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Remarks on the existence of infinitely many solutions for a p-Laplacian equation involving oscillatory nonlinearities

Robert Stegliński ✉

Institute of Mathematics, Technical University of Lodz, Wolczanska 215, 90-924 Lodz, Poland

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Abstract. In this paper, we study the existence of infinitely many solutions for an elliptic problem with the nonlinearity having an oscillatory behavior. We propose more general assumptions on the nonlinear term which improve the results occurring in the literature.

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1 Introduction

The following problem

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = f(x,u) & \text{in } \Omega \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{1.1}$$

has been considered under many assumptions on $p, f, \Omega \subset \mathbb{R}^N$. One of the questions is under what assumptions does the problem have infinitely many non-negative solutions. The answer is obtained by means of various methods; for instance: sub-super solution arguments; the general variational principle of Ricceri; the fountain theorems; the Nehari manifold method; continuity of certain superposition operators. When the nonlinear term has an appropriate oscillatory behavior at zero or at infinity, the existence of infinitely many solutions can be shown in two steps.

1. Firstly, by showing that there exists a sequence $\{u_k\}$ of critical points of an energy functional I corresponding to the problem.
2. Secondly, by showing that the sequence contains infinitely many distinct elements.

✉ Email: robert.steglinski@p.lodz.pl

There are at least two ways to obtain the first step. The existence of the sequence of critical points can be obtained by showing that

1A. the global minima of the energy functional restricted to suitable chosen sets are local minima of I (see [1,4–6,8]); or

1B. the global minima of suitable truncated problems are local minima of I (see [2]).

To carry out the second step, i.e. to show that there are infinitely many distinct u_k, it is enough to obtain $I(u_k) < 0$ and $\lim_{k\to+\infty} I(u_k) = 0$ in the case of oscillatory behavior at zero or $\lim_{k\to+\infty} I(u_k) = -\infty$ in the case of oscillatory behavior at infinity. For this, the above mentioned papers use the following assumptions: for $a = 0^+$ or $a = +\infty$

(U^a) there exists an open bounded set $\Omega' \subset \Omega$ such that

$$\liminf_{s\to a} \frac{F(x,s)}{s^p} \geq -l \quad \text{and} \quad \limsup_{s\to a} \frac{F(x,s)}{s^p} \geq L$$

uniformly in $x \in \Omega'$, where l, L are some positive constants or $L = +\infty$ and $F(x,s) = \int_0^s f(x,t)dt$.

Let us note here that [1] assume $\Omega' = \Omega = \mathbb{R}^N$. But this is contradictory with other assumptions in this paper (see below).

In [9] we have attempted to translate the above mentioned results into the discrete case on integers. It has emerged that the condition (U^a) corresponds to a condition in which the oscillatory behavior of nonlinearity $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ occurs on a finite number of integers. Consequently, the problem is essentially reduced to a finite dimensional one. In [9] we found another condition, which uses infinite number of elements of $\mathbb{Z}$ and which has not its counterpart in the continuous case.

In the present paper we find conditions on nonlinearity f, which are more general than condition (U^a). We give easy verifiable examples of such nonlinearities and we show that for some of them we have

$$\lim_{s\to a} \frac{F(x,s)}{s^p} = 0$$

for all $x \in \Omega$ and so (U^a) is not satisfied.

The paper is organized as follows. In Section 2 we follow [1], where the strategy 1A is used. In Section 3 we follow [2], where the strategy 1B is used. Here we observe that, if Ω is a bounded subset of $\mathbb{R}^N$, the strategy 1B provides us with alternative proofs of results obtained in [6,8]. The examples are also given.

2 The strategy 1A

In this section we assume $\Omega = \mathbb{R}^N$. From the variational viewpoint, one of the difficulties in addressing problem (1.1) in $\mathbb{R}^N$ arises from the lack of compactness of the Sobolev embeddings: $W^{1,p}(\mathbb{R}^N)$ cannot be embedded compactly into $L^q(\mathbb{R}^N)$, $q > 1$. In [3], Kristály showed that $W_r^{1,p}(\mathbb{R}^N)$, the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$, can be embedded compactly into $L^\infty(\mathbb{R}^N)$ whenever $2 \leq N < p < +\infty$.

Let $2 \leq N < p < +\infty$. The energy functional $I : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ associated with problem (1.1)

$$I(u) = \int_{\mathbb{R}^N} \frac{1}{p} \left(|\nabla u|^p + |u|^p \right) dx - \int_{\mathbb{R}^N} F(x,u)dx,$$

is well defined, by the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ and the condition (F_2) below. Moreover, standard arguments show that I is of class C^1 on $W^{1,p}(\mathbb{R}^N)$ (see [7] for a similar proof).

For $0 \leq r < R$ define $A(r,R) = \{x \in \mathbb{R}^N : r \leq |x| \leq R\} = \{x \in \mathbb{R}^N : \left||x| - \frac{r+R}{2}\right| \leq \frac{R-r}{2}\}$ and $A'(r,R) = \{x \in \mathbb{R}^N : \left||x| - \frac{r+R}{2}\right| \leq \frac{R-r}{4}\}$. Now, we make the following assumptions on function f.

(F_1) $f : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$ satisfies the Carathéodory condition and is radial with respect to the first argument, with $f(x,0) = 0$ for almost all $x \in \mathbb{R}^N$;

(F_2) $\sup_{|t|\leq s} |f(\cdot,t)| \in L^1(\mathbb{R}^N)$ for each $s > 0$;

(F_3^0) there are two sequences $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$ such that $0 < b_{k+1} < a_k < b_k$, $\lim_{k\to+\infty} b_k = 0$, and $f(x,s) \leq 0$ for almost all $x \in \mathbb{R}^N$, and $s \in [a_k, b_k], k \in \mathbb{N}$;

(F_4^0) there exist $\gamma > 0$, $s_0 > 0$, $l \geq 0$, $L > 2^{N+1}\left(\frac{1}{p}\left(\frac{4}{\gamma}\right)^p + \frac{1}{p} + l\right)$, a sequence of nonnegative numbers $\{r_k\}_{k\in\mathbb{N}}$ and sequences of positive numbers $\{R_k\}_{k\in\mathbb{N}}$, $\{\eta_k\}_{k\in\mathbb{N}}$ such that

