tilde{a}(x) = \begin{cases} a(x), & x \in [0,1], \\ a(-x), & x \in [-1,0]. \end{cases} \tag{3.6}$$

A more precise restriction on d_1 and d_2 will be needed later. Observe that $\theta(t) \to +\infty$ as $t \to 0^+, T^-$ and clearly

$$-d_1 d_2 \leq \psi(x) < 0 \quad \text{for every } x \in [-1,1].$$

The main result of this section is the following:

Theorem 3.2. *Let $T > 0$ be given. Assume Hypothesis 3.1 is satisfied. Then there exist two positive constants C and s_0 such that every solution (U, V) of (3.1)–(3.4) in*

$$\mathcal{V} = \mathrm{L}^2\Big(0,T;\mathcal{D}(\mathbb{A})\Big) \cap \mathrm{H}^1\Big(0,T;\mathcal{K}_1^1 \times \mathcal{K}_2^1\Big) \tag{3.7}$$

satisfies, for all $s \geq s_0$*,*

$$\int_0^T\int_0^1 \Big[s\theta(U_x^2+V_x^2)+s^3\theta^3\Big(\frac{x-x_0}{a}\Big)^2(U^2+V^2)\Big]e^{2s\varphi(t,x)}\,dx\,dt$$
$$\leq C\Big(\int_0^T\int_0^1 \Big[h_1^2+h_2^2\Big]\frac{e^{2s\varphi}}{a}\,dx\,dt+sd_1\int_0^T\theta\big[(x-x_0)e^{R(x-x_0)^2}\big(U_x^2+V_x^2\big)e^{2s\varphi}\big]_{x=0}^{x=1}dt\Big).$$

Remark 3.3. We underline that Theorem 3.2 still holds if we substitute the spatial domain $[0,1]$ with a general interval $[A,B]$ where the functions a, b_i and d satisfy Hypothesis 3.1.

Proof of Theorem 3.2. First of all, observe that that system (3.1)–(3.4) can be written in the following form:

$$\begin{aligned} &Y_t+\mathcal{A}Y+\mathcal{B}Y=H,\\ &Y(t,0)=Y(t,1)=\begin{pmatrix}0\\0\end{pmatrix}\end{aligned} \tag{3.8}$$

where $Y=\binom{U}{V}$ and $H=\binom{h_1}{h_2}$. Now, for $s>0$, define the function

$$Z(t,x)=e^{s\varphi(t,x)}Y(t,x):=\begin{pmatrix}w\\z\end{pmatrix},$$

where Y is any solution of (3.8). Observe that, since $Y\in\mathcal{V}$ and $\varphi<0$, then $Z\in\mathcal{V}$ and satisfies

$$\begin{aligned} &\mathcal{L}_s^+Z+\mathcal{L}_s^-Z=e^{s\varphi}H, \quad (t,x)\in Q,\\ &Z(t,0)=Z(t,1)=\begin{pmatrix}0\\0\end{pmatrix}, \quad t\in(0,T),\\ &Z(T,x)=Z(0,x)=\begin{pmatrix}0\\0\end{pmatrix}, \quad x\in(0,1),\end{aligned}$$

where

$$\mathcal{L}_s^+=\begin{pmatrix}L_s^{1+}&0\\0&L_s^{2+}\end{pmatrix}+\mathcal{B}\quad\text{and}\quad\mathcal{L}_s^-=\begin{pmatrix}L_s^-&0\\0&L_s^-\end{pmatrix},$$

with

$$\begin{aligned} &L_s^{i+}\bar{u}:=a\bar{u}_{xx}+\lambda_i\frac{\bar{u}}{b_i}-s\varphi_t\bar{u}+s^2a\varphi_x^2\bar{u},\\ &L_s^-\bar{u}:=\bar{u}_t-2sa\varphi_x\bar{u}_x-sa\varphi_{xx}\bar{u}.\end{aligned}$$

Moreover,

$$\begin{aligned} 2\langle\mathcal{L}_s^+Z,\mathcal{L}_s^-Z\rangle_{\mathbb{H}_{1/a}^T}&\leq 2\langle\mathcal{L}_s^+Z,\mathcal{L}_s^-Z\rangle_{\mathbb{H}_{1/a}^T}+\|\mathcal{L}_s^+Z\|^2_{\mathbb{H}_{1/a}^T}+\|\mathcal{L}_s^-Z\|^2_{\mathbb{H}_{1/a}^T}\\ &=\|e^{s\varphi}H\|^2_{\mathbb{H}_{1/a}^T}.\end{aligned}\tag{3.9}$$

Here $\mathbb{H}_{1/a}^T$ is the Hilbert space $L^2_{1/a}(Q)\times L^2_{1/a}(Q)$, equipped with the norm

$$\|X\|_{\mathbb{H}_{1/a}^T}=\big(\|u\|^2_{L^2_{1/a}(Q)}+\|v\|^2_{L^2_{1/a}(Q)}\big)^{\frac{1}{2}}$$

and $\langle\cdot,\cdot\rangle_{\mathbb{H}^T_{1/a}}$ the corresponding scalar product. Of course,

$$\begin{aligned}
\langle \mathcal{L}_s^+ Z, \mathcal{L}_s^- Z\rangle_{\mathbb{H}^T_{1/a}} &= \left\langle \begin{pmatrix} L_s^{1+} & \frac{\mu}{d} \\ \frac{\mu}{d} & L_s^{2+} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}, \begin{pmatrix} L_s^- & 0 \\ 0 & L_s^- \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle_{\mathbb{H}^T_{1/a}}, \\
&= \left\langle \begin{pmatrix} L_s^{1+} w + \frac{\mu}{d} z \\ L_s^{2+} z + \frac{\mu}{d} w \end{pmatrix}, \begin{pmatrix} L_s^- w \\ L_s^- z \end{pmatrix} \right\rangle_{\mathbb{H}^T_{1/a}}, \\
&= \langle L_s^+ w, L_s^- w\rangle_{L^2_{1/a}(Q)} + \langle L_s^+ z, L_s^- z\rangle_{L^2_{1/a}(Q)} \\
&\quad + \mu \left\langle \frac{z}{d}, L_s^- w\right\rangle_{L^2_{1/a}(Q)} + \mu \left\langle \frac{w}{d}, L_s^- z\right\rangle_{L^2_{1/a}(Q)}.
\end{aligned}$$

Observe that the operators L_s^{i+} and L_s^- are exactly the ones of [25]. Using [25, Lemma 4.2 and 4.3], we deduce immediately that there exist two positive constants C and s_0, such that for all $s \geq s_0$,

$$\begin{aligned}
\langle \mathcal{L}_s^+ Z, \mathcal{L}_s^- Z\rangle_{\mathbb{H}^T_{1/a}} \geq\, & C \int_0^T\!\!\int_0^1 \left[s\theta w_x^2 + s^3\theta^3 \left(\frac{x-x_0}{a}\right)^2 w^2 \right] dx\, dt \\
& + C \int_0^T\!\!\int_0^1 \left[s\theta z_x^2 + s^3\theta^3 \left(\frac{x-x_0}{a}\right)^2 z^2 \right] dx\, dt \\
& - s \int_0^T \theta \left[a \left(w_x^2 + z_x^2\right) \psi' \right]_{x=0}^{x=1} dt \\
& + \underbrace{\mu \left(\left\langle \frac{z}{d}, L_s^- w\right\rangle_{L^2_{1/a}(Q)} + \left\langle \frac{w}{d}, L_s^- z\right\rangle_{L^2_{1/a}(Q)} \right)}_{I}.
\end{aligned} \tag{3.10}$$

Integrating by parts, we decompose the term I into a sum of a distributed term I_d and a boundary term I_b where

$$\begin{aligned}
I_d &= -2s\mu \int_0^T\!\!\int_0^1 \frac{\varphi_x d'}{d^2} wz\, dx\, dt, \\
I_b &= \mu \int_0^1 \frac{1}{ad} [wz]_{t=0}^{t=T} dx - 2s\mu \int_0^T \left[\frac{\varphi_x}{d} wz \right]_{x=0}^{x=1} dt.
\end{aligned}$$

As in [25, Lemma 4.2], using the definition of φ and the boundary conditions on (w, z), the boundary terms reduce to 0.

