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On unbounded solutions of singular IVPs with ϕ-Laplacian

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Abstract. The paper deals with a singular nonlinear initial value problem with a ϕ-Laplacian

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0,\ t > 0, \quad u(0) = u_0 \in [L_0, L],\ u'(0) = 0.$$

Here, f is a continuous function with three roots $\phi(L_0) < 0 < \phi(L)$, $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and function p is positive and increasing on $(0, \infty)$. The problem is singular in the sense that $p(0) = 0$ and $1/p$ may not be integrable in a neighbourhood of the origin. The goal of this paper is to prove the existence of unbounded solutions. The investigation is held in two different ways according to the Lipschitz continuity of functions ϕ^{-1} and f. The case when those functions are not Lipschitz continuous is more involved that the opposite case and it is managed by means of the lower and upper functions method. In both cases, existence criteria for unbounded solutions are derived.

Keywords: second order ODE, time singularity, ϕ-Laplacian, unbounded solution, escape solution, lower and upper functions method.

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1 Introduction

The aim of this paper is to analyse the singular nonlinear equation

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0, \quad t > 0, \tag{1.1}$$

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with the initial conditions

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, L]. \tag{1.2}$$

Here we focus our attention on unbounded solutions of problem (1.1), (1.2) and provide sufficient conditions for their existence, while in [6] we discussed the existence and properties of bounded solutions of problem (1.1), (1.2). So, in a way, this paper completes results obtained in [6].

Problem (1.1), (1.2) is investigated under the basic assumptions

$$\phi \in C^1(\mathbb{R}), \quad \phi'(x) > 0 \text{ for } x \in (\mathbb{R} \setminus \{0\}), \tag{1.3}$$

$$\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0, \tag{1.4}$$

$$L_0 < 0 < L, \quad f(\phi(L_0)) = f(0) = f(\phi(L)) = 0, \tag{1.5}$$

$$f \in C[\phi(L_0), \infty), \quad xf(x) > 0 \text{ for } x \in ((\phi(L_0), \phi(L)) \setminus \{0\}), \quad f(x) \le 0 \text{ for } x > \phi(L), \tag{1.6}$$

$$p \in C[0, \infty) \cap C^1(0, \infty), \quad p'(t) > 0 \text{ for } t \in (0, \infty), \quad p(0) = 0. \tag{1.7}$$

As a model example, we can consider problem (1.1), (1.2) with α-Laplacian $\phi(x) = |x|^\alpha \operatorname{sgn} x$, $\alpha \ge 1$, $x \in \mathbb{R}$, and with a three degree polynomial $f(x) = x(x - \phi(L_0))(\phi(L) - x)$, $x \in \mathbb{R}$. For simplicity we can consider function p as a power function $p(t) = t^\beta$, $\beta > 0$, $t \ge 0$.

Definition 1.1. Let $[0, b) \subset [0, \infty)$ be a maximal interval such that a function $u \in C^1[0, b)$ with $\phi(u') \in C^1(0, b)$ satisfies equation (1.1) for every $t \in (0, b)$. Then u is called a *solution of equation* (1.1) on $[0, b)$. If u is a solution of equation (1.1) on $[0, \infty)$, then u is called a *solution of equation* (1.1). A solution u of equation (1.1) on $[0, b)$ which satisfies the initial conditions (1.2) is called a *solution of problem* (1.1), (1.2) *on* $[0, b)$. If u is a solution of problem (1.1), (1.2) on $[0, \infty)$, then u is called a *solution of problem* (1.1), (1.2).

Definition 1.2. Consider a solution of problem (1.1), (1.2) with $u_0 \in (L_0, L)$ and denote

$$u_{\sup} = \sup\{u(t) \colon t \in [0, \infty)\}.$$

If $u_{\sup} = L$, then u is called a *homoclinic solution* of problem (1.1), (1.2).

If $u_{\sup} < L$, then u is called a *damped solution* of problem (1.1), (1.2).