- (i) $\lim_{n+\infty} \eta_k = 0$ and $R_k - r_k \geq \gamma$ for all $k \in \mathbb{N}$;
- (ii) $F(x,s) \geq -ls^p$ for $s \in (0, s_0)$ and a.e. $x \in A(r_k,R_k) \setminus A'(r_k,R_k)$, $k \in \mathbb{N}$;
- (iii) $F(x,\eta_k) \geq L\eta_k^p$ for a.e. $x \in A'(r_k,R_k)$ and every $k \in \mathbb{N}$;

(F_3^∞) there are two sequences $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$ such that $0 < a_k < b_k < a_{k+1}$, $\lim_{k\to+\infty} a_k = +\infty$, and $f(x,s) \leq 0$ for almost all $x \in \mathbb{R}^N$ and $s \in [a_k,b_k], k \in \mathbb{N}$;

(F_4^∞) there exist $\gamma > 0$, $s_\infty > 0$, $l \geq 0$, $L > 2^{N+1}\left(\frac{1}{p}\left(\frac{4}{\gamma}\right)^p + \frac{1}{p} + l\right)$, a sequence of nonnegative numbers $\{r_k\}_{k\in\mathbb{N}}$ and sequences of positive numbers $\{R_k\}_{k\in\mathbb{N}}$, $\{\eta_k\}_{k\in\mathbb{N}}$ such that

- (i) $\lim_{n+\infty} \eta_k = +\infty$ and $R_k - r_k \geq \gamma$ for all $k \in \mathbb{N}$;
- (ii) $F(x,s) \geq -ls^p$ for $s \in (s_\infty, +\infty)$ and a.e. $x \in A(r_k,R_k) \setminus A'(r_k,R_k)$, $k \in \mathbb{N}$;
- (iii) $F(x,\eta_k) \geq L\eta_k^p$ for a.e. $x \in A'(r_k,R_k)$ and every $k \in \mathbb{N}$.

In the sequel we extend function f on the whole $\mathbb{R}^N \times \mathbb{R}$ by taking $f(x,s) = 0$ for a.e. $x \in \mathbb{R}^N$ and $s < 0$. Observe that (U^a) implies (F_4^a), $a = 0$ or $a = \infty$, when $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is radial with respect to the first argument.

Now we are ready to state our first results.

Theorem 2.1. *Let $2 \leq N < p < +\infty$. Let f satisfy (F_1), (F_2), (F_3^0) and (F_4^0). Then there exists a sequence $\{u_k\} \subset X$ of distinct radially symmetric, nonnegative weak solutions of (1.1) such that*

$$\lim_{k\to+\infty} I(u_k) = 0 \quad \textit{and} \quad \lim_{k\to+\infty} \|u_k\|_{W^{1,p}(\mathbb{R}^N)} = 0.$$

Theorem 2.2. *Let $2 \leq N < p < +\infty$. Let f satisfy (F_1), (F_2), (F_3^∞) and (F_4^∞). Then there exists a sequence $\{u_k\} \subset X$ of distinct radially symmetric, nonnegative weak solutions of (1.1) such that*

$$\lim_{k\to+\infty} I(u_k) = -\infty \quad \textit{and} \quad \lim_{k\to+\infty} \|u_k\|_{W^{1,p}(\mathbb{R}^N)} = +\infty.$$

In [1] the author considers problem (1.1) with a variable exponent $p \in C(\mathbb{R}^N)$ which is radial and $2 \leq N < p^- := \inf_{\mathbb{R}^N} p(x) \leq p^+ := \sup_{\mathbb{R}^N} p(x) < +\infty$. The only difference in hypotheses concerns (F_4^0) and (F_4^∞), The author assumes that there exist $h_0 > 0$ and a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ such that $F(x,\eta_k) \geq h_0\eta_k^p$ for almost all $x \in \mathbb{R}^N$. But this stands in contradiction with hypothesis (F_2). Indeed, (F_2) gives $\beta \in L^1(\mathbb{R}^N)$ such that $\sup_{|t|\leq\eta_1} |f(x,t)| \leq \beta(x)$ for all $x \in \mathbb{R}^N$. Then $|F(x,\eta_1)| = \left|\int_0^{\eta_1} f(x,s)ds\right| \leq \int_0^{\eta_1} \sup_{|t|\leq\eta_1} |f(x,t)|\, ds \leq \beta(x)\eta_1$ for all $x \in \mathbb{R}^N$ and so $F(\cdot,\eta_1) \in L^1(\mathbb{R}^N)$. On the other hand, the inequality $F(x,\eta_1) \geq h_0\eta_1^p$ for almost all $x \in \mathbb{R}^N$ gives $F(\cdot,\eta_1) \notin L^1(\mathbb{R}^N)$ and we obtain a contradiction.

Sketch of the proofs of Theorem 2.1 and Theorem 2.2. The beginning is the same in both proofs. Let I_G stand for the restriction of I to $W_r^{1,p}(\mathbb{R}^N)$. Due to the principle of symmetric criticality of Palais (see [10]), the critical points of I_G are critical points of I as well. By the compactness embedding of $W_r^{1,p}(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$, the functional I_G is sequentially weakly lower semicontinuous on $W_r^{1,p}(\mathbb{R}^N)$ [1, Proposition 3.1]. Let us fix number $r < 0$ arbitrarily, and for every $k \in \mathbb{N}$, consider the set

$$S_k = \left\{ u \in W_r^{1,p}(\mathbb{R}^N) : r \leq u(x) \leq b_k \text{ a.e. } x \in \mathbb{R}^N \right\}. \tag{2.1}$$

Then S_k is convex and closed in $W_r^{1,p}(\mathbb{R}^N)$, by Morrey inequality, and so weakly closed. Next, we show that the functional I_G is bounded from below on S_k and its infimum on S_k is attained at $u_k \in S_k$, which satisfies $0 \leq u_k(x) \leq a_k$ for almost all $x \in \mathbb{R}^N$ [1, Proposition 3.2 and Proposition 3.3] and we conclude that u_k is also a local minimum point of I_G in $W_r^{1,p}(\mathbb{R}^N)$ [1, Proposition 3.4].