On the other hand, by definition of φ and by the assumption on d, one has

$$\begin{aligned}
I_d &= -2s\mu d_1 \int_0^T\!\!\int_0^1 \theta \frac{(x-x_0)d'}{ad^2} e^{R(x-x_0)^2} wz\; dx\, dt \\
&\geq -2s\mu d_1 M \int_0^T\!\!\int_0^1 \frac{\theta}{ad} e^{R(x-x_0)^2} wz\, dx\, dt.
\end{aligned}$$

Next, using Young inequality, one can estimate $\int_0^T\int_0^1 \frac{\theta}{ad} e^{R(x-x_0)^2} wz\, dx\, dt$ as

$$\begin{aligned}
\left| \int_0^T\!\!\int_0^1 \frac{\theta}{ad} e^{R(x-x_0)^2} wz\, dx\, dt \right| &\leq C \int_0^T\!\!\int_0^1 \frac{\theta}{ad} |wz|\, dx\, dt \\
&= C \int_0^T\!\!\int_0^1 \left(\sqrt{\theta} \frac{|w|}{\sqrt{ad}} \right) \left(\sqrt{\theta} \frac{|z|}{\sqrt{ad}} \right) dx\, dt, \\
&\leq \frac{C}{2} \int_0^T\!\!\int_0^1 \theta \frac{w^2}{ad}\, dx\, dt + \frac{C}{2} \int_0^T\!\!\int_0^1 \theta \frac{z^2}{ad}\, dx\, dt,
\end{aligned}$$

and therefore from the Hardy–Poincaré inequality (2.4) we get

$$\left| \int_0^T \int_0^1 \frac{\theta}{ad} e^{R(x-x_0)^2} wz \, dx \, dt \right| \leq \frac{CC_d^\star}{2} \int_0^T \int_0^1 \theta w_x^2 \, dx \, dt + \frac{CC_d^\star}{2} \int_0^T \int_0^1 \theta z_x^2 \, dx \, dt.$$

Hence,

$$I_d \geq -s\mu d_1 MCC_d^\star \left(\int_0^T \int_0^1 \theta (w_x^2 + z_x^2) \, dx \, dt \right).$$

Proceeding as in [27, Lemma 3.7], we can choose C as large as desired, provided that s_0 increases as well, obtaining

$$I_d \geq -s\frac{C}{2} \left(\int_0^T \int_0^1 \theta (w_x^2 + z_x^2) \, dx \, dt \right).$$

Going back to (3.10) and taking into account the previous inequality, we deduce that there exist two positive constants C and s_0 such that for all $s \geq s_0$,

$$\begin{aligned} \langle \mathcal{L}_s^+ Z, \mathcal{L}_s^- Z \rangle_{\mathbb{H}^T_{1/a}} \geq{} & C \int_0^T \int_0^1 s\theta \left[w_x^2 + z_x^2 \right] \, dx \, dt \\ & + C \int_0^T \int_0^1 s^3 \theta^3 \left(\frac{x - x_0}{a} \right)^2 \left[w^2 + z^2 \right] \, dx \, dt \\ & - s \int_0^T \theta \left[a \left(w_x^2 + z_x^2 \right) \psi' \right]_{x=0}^{x=1} dt. \end{aligned} \tag{3.11}$$

Combining (3.9) and (3.11), we obtain

$$\begin{aligned} & \int_0^T \int_0^1 s\theta \left[w_x^2 + z_x^2 \right] + s^3 \theta^3 \left(\frac{x - x_0}{a} \right)^2 \left[w^2 + z^2 \right] \, dx \, dt \\ & \quad \leq C \bigg(\int_0^T \int_0^1 \left[h_1^2 + h_2^2 \right] \frac{e^{2s\varphi}}{a} \, dx \, dt + s \int_0^T \theta \left[a \left(w_x^2 + z_x^2 \right) \psi' \right]_{x=0}^{x=1} dt \bigg). \end{aligned}$$

Recall that $U = e^{-s\varphi} w$ and $V = e^{-s\varphi} z$. So, we have

$$\begin{aligned} U_x &= -s\theta \psi_x e^{-s\varphi} w + e^{-s\varphi} w_x, \\ V_x &= -s\theta \psi_x e^{-s\varphi} z + e^{-s\varphi} z_x. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left[s\theta (U_x^2 + V_x^2) + s^3 \theta^3 \left(\frac{x - x_0}{a} \right)^2 (U^2 + V^2) \right] e^{2s\varphi(t,x)} \\ & \quad \leq s\theta \left[2s^2 \theta^2 \psi_x^2 (w^2 + z^2) + 2(w_x^2 + z_x^2) \right] + s^3 \theta^3 \left(\frac{x - x_0}{a} \right)^2 (w^2 + z^2) \\ & \quad \leq C \left[s\theta (w_x^2 + z_x^2) + s^3 \theta^3 \left(\frac{x - x_0}{a} \right)^2 (w^2 + z^2) \right]. \end{aligned}$$

One thus obtains the asserted Carleman estimate for our original variables. $\square$

3.2 Carleman estimate with distributed observation for the homogeneous adjoint system

By the HUM method introduced by J.-L. Lions, the null controllability of problem (1.1)–(1.4) is equivalent to an observability estimate for the homogeneous backward system

$$U_t + a(x)U_{xx} + \frac{\lambda_1}{b_1}U + \frac{\mu}{d}V = 0, \quad (t,x) \in Q, \tag{3.12}$$
$$V_t + a(x)V_{xx} + \frac{\lambda_2}{b_2}V + \frac{\mu}{d}U = 0, \quad (t,x) \in Q, \tag{3.13}$$
$$U(t,1) = U(t,0) = V(t,1) = V(t,0) = 0, \quad t \in (0,T), \tag{3.14}$$
$$U(T,x) = U_T(x), \quad V(T,x) = V_T(x), \quad x \in (0,1). \tag{3.15}$$

To show that the adjoint system (3.12)–(3.15) is observable, we first derive an interesting Carleman estimate which could be used to show the null controllability for parabolic systems with two control forces. As a first step, consider the adjoint problem with more regular final datum

$$U_t + a(x)U_{xx} + \frac{\lambda_1}{b_1}U + \frac{\mu}{d}V = 0, \quad (t,x) \in Q, \tag{3.16}$$
$$V_t + a(x)V_{xx} + \frac{\lambda_2}{b_2}V + \frac{\mu}{d}U = 0, \quad (t,x) \in Q, \tag{3.17}$$
$$U(t,1) = U(t,0) = V(t,1) = V(t,0) = 0, \quad t \in (0,T), \tag{3.18}$$
$$\big(U(T,x) = U_T(x), \quad V(T,x) = V_T(x)\big) \in \mathcal{D}(\mathbb{A}^2), \quad x \in (0,1). \tag{3.19}$$

where $\mathcal{D}(\mathbb{A}^2) = \{X^T \in \mathcal{D}(\mathbb{A}) : (\mathbb{A}X)^T \in \mathcal{D}(\mathbb{A})\}$. Observe that $\mathcal{D}(\mathbb{A}^2)$ is densely defined in $\mathcal{D}(\mathbb{A})$ for the graph norm (see, e.g., [12, Lemma 7.2]) and hence in $\mathbb{H}_{1/a}$. As in [28] or [25], define

$$\mathcal{W} := \big\{(U,V) \text{ is a solution of (3.16)–(3.19)}\big\}.$$

Obviously (see, e.g., [12, Theorem 7.5]) $\mathcal{W} \subset C^1\big([0,T];\mathcal{D}(\mathbb{A})\big) \subset \mathcal{V} \subset \mathcal{U}$, where $\mathcal{V}$ is defined in (3.7) and

$$\mathcal{U} := C\big([0,T];\mathbb{H}_{1/a}\big) \cap L^2\big(0,T;\mathcal{K}_1^1 \times \mathcal{K}_2^1\big).$$

In order to prove the next result, we shall use the following non degenerate non singular classical Carleman estimate in suitable interval (A,B) (see [25, Proposition 4.1]).

Proposition 3.4. *Let z be the solution of*

$$z_t + az_{xx} + \frac{\lambda}{b(x)}z = h \in L^2((0,T)\times(A,B)), \quad x \in (A,B),\ t \in (0,T),$$
$$z(t,A) = z(t,B) = 0, \quad t \in (0,T),$$

where $a \in C^1([A,B])$, $b \in C([A,B])$ are in such a way that there exist two strictly positives constants a_0, b_0 such that $a \geq a_0$ and $b \geq b_0$ in $[A,B]$. Then there exist two positive constants r and s_0 such that for any $s > s_0$

$$\begin{aligned}\int_0^T\int_A^B \Big(s\theta z_x^2 + s^3\theta^3 z^2\Big)e^{2s\Phi}\,dx\,dt \\ \leq C\bigg(\int_0^T\int_A^B h^2 e^{2s\Phi}\,dx\,dt - sr\int_0^T \Big[a e^{2s\Phi(t,\cdot)}\theta e^{r\zeta_A} z_x^2(t,\cdot)\Big]_{x=A}^{x=B} dt\bigg),\end{aligned} \tag{3.20}$$

for some positive constant C. Here the functions Φ and ζ_A are defined as follows: For $x \in [A, B]$:

$$\begin{aligned} &\Phi(t,x) = \theta(t)\Psi(x), \quad \Psi(x) = e^{r\zeta_B(x)} - e^{2\rho}, \\ &\textit{where} \quad \zeta_B(x) = \mathfrak{d}\int_x^B \frac{dy}{a(y)}, \quad \rho = r\zeta_B(A), \end{aligned} \tag{3.21}$$

where $\mathfrak{d} = \|a'\|_{L^\infty(A,B)}$.

In the following we will assume that the parameters d_2, ρ and d_1 satisfy the following assumptions

$$d_2 > 16\tilde{d}_2^\star, \quad \rho > 2\ln(2), \tag{3.22}$$

and

$$d_1 \in I = \left[\frac{e^{2\rho}-1}{d_2 - \tilde{d}_2^\star}, \frac{4}{3d_2}(e^{2\rho} - e^{\rho})\right) \tag{3.23}$$

which can be shown not empty.

We shall begin by proving a simple but fundamental lemma concerning some properties that must be satisfied by the weight functions.

Lemma 3.5. *By (3.22)–(3.23), we have*

(i) For $(t, x) \in [0, T] \times [0, 1]$,

$$\begin{aligned} \varphi(t,x) \le \Phi(t,x) \quad &\textit{and} \\ -4\Phi(t,x) + 3\varphi(t,x) > 0. \end{aligned} \tag{3.24}$$

(ii) For $(t, x) \in [0, T] \times [0, 1]$,

$$\varphi(t,-x) \le \Phi(t,x). \tag{3.25}$$

Proof. First, let us set $d_2^\star := \max_{x\in[0,1]} \int_{x_0}^x \frac{y-x_0}{a(y)} e^{R(y-x_0)^2} dy$.

(i) 1. $\varphi \le \Phi$: since $d_1 \ge \frac{e^{2\rho}-1}{d_2 - \tilde{d}_2^\star} \ge \frac{e^{2\rho}-1}{d_2 - d_2^\star}$, we have $\max\{\psi(0), \psi(1)\} \le \Psi(1)$ and the conclusion follows immediately.

2. $-4\Phi(t, x) + 3\varphi(t, x) > 0$: this follows easily by the assumption $d_1 d_2 < -\frac{4}{3}\Psi(0)$.

(ii) $\varphi(t, -x) \le \Phi(t, x)$: since $d_1 \ge \frac{e^{2\rho}-1}{d_2 - \tilde{d}_2^\star}$, then $\max\{\psi(-1), \psi(0)\} \le \Psi(1)$ which completes the proof of the desired result. □

Now, we shall apply the just established Carleman inequalities with boundary observation to obtain a Carleman estimate with locally distributed observation. For this, we assume that the control set ω satisfies the following assumption:

Hypothesis 3.6. *The control set ω is such that*

$$\omega = \omega_1 \cup \omega_2,$$

where ω_i $(i = 1, 2)$ are intervals with $\omega_1 \subset\subset (0, x_0)$, $\omega_2 \subset\subset (x_0, 1)$, and $x_0 \notin \bar{\omega}$.

We claim the following.

Theorem 3.7. *Let $T > 0$ be given. Assume Hypotheses 3.1 and 3.6. Then there exist two positive constants C and s_0 such that every solution $(U, V) \in \mathcal{W}$ of (3.16)–(3.19) satisfies, for all $s \geq s_0$,*

$$\begin{aligned}\int_0^T\int_0^1 &\left[s\theta(U_x^2+V_x^2)+s^3\theta^3\left(\frac{x-x_0}{a}\right)^2(U^2+V^2)\right]e^{2s\varphi(t,x)}\,dx\,dt\\ &\leq C\int_0^T\int_\omega s^2\theta^2[U^2+V^2]\frac{e^{2s\Phi}}{a}\,dx\,dt.\end{aligned}\tag{3.26}$$

We remark that Theorem 3.7 has an immediate application also in the case in which the control set ω is an interval containing the degeneracy point. Indeed, if $x_0 \in \omega$ one can always find two subintervals $\omega_1 \subset (0, x_0)$, $\omega_2 =\subset (x_0, 1)$ such that $(\omega_1 \cup \omega_2) \subset\subset \omega \setminus \{x_0\}$.

Proof of Theorem 3.7. The statement is obtained by studying some auxiliary problems, introduced with suitable cut-off functions and a reflection procedure already introduced in [29]. First, by the assumption on the control set, we have $\omega_1 := (\alpha_1, \beta_1) \subset\subset (0, x_0)$, $\omega_2 := (\alpha_2, \beta_2) \subset\subset (x_0, 1)$. Let us fix four points $\gamma_i = \frac{2\alpha_i+\beta_i}{3}$ and $\gamma_i' = \frac{\alpha_i+2\beta_i}{3}$, so that $\alpha_i < \gamma_i < \gamma_i' < \beta_i$, for $i = 1, 2$. Then, fix $\tilde\beta_2 \in (\alpha_2, \gamma_2)$ and consider a smooth function $\eta : [0, 1] \to [0, 1]$ such that

$$\eta(x) = \begin{cases}1, & x \in [\gamma_2, 1],\\ 0, & x \in [0, \tilde\beta_2].\end{cases}$$

Define $(\hat p, \hat q) := (\eta U, \eta V)$, where (U, V) is any fixed solution of (3.16)–(3.19). Hence, fixed $\tilde\alpha_2 \in (\alpha_2, \tilde\beta_2)$, $(\hat p, \hat q)$ satisfies

$$\begin{aligned}\hat p_t + a\hat p_{xx} + \frac{\lambda_1}{b_1}\hat p &= -\frac{\mu}{d}\hat q + a(\eta_{xx}U + 2\eta_x U_x) := H_1, \quad (t,x) \in (0,T)\times(\tilde\alpha_2, 1),\\ \hat q_t + a\hat q_{xx} + \frac{\lambda_2}{b_2}\hat q &= -\frac{\mu}{d}\hat p + a(\eta_{xx}V + 2\eta_x V_x) := H_2, \quad (t,x) \in (0,T)\times(\tilde\alpha_2, 1),\\ \hat p(t,\tilde\alpha_2) = \hat p(t,1) &= \hat q(t,\tilde\alpha_2) = \hat q(t,1) = 0, \quad t \in (0,T)\end{aligned}$$

with $H_1, H_2 \in L^2((0, T) \times (\tilde\alpha_2, 1))$.

Since $x \in (\tilde\alpha_2, 1)$, observe that the system above is a nondegenerate and nonsingular problem. Thus, we can apply the analogue of Proposition 3.4 for the first component $\hat p$ in $(\tilde\alpha_2, 1)$ place of (A, B), obtaining that there exist two positive constants C and s_0 (s_0 sufficiently large), such that $\hat p$ satisfies, for all $s \geq s_0$,

$$\int_0^T\int_{\tilde\alpha_2}^1\left[s\theta\hat p_x^2+s^3\theta^3\hat p^2\right]e^{2s\Phi}\,dx\,dt \leq C\int_0^T\int_{\tilde\alpha_2}^1 H_1^2e^{2s\Phi}\,dx\,dt.$$

Let us remark that the boundary term in $x = 1$ is nonpositive, while the one in $x = \tilde\alpha_2$ is 0, so that they can be neglected in the classical Carleman estimate.

Then using the definition of η and in particular the fact that η_x and η_{xx} are supported in $\breve\omega := (\tilde\beta_2, \gamma_2) \subset\subset \omega_2$ where a, and $\frac{1}{d}$ are bounded, we can write

$$H_1^2 \leq c\hat q^2 + C(U^2 + U_x^2)1_{\breve\omega},$$

for positive constants c and C.