Now, let us continue with the proof of Theorem 2.1. Since $\|u_k\|_{L^\infty(\mathbb{R}^N)} \leq a_k$ for all $k \in \mathbb{N}$ and $\lim_{k\to+\infty} a_k = 0$, we have $\lim_{k\to\infty} \|u_k\|_{L^\infty(\mathbb{R}^N)} = 0$. To show that the sequence $\{u_k\}_{k\in\mathbb{N}}$ contains infinitely many distinct elements, it is enough to show that $I_G(u_k) < 0$, which gives the nontriviality of u_k. Let γ, s_0, l, L, $\{r_k\}_{k\in\mathbb{N}}$, $\{R_k\}_{k\in\mathbb{N}}$, $\{\eta_k\}_{k\in\mathbb{N}}$ be such as in (F_4^0). Up to extracting a subsequence, we may assume that $\{\eta_k\}_{k\in\mathbb{N}}$ satisfies $\eta_k \leq b_k$ for all $k \in \mathbb{N}$. Write $A_k = A(r_k,R_k)$, $A'_k = A'(r_k,R_k)$. It is easy to check that

$$\operatorname{meas} A'_k \geq \frac{1}{2^{N+1}} \operatorname{meas} A_k \tag{2.2}$$

and $\operatorname{meas} A_k \geq \omega\gamma^N$, where ω is the volume of the unit ball in $\mathbb{R}^N$. Define for every $k \in \mathbb{N}$ the function $w_k : \mathbb{R}^N \to \mathbb{R}$ by

$$w_k(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus A_k \\ \eta_k & \text{if } x \in A'_k \\ \frac{4\eta_k}{R_k-r_k}\left(\frac{R_k-r_k}{2} - \left||x| - \frac{R_k+r_k}{2}\right|\right) & \text{if } x \in A_k \setminus A'_k. \end{cases} \tag{2.3}$$

Then $w_k \in S_k$ and

$$\begin{aligned} I_G(w_k) &= \frac{1}{p}\|w_k\|^p_{W^{1,p}(\mathbb{R}^N)} - \int_{A'_k} F(x,w_k(x))dx - \int_{(A_k\setminus A'_k)} F(x,w_k(x))dx \\ &\leq \frac{1}{p}\left(\frac{4\eta_k}{R_k-r_k}\right)^p \operatorname{meas}(A_k\setminus A'_k) + \frac{1}{p}\eta_k^p \operatorname{meas} A_k - \int_{A'_k} F(x,\eta_k)dx + \int_{A_k\setminus A'_k} l\,(w_k(x))^p dx \\ &\leq \frac{1}{p}\left(\frac{4}{\gamma}\right)^p \eta_k^p \operatorname{meas} A_k + \frac{1}{p}\eta_k^p \operatorname{meas} A_k - L\eta_k^p \operatorname{meas} A'_k + l\eta_k^p \operatorname{meas} A_k \\ &\leq \eta_k^p \operatorname{meas} A_k \left[\frac{1}{p}\left(\frac{4}{\gamma}\right)^p + \frac{1}{p} - \frac{1}{2^{N+1}}L + l\right] \end{aligned}$$

where in the last inequality we have used (2.2). Since $L > 2^{N+1}\left(\frac{1}{p}\left(\frac{4}{\gamma}\right)^p + \frac{1}{p} + l\right)$, this forces $I_G(w_k) < 0$, which gives $I_G(u_k) < 0$. Moreover,

$$0 > I_G(u_k) \geq -\int_{\mathbb{R}^N} F(x, u_k(x))dx \geq -a_k \int_{\mathbb{R}^N} \sup_{|t|\leq a_1} |f(x,t)|\, dx$$

and as $\lim_{k\to+\infty} a_k = 0$, we have $\lim_{k\to\infty} I_G(u_k) = 0$. Further, we have

$$\frac{1}{p}\|u_k\|^p_{W^{1,p}(\mathbb{R}^N)} = I_G(u_k) + \int_{\mathbb{R}^N} F(x, u_k(x))dx < a_k \int_{\mathbb{R}^N} \sup_{|t|\leq a_1} |f(x,t)|\, dx,$$

so $\lim_{k\to+\infty} \|u_k\|_{W^{1,p}(\mathbb{R}^N)} = 0$.

Now, let us continue with the proof of Theorem 2.2. In this case, to show that the sequence $\{u_k\}_{k\in\mathbb{N}}$ contains infinitely many distinct elements, it is enough to show that $\lim_{k\to\infty} I_G(u_k) = -\infty$. Let γ, s_∞, l, L, $\{r_k\}_{k\in\mathbb{N}}$, $\{R_k\}_{k\in\mathbb{N}}$, $\{\eta_k\}_{k\in\mathbb{N}}$ be such as in (F_4^∞). Up to extracting a subsequence, we may assume that $\{\eta_k\}_{k\in\mathbb{N}}$ satisfies $\eta_k \leq b_k$ for all $k\in\mathbb{N}$ and $\eta_1 \geq s_\infty$. Taking w_k from (2.3) and using (F_4^∞), we obtain

$$\begin{aligned}
I_G(w_k) &= \frac{1}{p}\|w_k\|^p_{W^{1,p}(\mathbb{R}^N)} - \int_{A_k'} F(x, w_k(x))dx - \int_{(A_k\setminus A_k')\cap\{w_k > s_\infty\}} F(x, w_k(x))dx \\
&\quad - \int_{(A_k\setminus A_k')\cap\{w_k\leq s_\infty\}} F(x, w_k(x))dx \\
&\leq \frac{1}{p}\left(\frac{4}{\gamma}\right)^p \eta_k^p \operatorname{meas} A_k + \frac{1}{p}\eta_k^p \operatorname{meas} A_k - L\eta_k^p \operatorname{meas} A_k' + l\eta_k^p \operatorname{meas} A_k \\
&\quad + s_\infty \left\| \sup_{t\in[0,s_\infty]} |f(\cdot,t)| \right\|_{L^1(\mathbb{R}^N)} \\
&\leq \eta_k^p \operatorname{meas} A_k \left[\frac{1}{p}\left(\frac{4}{\gamma}\right)^p + \frac{1}{p} - \frac{1}{2^{N+1}}L + l\right] + s_\infty \left\| \sup_{t\in[0,s_\infty]} |f(\cdot,t)| \right\|_{L^1(\mathbb{R}^N)}.
\end{aligned}$$

Since $L > 2^{N+1}\left(\frac{1}{p}\left(\frac{4}{\gamma}\right)^p + \frac{1}{p} + l\right)$, $\operatorname{meas} A_k \geq \omega\gamma^N$, $\lim_{n\to+\infty} \eta_k = +\infty$ and $I_G(u_k) \leq I_G(w_k)$, we conclude that $\lim_{k\to\infty} I_G(u_k) = -\infty$.