Hence, we find

$$\int_0^T\int_{\tilde\alpha_2}^1\left[s\theta\hat p_x^2+s^3\theta^3\hat p^2\right]e^{2s\Phi}\,dx\,dt \leq c\int_0^T\int_{\tilde\alpha_2}^1\hat q^2e^{2s\Phi}\,dx\,dt + C\int_0^T\int_{\breve\omega}[U^2+U_x^2]e^{2s\Phi}\,dx\,dt.$$

Analogously, one can prove that $\hat{q}$ satisfies

$$\int_0^T\int_{\tilde{\alpha}_2}^1 \left[s\theta\hat{q}_x^2 + s^3\theta^3\hat{q}^2\right]e^{2s\Phi}\,dx\,dt \leq c\int_0^T\int_{\tilde{\alpha}_2}^1 \hat{p}^2 e^{2s\Phi}\,dx\,dt + C\int_0^T\int_{\breve{\omega}}[V^2+V_x^2]e^{2s\Phi}\,dx\,dt.$$

Thus, summing the last two inequalities, it follows that

$$\begin{aligned}
&\int_0^T\int_{\tilde{\alpha}_2}^1 \left[s\theta(\hat{p}_x^2+\hat{q}_x^2) + s^3\theta^3(\hat{p}^2+\hat{q}^2)\right]e^{2s\Phi}\,dx\,dt \\
&\quad \leq \tilde{C}\int_0^T\int_{\tilde{\alpha}_2}^1 [\hat{p}^2+\hat{q}^2]e^{2s\Phi}\,dx\,dt + C\int_0^T\int_{\breve{\omega}}[(U^2+V^2)+(U_x^2+V_x^2)]e^{2s\Phi}\,dx\,dt,
\end{aligned}$$

where $\tilde{C}$ and C are some universal positive constants.

Taking s such that $\tilde{C} \leq \frac{1}{2}s^3\theta^3$, we obtain

$$\int_0^T\int_{\tilde{\alpha}_2}^1 \left[s\theta(\hat{p}_x^2+\hat{q}_x^2) + s^3\theta^3(\hat{p}^2+\hat{q}^2)\right]e^{2s\Phi}\,dx\,dt \leq C\int_0^T\int_{\breve{\omega}}[U^2+V^2+U_x^2+V_x^2]e^{2s\Phi}\,dx\,dt.$$

Now, by the first inequality in (3.24), one can prove that there exists a positive constant k, such that for every $(t,x) \in [0,T]\times[\tilde{\alpha}_2, 1]$

$$e^{2s\varphi(t,x)} \leq k e^{2s\Phi(t,x)}, \quad \left(\frac{x-x_0}{a(x)}\right)^2 e^{2s\varphi(t,x)} \leq k e^{2s\Phi(t,x)}. \tag{3.27}$$

Hence, by (3.27) and using the definitions of $\hat{p}$ and $\hat{q}$, it results

$$\begin{aligned}
&\int_0^T\int_{\gamma_2}^1 \left[s\theta(U_x^2+V_x^2) + s^3\theta^3\left(\frac{x-x_0}{a}\right)^2(U^2+V^2)\right]e^{2s\varphi}\,dx\,dt \\
&\quad = \int_0^T\int_{\gamma_2}^1 \left[s\theta(\hat{p}_x^2+\hat{q}_x^2) + s^3\theta^3\left(\frac{x-x_0}{a}\right)^2(\hat{p}^2+\hat{q}^2)\right]e^{2s\varphi}\,dx\,dt \\
&\quad \leq \int_0^T\int_{\tilde{\alpha}_2}^1 \left[s\theta(\hat{p}_x^2+\hat{q}_x^2) + s^3\theta^3\left(\frac{x-x_0}{a}\right)^2(\hat{p}^2+\hat{q}^2)\right]e^{2s\varphi}\,dx\,dt \\
&\quad \leq k\int_0^T\int_{\tilde{\alpha}_2}^1 \left[s\theta(\hat{p}_x^2+\hat{q}_x^2) + s^3\theta^3(\hat{p}^2+\hat{q}^2)\right]e^{2s\Phi}\,dx\,dt \\
&\quad \leq C\int_0^T\int_{\breve{\omega}}[U^2+V^2+U_x^2+V_x^2]e^{2s\Phi}\,dx\,dt \\
&\quad \leq C\int_0^T\int_{\breve{\omega}}\left[(U^2+V^2)\frac{1}{a} + (U_x^2+V_x^2)\right]e^{2s\Phi}\,dx\,dt \\
&\quad \leq C\int_0^T\int_{\omega}(U^2+V^2)\frac{e^{2s\Phi}}{a}\,dx\,dt + C\int_0^T\int_{\breve{\omega}}(U_x^2+V_x^2)e^{2s\Phi}\,dx\,dt.
\end{aligned}$$

Consequently, by Lemma 5.1 and by the inequality above, we get

$$\begin{aligned}
&\int_0^T\int_{\gamma_2}^1 \left[s\theta(U_x^2+V_x^2) + s^3\theta^3\left(\frac{x-x_0}{a}\right)^2(U^2+V^2)\right]e^{2s\varphi}\,dx\,dt \\
&\quad \leq C\int_0^T\int_{\omega}(U^2+V^2)\frac{e^{2s\Phi}}{a}\,dx\,dt + C\int_0^T\int_{\omega_2} s^2\theta^2[U^2+V^2]\frac{e^{2s\Phi}}{a}\,dx\,dt \\
&\quad \leq C\int_0^T\int_{\omega} s^2\theta^2[U^2+V^2]\frac{e^{2s\Phi}}{a}\,dx\,dt,
\end{aligned} \tag{3.28}$$

for a positive constant C.

To complete the proof it is sufficient to prove a similar inequality on the interval $[0,\gamma_2]$. To this aim, we perform a reflection procedure already introduced in [29], considering the functions

$$W(t,x) := \begin{cases} U(t,x), & x \in [0,1], \\ -U(t,-x), & x \in [-1,0], \end{cases} \qquad Z(t,x) := \begin{cases} V(t,x), & x \in [0,1], \\ -V(t,-x), & x \in [-1,0], \end{cases}$$

where (U,V) solves (3.16)–(3.19). Therefore, (W,Z) solves the system

$$\begin{aligned} &W_t + \tilde{a}W_{xx} + \frac{\lambda_1}{\tilde{b}_1}W + \frac{\mu}{\tilde{d}}Z = 0, \quad (t,x) \in (0,T)\times(-1,1), \\ &Z_t + \tilde{a}Z_{xx} + \frac{\lambda_2}{\tilde{b}_2}Z + \frac{\mu}{\tilde{d}}W = 0, \quad (t,x) \in (0,T)\times(-1,1), \\ &W(t,-1) = W(t,1) = Z(t,-1) = Z(t,1) = 0, \quad t \in (0,T), \end{aligned} \tag{3.29}$$

being

$$\tilde{b}_i(x) := \begin{cases} b_i(x), & x \in [0,1], \\ b_i(-x), & x \in [-1,0], \end{cases} \qquad \tilde{d}(x) = \begin{cases} d(x), & x \in [0,1], \\ d(-x), & x \in [-1,0], \end{cases}$$

and $\tilde{a}$ is already defined in (3.6).

Now, consider a smooth function $\tau : [-1,1] \to [0,1]$ such that

$$\tau(x) = \begin{cases} 1, & x \in [-\gamma_1,\gamma_2], \\ 0, & x \in [-1,-\gamma_1'] \cup [\gamma_2',1], \end{cases}$$

and define the functions $\check{p} = \tau W$ and $\check{q} = \tau Z$, where (W,Z) is the solution of (3.29). Then $(\check{p},\check{q})$ satisfies

$$\begin{aligned} &\check{p}_t + \tilde{a}\check{p}_{xx} + \frac{\lambda_1}{\tilde{b}_1}\check{p} + \frac{\mu}{\tilde{d}}\check{q} = \tilde{a}(\tau_{xx}W + 2\tau_x W_x) := F_1, \quad (t,x) \in (0,T)\times(-\beta_1,1), \\ &\check{q}_t + \tilde{a}\check{q}_{xx} + \frac{\lambda_2}{\tilde{b}_2}\check{q} + \frac{\mu}{\tilde{d}}\check{p} = \tilde{a}(\tau_{xx}Z + 2\tau_x Z_x) := F_2, \quad (t,x) \in (0,T)\times(-\beta_1,1), \\ &(\check{p},\check{q})(t,-\beta_1) = (\check{p},\check{q})(t,1) = 0, \quad t \in (0,T). \end{aligned}$$

Observe that $\check{p}_x(t,-\beta_1) = \check{p}_x(t,1) = \check{q}_x(t,-\beta_1) = \check{q}_x(t,1) = 0$ and, by the assumption on a and the fact that τ_x, τ_{xx} are supported in $[-\gamma_1',-\gamma_1] \cup [\gamma_2,\gamma_2']$, $F_1, F_2 \in L^2_{1/\tilde{a}}((0,T)\times I)$, where $I := (-\beta_1,1)$. Thus, we can apply the analogue of Theorem 3.2 (which still holds true, since $\tilde{a}$ belongs to $W^{1,1}(-1,1)$ in the weakly degenerate case and to $W^{1,\infty}(-1,1)$ in the strongly degenerate one, see [12, Lemma 9.2]) on $(-\beta_1,1)$ in place of $(0,1)$, obtaining that there exist two positive constants C and s_0 (s_0 sufficiently large), such that $(\check{p},\check{q})$ satisfies, for all $s \geq s_0$,

$$\int_0^T\int_{-\beta_1}^1 \left[s\theta(\check{p}_x^2 + \check{q}_x^2) + s^3\theta^3\left(\frac{x-x_0}{\tilde{a}}\right)^2(\check{p}^2 + \check{q}^2)\right]e^{2s\varphi}\,dx\,dt \leq C\int_0^T\int_{-\beta_1}^1 (F_1^2 + F_2^2)\frac{e^{2s\varphi}}{\tilde{a}}\,dx\,dt.$$