To show that $\lim_{k\to+\infty} \|u_k\|_{W^{1,p}(\mathbb{R}^N)} = +\infty$, we argue by contradiction. Let us assume that there exists a subsequence $\{u_{k_l}\}$ of $\{u_k\}$ which is bounded in E. Thus, it is also bounded in $L^\infty(\mathbb{R}^N)$, by Morrey inequality. As $\lim_{k\to+\infty} b_k = +\infty$, there exists $k_0\in\mathbb{N}$ such that $u_{k_l} \in S_{k_0}$ for all $l\in\mathbb{N}$. Since $\{I(u_k)\}$ is nonincreasing, we have for all $k_l \geq k_0$

$$I(u_{k_0}) = \inf_{u\in S_{k_0}} I(u) \leq I(u_{k_l}) \leq I(u_{k_0}),$$

i.e. $I(u_{k_l}) = I(u_{k_0})$ for all $k_l \geq k_0$. But this fact contradics with $\lim_{l\to+\infty} I(u_{k_l}) = -\infty$. □

Now we will give couple of examples.

Example 2.3. Let us start with an example of function which satisfies (F_1), (F_2), (F_3^0), (U^0). Let $\{\eta_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $\eta_1 \leq 1$, $\eta_{k+1} < \frac{1}{2}\eta_k$ for all $k\in\mathbb{N}$. Let $\hat{f}:\mathbb{R}\to\mathbb{R}$ be defined by

$$\hat{f}(s) = 8L\sum_{k\in\mathbb{N}} \frac{\eta_k^p - \eta_{k+1}^p}{\eta_k^2}\left(\frac{1}{2}\eta_k - 2\left|s - \frac{3}{4}\eta_k\right|\right)\cdot \mathbf{1}_{[\frac{1}{2}\eta_k,\eta_k]}(s),$$

where $\mathbf{1}_A$ is the indicator of A and $L > 2^{N+1}\left(\frac{1}{p}4^p + \frac{1}{p} + 1\right)$. Obviously, $\hat{f}$ is continuous. Let $Q \in L_1(\mathbb{R}^N)$ be radially symmetric and $Q \geq 1$ on B_1, the unit ball in $\mathbb{R}^N$. Now, let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x,s) = Q(x)\hat{f}(s)$. Then f satisfies $(F_1), (F_2)$ and (F_3^0) with $a_k = \eta_{k+1}$ and $b_k = \eta_k$ for all $k \in \mathbb{N}$. Since $F \geq 0$ and $F(x,\eta_k) = Q(x)L\sum_{l=k}^{+\infty}(\eta_l^p - \eta_{l+1}^p) \geq L\eta_k^p$ for all $x \in B_1$ and $k \in \mathbb{N}$, the condition (U^0) is satisfied with $\Omega' = B_1$.

Example 2.4. Now we give an example of function which satisfies (F_1), (F_2), (F_3^0), (F_4^0) and does not satisfy (U^0). Let $\{r_k\}_{k\in\mathbb{N}}$ be an increasing sequence such that $r_1 > 1$ and $r_{k+1} > r_k + 1$ for every $k \in \mathbb{N}$. Let $\{a_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $a_1 \leq 1$, $a_{k+1} < \frac{1}{2}a_k$ for all $k \in \mathbb{N}$, and $\sum_{k\in\mathbb{N}} a_k^{p-1}\text{meas}\, A_k < \infty$, where $A_k = A(r_k, r_k+1)$. Let $l = 0$ and $L > 2^{N+1}\left(\frac{4^p}{p} + \frac{1}{p}\right)$. Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x,s) = 8L\sum_{k\in\mathbb{N}} a_k^{p-2}\left(\frac{1}{2}a_k - 2\left|s - \frac{3}{4}a_k\right|\right) \cdot \mathbf{1}_{[r_k,r_k+1]\times[\frac{1}{2}a_k,a_k]}(|x|, s),$$

where $\mathbf{1}_{A\times B}$ is the indicator of $A \times B$. Obviously, f satisfies (F_1), and (F_3^0). Since for all $s \geq 0$ and $x \in \mathbb{R}^N$

$$\sup_{|t|\leq s}|f(x,t)| \leq \sup_{|t|\leq a_1}|f(x,t)| = 4L\sum_{k\in\mathbb{N}} a_k^{p-1} \cdot \mathbf{1}_{[r_k,r_k+1]}(|x|),$$

the condition (F_2) is satisfied. Moreover, $F \geq 0$ and $F(x,a_k) = La_k^p$ for all $x \in A_k$ and $k \in \mathbb{N}$, which gives (F_4^0). Now, for any $x \in \mathbb{R}^N$ there is k_0 such that for all $0 < s < a_{k_0}$ we have $F(x,s) = 0$. This means that $\lim_{s\to 0^+}\frac{F(x,s)}{s^p} = 0$ and f does not satisfy condition (U^0).