Using again the fact that τ_x, τ_{xx} are supported in $[-\gamma_1',-\gamma_1] \cup [\gamma_2,\gamma_2']$, it follows that

$$\begin{aligned} &\int_0^T\int_{-\beta_1}^1 \left[s\theta(\check{p}_x^2 + \check{q}_x^2) + s^3\theta^3\left(\frac{x-x_0}{\tilde{a}}\right)^2(\check{p}^2 + \check{q}^2)\right]e^{2s\varphi}\,dx\,dt \\ &\leq C\left[\int_0^T\int_{-\gamma_1'}^{-\gamma_1}[W^2 + W_x^2 + Z^2 + Z_x^2]e^{2s\varphi}\,dx\,dt \right.\\ &\qquad \left. + \int_0^T\int_{\gamma_2}^{\gamma_2'}[W^2 + W_x^2 + Z^2 + Z_x^2]e^{2s\varphi}\,dx\,dt\right]. \end{aligned} \tag{3.30}$$

Now, by the definitions of W and Z, we note that

$$\begin{aligned}
&\int_0^T\int_{-\gamma_1'}^{-\gamma_1}[W^2+W_x^2+Z^2+Z_x^2]e^{2s\varphi(x)}\,dx\,dt\\
&\quad=\int_0^T\int_{-\gamma_1'}^{-\gamma_1}[U^2(-x)+U_x^2(-x)+V^2(-x)+V_x^2(-x)]e^{2s\varphi(x)}\,dx\,dt.
\end{aligned}\tag{3.31}$$

On the other hand, using a change of variable, one has

$$\begin{aligned}
&\int_0^T\int_{-\gamma_1'}^{-\gamma_1}[U^2(-x)+U_x^2(-x)+V^2(-x)+V_x^2(-x)]e^{2s\varphi(x)}\,dx\,dt\\
&\quad=\int_0^T\int_{\gamma_1}^{\gamma_1'}[U^2(x)+U_x^2(x)+V^2(x)+V_x^2(x)]e^{2s\varphi(-x)}\,dx\,dt.
\end{aligned}$$

At this point, we use (3.25) to deduce that

$$\begin{aligned}
&\int_0^T\int_{-\gamma_1'}^{-\gamma_1}[U^2(-x)+U_x^2(-x)+V^2(-x)+V_x^2(-x)]e^{2s\varphi(x)}\,dx\,dt\\
&\quad\le\int_0^T\int_{\gamma_1}^{\gamma_1'}[U^2+U_x^2+V^2+V_x^2]e^{2s\Phi(x)}\,dx\,dt.
\end{aligned}\tag{3.32}$$

Combining (3.31) and (3.32), it follows that

$$\int_0^T\int_{-\gamma_1'}^{-\gamma_1}[W^2+W_x^2+Z^2+Z_x^2]e^{2s\varphi(x)}\,dx\,dt\le\int_0^T\int_{\gamma_1}^{\gamma_1'}[U^2+U_x^2+V^2+V_x^2]e^{2s\Phi(x)}\,dx\,dt.\tag{3.33}$$

Going back to (3.30), by (3.33) and using the fact that $\varphi\le\Phi$, we obtain

$$\begin{aligned}
&\int_0^T\int_{-\beta_1}^1\left(s\theta(\check{p}_x^2+\check{q}_x^2)e^{2s\varphi}+s^3\theta^3\Big(\frac{x-x_0}{\tilde a}\Big)^2(\check{p}^2+\check{q}^2)\right)e^{2s\varphi}\,dx\,dt\\
&\quad\le C\left[\int_0^T\int_{\gamma_1}^{\gamma_1'}[U^2+U_x^2+V^2+V_x^2]e^{2s\Phi}\,dx\,dt+\int_0^T\int_{\gamma_2}^{\gamma_2'}[U^2+U_x^2+V^2+V_x^2]e^{2s\Phi}\,dx\,dt\right]\\
&\quad\le C\left[\int_0^T\left(\int_{\gamma_1}^{\gamma_1'}+\int_{\gamma_2}^{\gamma_2'}\right)[U^2+V^2]\frac{1}{a}e^{2s\Phi}\,dx\,dt+\int_0^T\left(\int_{\gamma_1}^{\gamma_1'}+\int_{\gamma_2}^{\gamma_2'}\right)[U_x^2+V_x^2]e^{2s\Phi}\,dx\,dt\right]\\
&\quad\le C\left[\int_0^T\left(\int_{\omega_1}+\int_{\omega_2}\right)[U^2+V^2]\frac{1}{a}e^{2s\Phi}\,dx\,dt+\int_0^T\left(\int_{\gamma_1}^{\gamma_1'}+\int_{\gamma_2}^{\gamma_2'}\right)[U_x^2+V_x^2]e^{2s\Phi}\,dx\,dt\right].
\end{aligned}$$

Thus, applying the Caccioppoli inequality given in Lemma 5.1, one gets

$$\begin{aligned}
&\int_0^T\int_{-\beta_1}^1\left(s\theta(\check{p}_x^2+\check{q}_x^2)e^{2s\varphi}+s^3\theta^3\Big(\frac{x-x_0}{\tilde a}\Big)^2(\check{p}^2+\check{q}^2)\right)e^{2s\varphi}\,dx\,dt\\
&\quad\le C\left[\int_0^T\left(\int_{\omega_1}+\int_{\omega_2}\right)[U^2+V^2]\frac{1}{a}e^{2s\Phi}\,dx\,dt+\int_0^T\left(\int_{\omega_1}+\int_{\omega_2}\right)s^2\theta^2[U^2+V^2]\frac{e^{2s\Phi}}{a}\,dx\,dt\right]\\
&\quad\le C\int_0^T\left(\int_{\omega_1}+\int_{\omega_2}\right)s^2\theta^2[U^2+V^2]\frac{e^{2s\Phi}}{a}\,dx\,dt\\
&\quad\le C\int_0^T\int_{\omega}s^2\theta^2[U^2+V^2]\frac{e^{2s\Phi}}{a}\,dx\,dt.
\end{aligned}$$

Hence, using the definitions of W, Z, $\check{p}$ and $\check{q}$, it results

$$\begin{aligned}
&\int_0^T\int_0^{\gamma_2}\left(s\theta(U_x^2+V_x^2)+s^3\theta^3C\Big(\frac{x-x_0}{a}C\Big)^2(U^2+V^2)\right)e^{2s\varphi}\,dx\,dt\\
&\quad=\int_0^T\int_0^{\gamma_2}\left(s\theta(W_x^2+Z_x^2)+s^3\theta^3C\Big(\frac{x-x_0}{\tilde a}C\Big)^2(W^2+Z^2)\right)e^{2s\varphi}\,dx\,dt\\
&\quad=\int_0^T\int_0^{\gamma_2}\left(s\theta(\check p_x^2+\check q_x^2)+s^3\theta^3C\Big(\frac{x-x_0}{\tilde a}C\Big)^2(\check p^2+\check q^2)\right)e^{2s\varphi}\,dx\,dt\\
&\quad\le\int_0^T\int_{-\beta_1}^{1}\left(s\theta(\check p_x^2+\check q_x^2)+s^3\theta^3C\Big(\frac{x-x_0}{\tilde a}C\Big)^2(\check p^2+\check q^2)\right)e^{2s\varphi}\,dx\,dt\\
&\quad\le C\int_0^T\int_\omega s^2\theta^2[U^2+V^2]\frac{e^{2s\Phi}}{a}\,dx\,dt, \qquad (3.34)
\end{aligned}$$

for all $s\ge s_0$ and for a positive constant C.

Finally adding up (3.28) and (3.34), Theorem 3.7 follows. □

The above Carleman estimate can be used to obtain null controllability of system (1.1)–(1.4) if we apply 2 control forces, but to obtain this aim only with one control force, we need to eliminate the second local term from the right side of (3.26). In order to carry this process out, we will need the following result.

Lemma 3.8. *Let $\varepsilon>0$ and consider an open set ω_0 such that $\omega_0\subset\subset\omega$. Then, there is $C_\varepsilon>0$ such that every solution (U,V) to (3.16)–(3.19) satisfies*

$$\int_0^T\int_{\omega_0}s^2\theta^2V^2\frac{e^{2s\Phi}}{a}\,dx\,dt\le\varepsilon J(V)+C_\varepsilon\int_0^T\int_\omega\frac{U^2}{a}\,dx\,dt,$$

where $\varepsilon>0$ is small enough, s is large enough and

$$J(V)=\int_0^T\int_0^1\left(s\theta V_x^2+s^3\theta^3\Big(\frac{x-x_0}{a}\Big)^2V^2\right)e^{2s\varphi}\,dx\,dt.$$

Proof. Let $\chi\in C^\infty(0,1)$, such that $0\le\chi\le1$ in $(0,1)$, $\mathrm{supp}(\chi)\subset\omega$ and $\chi\equiv1$ on ω_0. Multiplying the first equation of system (3.16)–(3.19) by $s^2\theta^2\chi\frac{e^{2s\Phi}}{a}V$ and integrating over Q, we have

$$\begin{aligned}
\iint_Q s^2\theta^2\frac{\mu}{d}\chi\frac{e^{2s\Phi}}{a}V^2\,dx\,dt=&-\iint_Q s^2\theta^2\chi\frac{e^{2s\Phi}}{a}U_tV\,dx\,dt\\
&-\iint_Q s^2\theta^2\chi e^{2s\Phi}U_{xx}V\,dx\,dt \qquad (3.35)\\
&-\iint_Q s^2\theta^2\frac{\lambda_1}{b_1}\chi\frac{e^{2s\Phi}}{a}UV\,dx\,dt.
\end{aligned}$$

After integrating in time and having in mind the equation satisfied by V, we get

$$\begin{aligned}
&-\iint_Q s^2\theta^2\chi\frac{e^{2s\Phi}}{a}U_tV\,dx\,dt\\
&\quad=\iint_Q s^2\theta^2\chi e^{2s\Phi}U_xV_x\,dx\,dt+\iint_Q s^2\theta^2(\chi e^{2s\Phi})_xUV_x\,dx\,dt\\
&\qquad+\iint_Q\left[-s^2\theta^2\frac{\lambda_2}{b_2}+2s^3\theta^2\dot\theta\Psi+2s^2\theta\dot\theta\right]\chi\frac{e^{2s\Phi}}{a}UV\,dx\,dt-\iint_Q s^2\theta^2\frac{\mu}{d}\chi\frac{e^{2s\Phi}}{a}U^2\,dx\,dt, \qquad (3.36)
\end{aligned}$$

and

$$\iint_Q s^2\theta^2\chi e^{2s\Phi}U_{xx}V\,dx\,dt = -\iint_Q s^2\theta^2\chi e^{2s\Phi}U_xV_x\,dx\,dt + \iint_Q s^2\theta^2(\chi e^{2s\Phi})_xUV_x\,dx\,dt + \iint_Q s^2\theta^2(\chi e^{2s\Phi})_{xx}UV\,dx\,dt. \tag{3.37}$$

Altogether from (3.35)-(3.37), we obtain

$$\iint_Q s^2\theta^2\frac{\mu}{d}\chi\frac{e^{2s\Phi}}{a}V^2\,dx\,dt = K_1+K_2+K_3,$$

where

$$\begin{aligned}
K_1 &= 2\iint_Q s^2\theta^2\chi e^{2s\Phi}U_xV_x\,dx\,dt,\\
K_2 &= -\iint_Q s^2\theta^2\frac{\mu}{d}\chi\frac{e^{2s\Phi}}{a}U^2\,dx\,dt,\\
K_3 &= \iint_Q\left[-s^2\theta^2\left(\frac{\lambda_1}{b_1}+\frac{\lambda_2}{b_2}\right)+2s^3\theta^2\dot\theta\Psi+2s^2\theta\dot\theta\right]\chi\frac{e^{2s\Phi}}{a}UV\,dx\,dt\\
&\quad-\iint_Q s^2\theta^2(\chi e^{2s\Phi})_{xx}UV\,dx\,dt.
\end{aligned}$$

For $\varepsilon>0$, using Young's inequality, we have

$$\begin{aligned}
|K_1| &= 2\left|\iint_Q(\sqrt{s\theta}e^{s\varphi}V_x)((s\theta)^{\frac{3}{2}}\chi e^{s(2\Phi-\varphi)}U_x)\,dx\,dt\right|\\
&\le \varepsilon\iint_Q s\theta e^{2s\varphi}V_x^2\,dx\,dt+\frac{1}{\varepsilon}\underbrace{\iint_Q s^3\theta^3\chi^2e^{2s(2\Phi-\varphi)}U_x^2\,dx\,dt}_{L}.
\end{aligned}$$

In the last inequality, we still have to estimate L by an integral in U^2. For this, we multiply the equation by U by $s^3\theta^3\chi^2\frac{e^{2s(2\Phi-\varphi)}}{a}U$ and integrate on Q to obtain

$$L = L_1+L_2+L_3+L_4,$$

where

$$\begin{aligned}
L_1 &= -\frac{1}{2}\iint_Q s^3(3\theta^2+2s\theta^3(2\Psi-\psi))\dot\theta\chi^2\times\frac{e^{2s(2\Phi-\varphi)}}{a}U^2\,dx\,dt,\\
L_2 &= \frac{1}{2}\iint_Q s^3\theta^3(\chi^2e^{2s(2\Phi-\varphi)})_{xx}U^2\,dx\,dt,\\
L_3 &= \iint_Q s^3\theta^3\chi^2\frac{\lambda_1}{b_1}\frac{e^{2s(2\Phi-\varphi)}}{a}U^2\,dx\,dt,\\
L_4 &= \iint_Q s^3\theta^3\chi^2\frac{\mu}{d}\frac{e^{2s(2\Phi-\varphi)}}{a}UV\,dx\,dt.
\end{aligned}$$

Since $\mathrm{supp}(\chi)\subset\omega$, we observe that the functions a, $\frac{1}{b_i}$, $\frac{1}{d}$, χ, ψ, Ψ and their derivatives are bounded on ω. Then, by the fact that $|\dot\theta|\le C\theta^2$, we deduce that, for $i\in\{1,2,3\}$

$$|L_i|\le C\int_0^T\!\!\int_\omega s^5\theta^5\frac{e^{2s(2\Phi-\varphi)}}{a}U^2\,dx\,dt.$$

For $i = 4$, one can see that

$$\begin{aligned}
|L_4| &= \left| \iint_Q \left[(s\theta)^{\frac{3}{2}} \frac{(x-x_0)}{a} e^{s\varphi} V \right] \left[(s\theta)^{\frac{3}{2}} \frac{\mu}{d} \chi^2 \frac{1}{(x-x_0)} e^{s(4\Phi-3\varphi)} U \right] \, dx\, dt \right| \\
&\le \varepsilon^2 \iint_Q s^3\theta^3 \left(\frac{x-x_0}{a} \right)^2 V^2 e^{2s\varphi} \, dx\, dt \\
&\quad + \frac{1}{4\varepsilon^2} \iint_Q s^3\theta^3 \left(\frac{\mu}{d} \right)^2 \chi^4 \frac{1}{(x-x_0)^2} e^{2s(4\Phi-3\varphi)} U^2 \, dx\, dt \\
&\le \varepsilon^2 \iint_Q s^3\theta^3 \left(\frac{x-x_0}{a} \right)^2 V^2 e^{2s\varphi} \, dx\, dt + C_\varepsilon \int_0^T\!\!\int_\omega s^3\theta^3 \frac{e^{2s(4\Phi-3\varphi)}}{a} U^2 \, dx\, dt.
\end{aligned}$$

Hence,

$$|L| \le C_\varepsilon \int_0^T\!\!\int_\omega s^5\theta^5 \frac{e^{2s(4\Phi-3\varphi)}}{a} U^2 \, dx\, dt + \varepsilon^2 \iint_Q s^3\theta^3 \left(\frac{x-x_0}{a} \right)^2 V^2 e^{2s\varphi} \, dx\, dt.$$

Therefore,

$$|K_1| \le C_\varepsilon \int_0^T\!\!\int_\omega s^5\theta^5 \frac{e^{2s(4\Phi-3\varphi)}}{a} U^2 \, dx\, dt + \varepsilon J(V).$$