Example 2.5. Now we will give an example of a function which satisfies (F_1), (F_2), (F_3^∞), (U^∞). Let $\{\eta_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers such that $\eta_k > \eta_{k-1} + 1$ for all $k \in \mathbb{N}$, where $\eta_0 = 0$. Let $\hat{f} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\hat{f}(s) = 2L\sum_{k\in\mathbb{N}}\left(\eta_k^p - \eta_{k-1}^p\right)\left(1 - 2\left|s - \eta_k - \frac{1}{2}\right|\right) \cdot \mathbf{1}_{[\eta_k,\eta_k+1]}(s),$$

where $L > 2^{N+1}\left(\frac{1}{p}4^p + \frac{1}{p} + 1\right)$. Obviously, $\hat{f}$ is continuous. Let $Q \in L_1(\mathbb{R}^N)$ be radially symmetric and $Q \geq 1$ on B_1. Now, let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x,s) = Q(x)\hat{f}(s)$. Then f satisfies $(F_1), (F_2)$ and (F_3^0) with $a_k = \eta_k + 1$ and $b_k = \eta_{k+1}$ for all $k \in \mathbb{N}$. Since $F \geq 0$ and $F(x,\eta_k) = Q(x)L\sum_{l=1}^{k}\left(\eta_l^p - \eta_{l-1}^p\right) \geq L\eta_k^p$ for all $x \in B_1$ and $k \in \mathbb{N}$, the condition (U^∞) is satisfied with $\Omega' = B_1$.

Example 2.6. Now we will give an example of a function which satisfies (F_1), (F_2), (F_3^∞), (F_4^∞) and does not satisfy (U^∞). Let $\{r_k\}_{k\in\mathbb{N}}$ be an increasing sequence such that $r_1 > 1$ and $r_{k+1} > r_k + 1$ for every $k \in \mathbb{N}$. Let $\{a_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers such that $a_1 \geq 1$, $a_{k+1} - 1 > a_k$ for all $k \in \mathbb{N}$. Let $l = 0$ and $L > 2^{N+1}\left(\frac{4^p}{p} + \frac{1}{p}\right)$. Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x,s) = 2L\sum_{k\in\mathbb{N}} a_k^p\left(1 - 2\left|s - a_k + \frac{1}{2}\right|\right) \cdot \mathbf{1}_{[r_k,r_k+1]\times[a_k-1,a_k]}(|x|, s).$$

Obviously, f satisfies (F_1), and (F_3^∞). Since for all $s \geq 0$ and $x \in \mathbb{R}^N$

$$\sup_{|t|\leq s}|f(x,t)| \leq 2L\sum_{k=1}^{\min\{l:s\leq a_l\}} a_k^p \cdot \mathbf{1}_{[r_k,r_k+1]}(|x|),$$

the condition (F_2) is satisfied. Moreover, $F \geq 0$ and $F(x,a_k) = La_k^p$ for all $x \in A_k$ and $k \in \mathbb{N}$, which gives (F_4^0). Now, for any $x \in \mathbb{R}^N$ there is k_0 such that for all $a_{k_0} < s < +\infty$ we have $F(x,s) \leq La_{k_0}^p$. This means that $\lim_{s\to+\infty}\frac{F(x,s)}{s^p} = 0$ and f does not satisfy condition (U^∞).

3 The strategy 1B

In this section we follow [2], where the strategy 1B was used. In this paper the perturbed quasilinear elliptic problem with oscillatory terms was investigated. The unperturbed version reads as follows

$$\begin{cases} -\text{div}\left(|\nabla u|^p \nabla u\right) + |u|^{p-2} u = Q(x) f(x,u), & x \in \Omega \\ u \in W_0^{1,p}(\Omega) \end{cases} \tag{P}$$

where $p > 1$ and Ω is a domain in $\mathbb{R}^N$ which may be unbounded. Let us enunciate the assumptions.

(Q) $Q : \Omega \to \mathbb{R}$ is a positive potential such that $Q \in L^1(\Omega) \cap L^{p'}(\Omega)$ where $\frac{1}{p'} + \frac{1}{p} = 1$;

(H_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition with $f(x,0) = 0$ for almost all $x \in \Omega$;

(H_2^0) $\sup_{t\in[0,T_0]} |f(\cdot,t)| \in L^\infty(\Omega)$ for some $T_0 > 0$;

(H_3^0) there are two sequences $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}}$ such that $0 < b_{k+1} < a_k < b_k$, $\lim_{k\to+\infty} b_k = 0$, and $f(x,s) \le 0$ for almost all $x \in \Omega$, and $s \in [a_k, b_k], k \in \mathbb{N}$;

(H_4^0) there exist $s_0 > 0, l_0 \ge 0$, a sequence $\{x_k\}_{k\in\mathbb{N}}$ in Ω and sequences of positive numbers $\{r_k\}_{k\in\mathbb{N}}, \{L_k\}_{k\in\mathbb{N}}, \{\eta_k\}_{k\in\mathbb{N}}$ such that

- (i) $\lim_{n+\infty} \eta_k = 0$ and

$$L_k > \frac{1}{\|Q\|_{L^1(B_{r_k/2}(x_k))}} \left[\frac{1}{p}\left(\frac{2}{r_k}\right)^p \omega r_k^N \left(1 - \frac{1}{2^N}\right) + \frac{1}{p2^N}\omega r_k^N + l_0 \|Q\|_{L^1(B_{r_k}(x_k)\setminus B_{r_k/2}(x_k))}\right]$$

  for all $k \in \mathbb{N}$;

- (ii) $F(x,s) \ge -l_0 s^p$ for $s \in (0, s_0)$ and a.e. $x \in B_{r_k}(x_k) \setminus B_{r_k/2}(x_k)$, $k \in \mathbb{N}$;
- (iii) $F(x,\eta_k) \ge L_k \eta_k^p$ for a.e. $x \in B_{r_k/2}(x_k)$ and every $k \in \mathbb{N}$;

(H_2^∞) $\sup_{t\in[0,T]} |f(\cdot,t)| \in L^\infty(\Omega)$ for any $T > 0$;

(H_3^∞) there are two sequences $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}}$ such that $0 < a_k < b_k < a_{k+1}$, $\lim_{k\to+\infty} a_k = +\infty$, and $f(x,s) \le 0$ for almost all $x \in \Omega$ and $s \in [a_k, b_k], k \in \mathbb{N}$;

(H_4^∞) there exist $s_\infty > 1, l_\infty \ge 0$, a sequence $\{x_k\}_{k\in\mathbb{N}}$ in Ω and sequences of positive numbers $\{r_k\}_{k\in\mathbb{N}}, \{L_k\}_{k\in\mathbb{N}}, \{\eta_k\}_{k\in\mathbb{N}}$ such that