Using the fact that χ' and χ are supported in ω and $x_0 \notin \omega$, proceeding as before, one obtains

$$\begin{aligned}
|K_2| &\le C \int_0^T\!\!\int_\omega s^2\theta^2 \frac{e^{2s\Phi}}{a} U^2 \, dx\, dt, \\
|K_3| &\le C \iint_Q s^4\theta^4 (\chi'' + \chi' + \chi) \frac{e^{2s\Phi}}{a} UV \, dx\, dt \\
&\le C \iint_Q \left((s\theta)^{\frac{3}{2}} \frac{x-x_0}{a} e^{s\varphi} V \right) \left((s\theta)^{\frac{5}{2}} \frac{1}{(x-x_0)} (\chi'' + \chi' + \chi) e^{s(2\Phi-\varphi)} U \right) dx\, dt \\
&\le \varepsilon \iint_Q s^3\theta^3 \left(\frac{x-x_0}{a} \right)^2 V^2 e^{2s\varphi} \, dx\, dt + C_\varepsilon \int_0^T\!\!\int_\omega s^5\theta^5 \frac{e^{2s(2\Phi-\varphi)}}{a} U^2 \, dx\, dt.
\end{aligned}$$

Furthermore, thanks to Lemma 3.5, we have

$$\begin{aligned}
&e^{2s\Phi} \le e^{2s(2\Phi-\varphi)} \le e^{2s(4\Phi-3\varphi)} \le 1, \\
&\sup_{(t,x)\in Q} s^r\theta^r(t) e^{2s(4\Phi-3\varphi)} < \infty, \quad r \in \mathbb{R}.
\end{aligned}$$

Then, for ε small enough and s large enough, we have

$$\left| \iint_Q s^2\theta^2 \frac{\mu}{d} \chi \frac{e^{2s\Phi}}{a} V^2 \, dx\, dt \right| \le C_\varepsilon \int_0^T\!\!\int_\omega \frac{U^2}{a} \, dx\, dt + 2\varepsilon J(V).$$

Finally, by the definition of χ and the previous inequality, it follows that

$$\begin{aligned}
\frac{\mu}{\max\limits_{x\in\omega_0} d(x)} \int_0^T\!\!\int_{\omega_0} s^2\theta^2 \frac{e^{2s\Phi}}{a} V^2 \, dx\, dt &\le \left| \int_0^T\!\!\int_{\omega_0} s^2\theta^2 \frac{\mu}{d} \chi \frac{e^{2s\Phi}}{a} V^2 \, dx\, dt \right| \\
&\le \left| \iint_Q s^2\theta^2 \frac{\mu}{d} \chi \frac{e^{2s\Phi}}{a} V^2 \, dx\, dt \right| \\
&\le C_\varepsilon \int_0^T\!\!\int_\omega \frac{U^2}{a} \, dx\, dt + \varepsilon J(V).
\end{aligned}$$

This ends the proof. □

Now, we apply inequality (3.26) with ω_0 to obtain the following main Carleman estimate for the adjoint system which bounds the global integrals of the variable (U,V) in terms of a unique localized variable.

Theorem 3.9. *Let $T>0$. Then there exist two positive constants C and s_0 such that, for all $s\geq s_0$, the solution $(U,V)\in\mathcal{W}$ of (3.16)–(3.19) satisfies*

$$\int_0^T\!\!\int_0^1\Big[s\theta(U_x^2+V_x^2)+s^3\theta^3\Big(\frac{x-x_0}{a}\Big)^2(U^2+V^2)\Big]e^{2s\varphi(t,x)}\,dx\,dt\leq C\int_0^T\!\!\int_\omega\frac{U^2}{a}\,dx\,dt. \tag{3.38}$$

4 Application to observability inequality

In this section, we investigate the observability inequality for the problem (3.12)–(3.15) and deduce the null controllability for the problem (1.1)–(1.4). In particular, using the local Carleman estimate in Theorem 3.9, we will prove the next observability inequality:

Theorem 4.1. *Assume Hypotheses 3.1 and 3.6. Then there exists a positive constant C_T such that every $(U,V)\in C([0,T];\mathbb{H}_{1/a})\cap L^2(0,T;\mathcal{K}_1^1\times\mathcal{K}_2^1)$ solution of (3.12)–(3.15) satisfies*

$$\int_0^1[U^2(0,x)+V^2(0,x)]\frac{1}{a}\,dx\leq C_T\int_0^T\!\!\int_\omega U^2(t,x)\frac{1}{a}\,dx\,dt.$$

The above theorem follows by a density argument as in [29, Proposition 4.1] as a consequence of the next observability inequality in the case of a regular final-time datum.

Lemma 4.2. *Assume Hypotheses 3.1 and 3.6. Then there exists a positive constant C_T such that every $(U,V)\in\mathcal{W}$ solution of (3.16)–(3.19) satisfies*

$$\int_0^1\left[U^2(0,x)+V^2(0,x)\right]\frac{1}{a}dx\leq C_T\int_0^T\!\!\int_\omega U^2(t,x)\frac{1}{a}dxdt.$$

Proof. Multiplying the first and the second equations in the system (3.16)–(3.19) respectively by $\frac{U_t}{a}$ and $\frac{V_t}{a}$, integrating over $(0,1)$, the sum of the new equations gives

$$\begin{aligned}0&=\int_0^1\left[U_t^2+V_t^2\right]\frac{1}{a}\,dx+[U_xU_t+V_xV_t]_0^1-\frac{1}{2}\frac{d}{dt}\int_0^1\left[U_x^2+V_x^2\right]\,dx\\&\quad+\int_0^1\left[\frac{\lambda_1}{ab_1}UU_t+\frac{\lambda_2}{ab_2}VV_t\right]\,dx+\int_0^1\frac{\mu}{ad}(UV_t+VU_t)\,dx\\&=\int_0^1\left[U_t^2+V_t^2\right]\frac{1}{a}\,dx-\frac{1}{2}\frac{d}{dt}\int_0^1\left[U_x^2+V_x^2\right]\,dx\\&\quad+\frac{1}{2}\frac{d}{dt}\int_0^1\left[\frac{\lambda_1}{ab_1}U^2+\frac{\lambda_2}{ab_2}V^2\right]\,dx+\mu\frac{d}{dt}\int_0^1\frac{UV}{ad}\,dx\\&\geq-\frac{1}{2}\frac{d}{dt}\int_0^1\left[U_x^2+V_x^2\right]\,dx+\frac{1}{2}\frac{d}{dt}\int_0^1\left[\frac{\lambda_1}{ab_1}U^2+\frac{\lambda_2}{ab_2}V^2\right]\,dx+\mu\frac{d}{dt}\int_0^1\frac{UV}{ad}\,dx.\end{aligned}$$

Hence, the function $t\longmapsto\int_0^1\left[U_x^2+V_x^2\right]dx-\int_0^1\left[\frac{\lambda_1}{ab_1}U^2+\frac{\lambda_2}{ab_2}V^2\right]dx-2\mu\int_0^1\frac{UV}{ad}\,dx$ is non decreasing for all $t\in[0,T]$. In particular, by Young and Hardy–Poincaré inequalities (2.2) and

(2.4), it results

$$\begin{aligned}
&\int_0^1 \left[U_x^2(0,x)+V_x^2(0,x)\right]\,dx-\int_0^1\left[\frac{\lambda_1}{ab_1}U^2(0,x)+\frac{\lambda_2}{ab_2}V^2(0,x)\right]\,dx-2\mu\int_0^1\frac{U(0,x)V(0,x)}{ad}\,dx\\
&\quad\le \int_0^1 \left[U_x^2(t,x)+V_x^2(t,x)\right]\,dx-\int_0^1\left[\frac{\lambda_1}{ab_1}U^2(t,x)+\frac{\lambda_2}{ab_2}V^2(t,x)\right]\,dx\\
&\qquad-2\mu\int_0^1\frac{U(t,x)V(t,x)}{ad}\,dx\\
&\quad\le \int_0^1 \left[U_x^2(t,x)+V_x^2(t,x)\right]\,dx+\lambda C^\star\int_0^1 \left[U_x^2(t,x)+V_x^2(t,x)\right]\,dx\\
&\qquad+\mu C_d^\star\int_0^1 \left[U_x^2(t,x)+V_x^2(t,x)\right]\,dx\\
&\quad=(1+\lambda C^\star+\mu C_d^\star)\int_0^1 \left[U_x^2(t,x)+V_x^2(t,x)\right]\,dx,
\end{aligned}$$

where $\lambda=\max\{|\lambda_1|,|\lambda_2|\}$ and $C^\star=\max\{C_1^\star,C_2^\star\}$.