- (i) $\lim_{n+\infty} \eta_k = +\infty$ and

$$L_k > \frac{1}{\|Q\|_{L^1(B_{r_k/2}(x_k))}} \left[\frac{1}{p}\left(\frac{2}{r_k}\right)^p \omega r_k^N \left(1 - \frac{1}{2^N}\right) + \frac{1}{p2^N}\omega r_k^N + l_\infty \|Q\|_{L^1(B_{r_k}(x_k)\setminus B_{r_k/2}(x_k))} + 1\right]$$

  for all $k \in \mathbb{N}$;

- (ii) $F(x,s) \ge -l_\infty s^p$ for $s \in (s_\infty, +\infty)$ and a.e. $x \in B_{r_k}(x_k) \setminus B_{r_k/2}(x_k)$, $k \in \mathbb{N}$;
- (iii) $F(x,\eta_k) \ge L_k \eta_k^p$ for a.e. $x \in B_{r_k/2}(x_k)$ and every $k \in \mathbb{N}$,

where $B_r(x) = \{y \in \mathbb{R}^N : |y - x| < r\}$.

Since we only search the solutions belonging to $W_0^{1,p}(\Omega)$, we may relax hypothesis (Q) and $(H)'$ in [2] to our one (see [2], Remark 3.1; see also [8]). In the sequel we extend the function f on the whole $\Omega \times \mathbb{R}$ by taking $f(x,s) = 0$ for a.e. $x \in \Omega$ and $s < 0$.

Now we can formulate the following theorems.

Theorem 3.1. *Let $p > 1$. Assume (Q), (H_1), (H_2^0), (H_3^0), (H_4^0). Then there exist infinitely many nonnegative weak solutions $\{u_k\}$ for (P) such that*

$$\lim_{k\to+\infty} \|u_k\|_{L^\infty(\Omega)} = \lim_{k\to+\infty} \|u_k\|_{W_0^{1,p}(\Omega)} = 0.$$

Theorem 3.2. *Let $p > 1$. Assume (Q), (H_1), (H_2^∞), (H_3^∞), (H_4^∞). Then there exist infinitely many nonnegative weak solutions $\{u_k\}$ for (P) such that*

$$\lim_{k\to+\infty} \|u_k\|_{L^\infty(\Omega)} = +\infty.$$

Sketch of the proofs of Theorem 3.1 and Theorem 3.2. The beginnings in both proofs are the same. For $k \in \mathbb{N}$, define the truncation function

$$f_k(x,s) = \begin{cases} 0, & (s \le 0 \text{ or } s \ge a_k + 1) \text{ and } x \in \Omega, \\ f(x,s), & 0 \le s \le a_k \text{ and } x \in \Omega, \\ f(x,a_k)\,(a_k + 1 - s)\,, & a_k \le s \le a_k + 1 \text{ and } x \in \Omega. \end{cases}$$

and consider the equation

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^p \, \nabla u\right) + |u|^{p-2}\, u = Q(x) f_k(x,u) & x \in \Omega \\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{P_k}$$

A weak solution of the problem (P_k) is a critical point of the energy functional $J_k : W_0^{1,p}(\Omega) \to \mathbb{R}$,

$$J_k(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |u|^p\right) - \int_\Omega Q(x) F_k(x,u),$$

where $F_k(x,s) = \int_0^s f_k(x,t)dt$. It is easy to check that the functional J_k is well defined and $J_k \in C^1(W_0^{1,p}(\Omega))$ (in the case of nonlinearity which oscillates at the origin, up to subsequence, we may assume that $a_1 \le T_0$). J_k satisfies the (PS) condition and is bounded from below [2, Lemma 2.3]. So, there exists $u_k \in W_0^{1,p}(\Omega)$ such that $J_k(u_k) = \inf_{u \in W_0^{1,p}(\Omega)} J_k(u)$. Hence u_k is a critical point of J_k and as such u_k is a weak solution for problem (P_k). Arguing as in Lemma 2.4 in [2] one can prove that $0 \le u_k(x) \le a_k$ for almost all $x \in \Omega$ (the proof works with our assumptions (H_3^0) and (H_3^∞), which are slightly weaker than $(f_2^0)'$ and $(f_2^\infty)'$, respectively). This means that $f(x,u_k(x)) = f_k(x,u_k(x))$ for almost all $x \in \Omega$, which implies that u_k is a weak solution for problem (P).

Now, let us continue with the proof of Theorem 3.1. Since $\|u_k\|_{L^\infty(\Omega)} \le a_k$ for all $k \in \mathbb{N}$, we have $\lim_{k\to\infty} \|u_k\|_{L^\infty(\Omega)} = 0$. To show that the sequence $\{u_k\}_{k\in\mathbb{N}}$ contains infinitely many distinct elements, it is enough to show that $J(u_k) < 0$, which gives the nontriviality of u_k. Let s_0, l_0, $\{x_k\}_{k\in\mathbb{N}}$, $\{r_k\}_{k\in\mathbb{N}}$, $\{L_k\}_{k\in\mathbb{N}}$, $\{\eta_k\}_{k\in\mathbb{N}}$ be such as in (H_4^0). Up to extracting a subsequence, we may assume that $\{\eta_k\}_{k\in\mathbb{N}}$ satisfies $\eta_k \le a_k$ for all $k \in \mathbb{N}$ and $\eta_1 \le s_0$. Let $B_k = B_{r_k}(x_k)$ and $B'_k = B_{\frac{r_k}{2}}(x_k)$. Define for every $k \in \mathbb{N}$ the function $w_k : \mathbb{R}^N \to \mathbb{R}$ by

$$w_k(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus B_k \\ \eta_k & \text{if } x \in B'_k \\ \frac{2\eta_k}{r_k}\left(r_k - |x - x_k|\right) & \text{if } x \in B_k \setminus B'_k. \end{cases} \tag{3.1}$$

Then $0 \le w_k(x) \le a_k$ for all $x \in \Omega$ and consequently $J(u_k) = J_k(u_k) \le J_k(w_k) = J(w_k)$. We have

$$\begin{aligned} J(w_k) &\le \frac{1}{p}\|w_k\|^p_{W_0^{1,p}(\Omega)} - \int_{B'_k} Q(x)F(x,\eta_k)dx + \int_{B_k\setminus B'_k} Q(x)l_0(w_k(x))^p dx \\ &\le \frac{1}{p}\left(\frac{2}{r_k}\right)^p \eta_k^p \operatorname{meas}(B_k \setminus B'_k) + \frac{1}{p}\eta_k^p \operatorname{meas}(B'_k) - L_k\eta_k^p \|Q\|_{L^1(B'_k)} + l_0\eta_k^p \|Q\|_{L^1(B_k\setminus B'_k)} \\ &= \eta_k^p\left[\frac{1}{p}\left(\frac{2}{r_k}\right)^p \omega r_k^N\left(1 - \frac{1}{2^N}\right) + \frac{1}{p2^N}\omega r_k^N - L_k\|Q\|_{L^1(B'_k)} + l_0\|Q\|_{L^1(B_k\setminus B'_k)}\right]. \end{aligned}$$

Since $L_k > \frac{1}{\|Q\|_{L^1(B'_k)}}\left[\frac{1}{p}\left(\frac{2}{r_k}\right)^p \omega r_k^N\left(1 - \frac{1}{2^N}\right) + \frac{1}{p2^N}\omega r_k^N + l_0\|Q\|_{L^1(B_k\setminus B'_k)}\right]$, this forces $J(w_k) < 0$. Moreover,

$$\frac{1}{p}\|u_k\|^p_{W_0^{1,p}(\Omega)} = J(u_k) + \int_\Omega Q(x)F(x,u_k(x))dx < a_k \left\|\sup_{t\in[0,s_0]}|f(\cdot,t)|\right\|_\infty \int_\Omega Q(x)dx.$$

As the sequence $\{a_k\}$ tends to zero we have $\|u_k\|_{W_0^{1,p}(\Omega)} \to 0$ as $k \to +\infty$.

Now, let us continue with the proof of Theorem 3.2. In this case, to show that the sequence $\{u_k\}_{k\in\mathbb{N}}$ contains infinitely many distinct elements, it is enough to show that $\lim_{k\to\infty} J(u_k) = -\infty$. Let s_∞, l_∞, $\{x_k\}_{k\in\mathbb{N}}$, $\{r_k\}_{k\in\mathbb{N}}$, $\{L_k\}_{k\in\mathbb{N}}$, $\{\eta_k\}_{k\in\mathbb{N}}$ be such as in (H_4^∞). Up to extracting a subsequence, we may assume that $\{\eta_k\}_{k\in\mathbb{N}}$ satisfies $\eta_k \le a_k$ for all $k \in \mathbb{N}$ and $\eta_1 \ge s_\infty$. Taking w_k from (3.1) and using (H_4^∞), we obtain

$$\begin{aligned} J(w_k) &\le \frac{1}{p}\|w_k\|^p_{W_0^{1,p}(\Omega)} - \int_{B'_k} Q(x)F(x,w_k(x))dx - \int_{(B_k\setminus B'_k)\cap\{w_k > s_\infty\}} Q(x)F(x,w_k(x))dx \\ &\quad - \int_{(B_k\setminus B'_k)\cap\{w_k \le s_\infty\}} Q(x)F(x,w_k(x))dx \\ &\le \frac{1}{p}\left(\frac{2}{r_k}\right)^p \eta_k^p \operatorname{meas}(B_k \setminus B'_k) + \frac{1}{p}\eta_k^p \operatorname{meas}(B'_k) - L_k\eta_k^p \|Q\|_{L^1(B'_k)} + l_\infty\eta_k^p \|Q\|_{L^1(B_k\setminus B'_k)} \\ &\quad + s_\infty\|Q\|_{L^1(B_k\setminus B'_k)} \left\|\sup_{t\in[0,s_\infty]}|f(\cdot,t)|\right\|_{L^\infty(\Omega)} \\ &\le \eta_k^p\left[\frac{1}{p}\left(\frac{2}{r_k}\right)^p \omega r_k^N\left(1 - \frac{1}{2^N}\right) + \frac{1}{p2^N}\omega r_k^N - L_k\|Q\|_{L^1(B'_k)} + l_0\|Q\|_{L^1(B_k\setminus B'_k)}\right] \\ &\quad + s_\infty\|Q\|_{L^1(\Omega)} \left\|\sup_{t\in[0,s_\infty]}|f(\cdot,t)|\right\|_{L^\infty(\Omega)} \\ &\le -\eta_k^p + s_\infty\|Q\|_{L^1(\Omega)} \left\|\sup_{t\in[0,s_\infty]}|f(\cdot,t)|\right\|_{L^\infty(\Omega)}. \end{aligned}$$

Combining this with $J(u_k) \le J(w_k)$ and $\lim_{k\to\infty} \eta_k = +\infty$ we conclude that $\lim_{k\to\infty} J(u_k) = -\infty$.

Arguing by contradiction as in the end of the proof of Theorem 1.3 in [2], we can show that $\lim_{k\to+\infty} \|u_k\|_{L^\infty(\Omega)} = +\infty$. □

Remark 3.3. If Ω is bounded and $Q \equiv 1$, the estimate on L_k in (H_4^a), $a = 0$ or $a = \infty$, is simpler: $L_k > \frac{1}{p}\left(\frac{2}{r_k}\right)^p \left(2^N - 1\right) + \frac{1}{p}2^N + l_0\left(2^N - 1\right) + b$ for all $k \in \mathbb{N}$, where $b = 0$, if $a = 0$ and $b = 1$, if $a = \infty$.

Remark 3.4. In [8] the following auxiliary problem was investigated

$$\begin{cases} -\operatorname{div} A\left(x, \nabla u\right) + K(x)\left|u\right|^{p-2} u = f(x,u) & x \in \Omega \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

Here Ω is a bounded domain of $\mathbb{R}^N$, $N \ge 3$, with smooth boundary, $K \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} K(x) > 0$ and $\operatorname{div} A\left(x, \nabla u\right)$ is a general operator in divergence form of p-Laplacian type with $p > 1$; its special case is p-Laplace operator $\operatorname{div}\left(\left|\nabla u\right|^p \nabla u\right)$. Now, if we change the estimate on L_k in (H_4^a), $a = 0$ or $a = \infty$, into the following estimate $L_k > \frac{1}{p}\left(\frac{2}{r_k}\right)^p\left(2^N - 1\right) + \frac{\|K\|_{L^\infty(\Omega)}}{p}2^N + l_0\left(2^N - 1\right) + b$ for all $k \in \mathbb{N}$, where $b = 0$ if $a = 0$ and $b = 1$ if $a = \infty$, we obtain analogous results, which improve results in [6,8] with different proofs. We omit details.