Integrating the previous inequality over $[\frac{T}{4},\frac{3T}{4}]$, θ being bounded therein, we find

$$\begin{aligned}
&\int_0^1 \left[U_x^2(0,x)-\frac{\lambda_1}{ab_1}U^2(0,x)\right]\,dx+\int_0^1\left[V_x^2(0,x)-\frac{\lambda_2}{ab_2}V^2(0,x)\right]\,dx-2\mu\int_0^1\frac{U(0,x)V(0,x)}{ad}\,dx\\
&\quad\le \frac{2}{T}(1+\lambda C^\star+\mu C_d^\star)\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 \left[U_x^2(t,x)+V_x^2(t,x)\right]\,dx\,dt\\
&\quad\le C_T\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 s\theta\left[U_x^2(t,x)+V_x^2(t,x)\right]e^{2s\varphi}\,dx\,dt.
\end{aligned}$$

Hence, by the Carleman estimate given in Theorem 3.9 and the previous inequality, there exists a positive constant C such that

$$\begin{aligned}
&\int_0^1 \left[U_x^2(0,x)-\frac{\lambda_1}{ab_1}U^2(0,x)\right]\,dx+\int_0^1\left[V_x^2(0,x)-\frac{\lambda_2}{ab_2}V^2(0,x)\right]\,dx-2\mu\int_0^1\frac{U(0,x)V(0,x)}{ad}\,dx\\
&\quad\le C\int_0^T\int_\omega U^2(t,x)\frac{1}{a}\,dx\,dt.
\end{aligned}$$

From the previous inequality and Propositions 2.6, for $\delta>0$, one has

$$\begin{aligned}
\Lambda_1&\int_0^1 U_x^2(0,x)\,dx+\Lambda_2\int_0^1 V_x^2(0,x)\,dx\\
&\le C\int_0^T\int_\omega U^2(t,x)\frac{1}{a}\,dx\,dt+2\mu\int_0^1\frac{U(0,x)V(0,x)}{ad}\,dx\\
&\le C\int_0^T\int_\omega U^2(t,x)\frac{1}{a}\,dx\,dt+2\mu\delta C_d^\star\int_0^1 U_x^2(0,x)\,dx+\mu\frac{C_d^\star}{2\delta}\int_0^1 V_x^2(0,x)\,dx.
\end{aligned}$$

Therefore,

$$(\Lambda_1-2\mu\delta C_d^\star)\int_0^1 U_x^2(0,x)\,dx+\left(\Lambda_2-\mu\frac{C_d^\star}{2\delta}\right)\int_0^1 V_x^2(0,x)\,dx\le C\int_0^T\int_\omega U^2(t,x)\frac{1}{a}\,dx\,dt.$$

Consequently, if we now choose δ satisfying (2.12), we readily deduce that there exists $C>0$ such that

$$\int_0^1 \left[U_x^2(0,x)+V_x^2(0,x)\right]\,dx\le C\int_0^T\int_\omega U^2(t,x)\frac{1}{a}\,dx\,dt. \tag{4.1}$$

Finally, applying the Hardy–Poincaré inequality (see [29, Proposition 2.6]) and (4.1), we have

$$\begin{aligned}\int_0^1 \left[U^2(0,x)+V^2(0,x)\right]\frac{1}{a}\,dx &= \int_0^1 \frac{p}{(x-x_0)^2}\left[U^2(0,x)+V^2(0,x)\right]\,dx\\ &\le C_{HP}\int_0^1 p\left[U_x^2(0,x)+V_x^2(0,x)\right]\,dx\\ &\le C_0C_{HP}\int_0^1 \left[U_x^2(0,x)+V_x^2(0,x)\right]\,dx\\ &\le C\int_0^T\int_\omega U^2(t,x)\frac{1}{a}\,dx\,dt,\end{aligned} \tag{4.2}$$

for a positive constant C. Here $p(x)=\frac{(x-x_0)^2}{a}$, C_{HP} is the Hardy–Poincaré constant and

$$C_0 := \max\left[\frac{x_0^2}{a(0)}, \frac{(1-x_0)^2}{a(1)}\right].$$

Hence, the conclusion follows. □

5 Appendix

The basic result to prove Theorem 3.7 is the following Caccioppoli's inequality for systems of degenerate singular parabolic equations, which is the counterpart of [33, Lemma 6.1] for the non divergence case.

Lemma 5.1 (Caccioppoli's inequality). *Let ω' and ω two open subintervals of $(0,1)$ such that $\omega' \subset\subset \omega \subset (0,1)$ and $x_0 \notin \overline{\omega}$. Then, there exist two positive constants C and s_0 such that every solution $(U,V)\in\mathcal{W}$ of the adjoint problem (3.16)–(3.19) satisfies*

$$\int_0^T\int_{\omega'}[U_x^2(t,x)+V_x^2(t,x)]e^{2s\Phi}dxdt \le C\int_0^T\int_\omega s^2\theta^2[U^2(t,x)+V^2(t,x)]\frac{e^{2s\Phi}}{a}dxdt, \tag{5.1}$$

for all $s\ge s_0$.

Observe that we require $x_0\notin\bar{\omega}$, since in the applications above the control region ω is assumed to satisfy 3.6.

Proof of Lemma 5.1. Let us consider a smooth function $\xi\in C^\infty(0,1)$ such that $0\le\xi\le 1$ in $(0,1)$, $\operatorname{supp}\xi\subset\omega$ and $\xi\equiv 1$ on ω'. Hence, by definition of Φ and having in mind the equations satisfied by (U,V), we have

$$\begin{aligned}0 &= \int_0^T \frac{d}{dt}\left[\int_0^1 \xi^2 e^{2s\Phi}(U^2+V^2)dx\right]dt\\ &= 2\int_0^T\int_0^1 s\dot{\Phi}\xi^2e^{2s\Phi}(U^2+V^2)\,dx\,dt + 2\int_0^T\int_0^1 \xi^2e^{2s\Phi}a(x)\left[U_x^2+V_x^2\right]dx\,dt\\ &\quad + 2\int_0^T\int_0^1 (a(x)\xi^2e^{2s\Phi})_x\left[UU_x+VV_x\right]\,dx\,dt - 2\int_0^T\int_0^1 \xi^2e^{2s\Phi}\left[\frac{\lambda_1}{b_1}U^2+\frac{\lambda_1}{b_1}U^2\right]\,dx\,dt\\ &\quad - 4\mu\int_0^T\int_0^1 \xi^2e^{2s\Phi}\frac{UV}{d}\,dx\,dt.\end{aligned}$$

Then an integration by parts leads to

$$\begin{aligned}
&\int_0^T\int_0^1 \xi^2 e^{2s\Phi} a(x)\left[U_x^2+V_x^2\right] dx\,dt \\
&\quad = -\int_0^T\int_0^1 s\dot{\Phi}\xi^2 e^{2s\Phi}(U^2+V^2)\,dx\,dt + \frac{1}{2}\int_0^T\int_0^1 (a(x)\xi^2 e^{2s\Phi})_{xx}(U^2+V^2)\,dx\,dt \\
&\qquad + \int_0^T\int_0^1 \xi^2 e^{2s\Phi}\left(\frac{\lambda_1}{b_1}U^2+\frac{\lambda_2}{b_2}V^2\right) dx\,dt + 2\mu\int_0^T\int_0^1 \xi^2 e^{2s\Phi}\frac{UV}{d}\,dx\,dt.
\end{aligned}$$

Since $\min_{x\in\omega'} a(x) > 0$ and $|\dot{\theta}| \le c\theta^2$, then, by the Young's inequality and by definition of ξ,

$$\begin{aligned}
\min_{x\in\omega'} a(x)\int_0^T\int_{\omega'} e^{2s\Phi}[U_x^2+V_x^2]\,dx\,dt &\le \int_0^T\int_0^1 \xi^2 e^{2s\Phi} a(x)[U_x^2+V_x^2]\,dx\,dt \\
&\le C\int_0^T\int_{\omega} (1+s^2\theta^2+s|\dot{\theta}|)[U^2+V^2]e^{2s\Phi}\,dx\,dt \\
&\le C\int_0^T\int_{\omega} s^2\theta^2[U^2+V^2]e^{2s\Phi}\,dx\,dt \\
&\le C\int_0^T\int_{\omega} s^2\theta^2[U^2+V^2]\frac{e^{2s\Phi}}{a}\,dx\,dt.
\end{aligned}$$

Thus, the claim follows. □

6 Conclusion

In this paper, we studied the null controllability for a coupled degenerate parabolic system with a symmetric singular coupling matrix $\mathcal{C}$, see (1.8). In particular, the question of well posedness of the problem is addressed. Then, thanks to Carleman estimates, an observability inequality with observation being made on only one of the components of the state is proved. The main restrictive assumption under which the results presented in this paper are valid is the symmetry of the singular coupling matrix. This mentioned assumption is required not only to obtain well-posedness result but also to get the observability estimate. It would be interesting to know if a more general singular coupling matrix can still lead to indirect observability and null controllability results.

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