An example of nonlinearity which satisfies (H_1), (H_2^0), (H_3^0), (U^0) or (H_1), (H_2^∞), (H_3^∞), (U^∞) we may take from Example 2.3 and Example 2.5, respectively, requiring only that $Q \in L^1(\Omega) \cap L^{p'}(\Omega)$. Additionally, we may remove the assumption about the radial symmetry of Q. Then, such a nonlinearity does not satisfy (F_1). On the other hand, if we choose $Q \in L^1(\Omega) \setminus L^{p'}(\Omega)$ in Example 2.3 or Example 2.5, the condition (Q) does not hold. This means that the hypotheses (F_1), (F_2), (F_3^a) and (F_4^a) are independent from the hypotheses (Q), (H_1), (H_2^a)–(H_4^a), where $a = 0$ or $a = \infty$.

Example 3.5. Now we will give an example of a function which satisfies (H_1), (H_2^0), (H_3^0), (H_4^0) and does not satisfy (U^0). Choose any function Q satisfying (Q). Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in Ω and let $\{r_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $B_{r_k}(x_k) \subset \Omega$ for $k \in \mathbb{N}$ and $B_{r_k}(x_k) \cap B_{r_l}(x_l) = \emptyset$ for $k \ne l$. Note that if Ω is bounded, then $\{r_k\}_{k\in\mathbb{N}}$ is a null sequence. Choose any $l_0 \ge 0$ and let $\{L_k\}_{k\in\mathbb{N}}$ be a sequence with

$$L_k > \frac{1}{\|Q\|_{L^1(B_k')}}\left[\frac{1}{p}\left(\frac{2}{r_k}\right)^p \omega r_k^N \left(1 - \frac{1}{2^N}\right) + \frac{1}{p2^N}\omega r_k^N + l_0 \|Q\|_{L^1(B_k \setminus B_k')}\right]$$

for all $k \in \mathbb{N}$. Let $\{a_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive numbers such that $a_{k+1} < \frac{1}{2}a_k$ for all $k \in \mathbb{N}$ and $\{L_k a_k^{p-1}\}_{k\in\mathbb{N}}$ is a bounded sequence. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x,s) = 8\sum_{k\in\mathbb{N}} L_k a_k^{p-2}\left(\frac{1}{2}a_k - 2\left|s - \frac{3}{4}a_k\right|\right) \cdot \mathbf{1}_{B_{r_k}(x_k)\times[\frac{1}{2}a_k, a_k]}(x,s),$$

where $\mathbf{1}_{A\times B}$ is the indicator of $A \times B$. Obviously, f satisfies (H_3^0). Since for all $x \in \Omega$

$$\sup_{|t|\le a_1} |f(x,t)| = 4\sum_{k\in\mathbb{N}} L_k a_k^{p-1} \cdot \mathbf{1}_{B_{r_k}(x_k)}(x),$$

we have $\sup_{t\in[0,a_1]} |f(\cdot,t)| \in L^\infty(\Omega)$. Moreover, $F \ge 0$ and $F(x, a_k) = L_k a_k^p$ for all $x \in B_{r_k}(x_k)$ and $k \in \mathbb{N}$, which gives (H_4^0). Now, for any $x \in \Omega$ there is at most one $k \in \mathbb{N}$ such that $x \in B_{r_k}(x_k)$. Then $F(x,s) = 0$ for all $s < \frac{1}{2}a_k$. This means that $\lim_{s\to 0^+} \frac{F(x,s)}{s^p} = 0$ for every $x \in \Omega$.

Example 3.6. Now we will give an example of a function which satisfies (H_1), (H_2^∞), (H_3^∞), (H_4^∞) and does not satisfy (U^∞). Let $Q, \{x_k\}_{k\in\mathbb{N}}$ and $\{r_k\}_{k\in\mathbb{N}}$ be such as in Example 3.5. Choose any $l_\infty \geq 0$ and let $\{L_k\}_{k\in\mathbb{N}}$ be a sequence with

$$L_k > \frac{1}{\|Q\|_{L^1(B'_k)}}\left[\frac{1}{p}\left(\frac{2}{r_k}\right)^p \omega r_k^N\left(1-\frac{1}{2^N}\right)+\frac{1}{p2^N}\omega r_k^N + l_\infty \|Q\|_{L^1(B_k\setminus B'_k)}+1\right]$$

for all $k \in \mathbb{N}$. Let $\{a_k\}_{k\in\mathbb{N}}$ be a increasing sequence of positive numbers such that $a_{k+1}-1 > a_k$ for all $k \in \mathbb{N}$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x,s) = 2\sum_{k\in\mathbb{N}} L_k a_k^p \left(1-2\left|s-a_k+\frac{1}{2}\right|\right)\cdot \mathbf{1}_{B_{r_k}(x_k)\times[a_k-1,a_k]}(x,s).$$

Obviously, f satisfies (H_3^∞). Since for all $s \geq 0$ and $x \in \Omega$

$$\sup_{|t|\leq s}|f(x,t)| \leq 2\sum_{k=1}^{\min\{l:s\leq a_l\}} L_k a_k^p \cdot \mathbf{1}_{B_{r_k}(x_k)}(x),$$

the condition (H_2^∞) is satisfied. Moreover, $F \geq 0$ and $F(x,a_k) = L_k a_k^p$ for all $x \in B_{r_k}(x_k)$ and $k \in \mathbb{N}$, which gives (H_4^∞). Now, choose any $x \in \Omega$. Then there is at most one $k \in \mathbb{N}$ such that $x \in B_{r_k}(x_k)$ and so $F(x,s) \leq L_k a_k^p$ for all $s > a_k$. This means that $\lim_{s\to+\infty}\frac{F(x,s)}{s^p} = 0$ for every $x \in \Omega$.

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