Remark 1.3. Assumption (1.5) yields that constant functions $u(t) \equiv L_0$, $u(t) \equiv 0$ and $u(t) \equiv L$ are solutions of problem (1.1), (1.2) on $[0, \infty)$ with $u_0 = L_0$, $u_0 = 0$ and $u_0 = L$, respectively. If $u(0) = 0$, then u' cannot be positive on $(0, \delta)$ for any $\delta > 0$, since then u is positive on $(0, \delta)$ and integrating equation (1.1) from 0 to $t \in (0, \delta)$, we get, by (1.6),

$$p(t)\phi(u'(t)) = -\int_0^t p(s) f(\phi(u(s)))\, \mathrm{d}s < 0,$$

a contradiction. Similarly, u' cannot be negative. Therefore, the solution $u(t) \equiv 0$ is the unique solution of problem (1.1), (1.2) with $u_0 = 0$ and clearly, it is a damped solution.

Solutions from Definition 1.2 are bounded. Therefore, we are mostly interested in another type of solutions specified in the next definition.

Definition 1.4. Let u be a solution of problem (1.1), (1.2) on $[0, b)$, where $b \in (0, \infty]$. If there exists $c \in (0, b)$ such that

$$u(c) = L, \quad u'(c) > 0, \tag{1.8}$$

then u is called an *escape solution* of problem (1.1), (1.2) on $[0, b)$.

A special case of equation (1.1) with $\phi(u) \equiv u$ and $p(t) = t^{n-1}$, $n \in \mathbb{N}$, $n \geq 2$

$$\left(t^{n-1}u'(t)\right)' + t^{n-1}f(u(t)) = 0, \quad t > 0,$$

arises in many areas. For example in the study of phase transition of Van der Waals fluids [11], in population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population [10], in the homogeneous nucleation theory [1], in the relativistic cosmology for description of particles which can be treated as domains in the universe [17], or in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [8]. The above nonlinear equation was replaced with its abstract and more general form

$$\left(p(t)u'(t)\right)' + q(t)f(u(t)) = 0, \quad t > 0,$$

which was investigated for $p \equiv q$ in [20–25] and for $p \not\equiv q$ in [5, 7, 26, 27]. Other problems without ϕ-Laplacian close to (1.1), (1.2) can be found in [2–4, 13–15] and those with ϕ-Laplacian in [9, 12, 16, 18, 19].

Analytical properties of solutions of problem (1.1), (1.2) with a ϕ-Laplacian have been already studied in [6] with a focus on existence of bounded solutions on $[0, \infty)$. In more details, the existence of damped solutions was proved for $u_0 \in [\bar{B}, L]$. Some results derived in [6] are also useful here when the existence and properties of unbounded solutions are of interest. Therefore, we recapitulate them in Section 2 for the reader's convenience.

The goal of this paper is to find conditions which guarantee the existence of escape solutions of problem (1.1), (1.2), which are unbounded. The analysis of problem (1.1), (1.2) with a general ϕ-Laplacian includes also $\phi(x) = |x|^\alpha \operatorname{sgn} x$, for $\alpha > 1$. Let us emphasise that in this case, $\phi^{-1}(x) = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x$ is not locally Lipschitz continuous. Since ϕ^{-1} is present in the integral form of (1.1), (1.2)

$$u(t) = u_0 + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)f(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \geq 0,$$

the standard technique based on the Lipschitz property is not applicable here and another approach needs to be developed. Therefore, we distinguish two cases.

- In the first case, where functions ϕ^{-1} and f are Lipschitz continuous, the uniqueness of a solution of problem (1.1), (1.2) is guaranteed. This considerably helps to derive conditions when a sequence of solutions contains an escape solution.

- In the second case, functions ϕ^{-1} and f do not have to be Lipschitz continuous. The lack of uniqueness causes difficulties and therefore is more challenging. The problems are overcome by means of the lower and upper functions method. Also here sufficient conditions for the existence of escape solutions are derived.

Since in general an escape solution needs not be unbounded, criteria for an escape solution to tend to infinity are derived. In this manner, we obtain new existence results for unbounded solutions of problem (1.1), (1.2). The aim of our further research is to analyse the existence of homoclinic solutions.

The paper is organised in the following manner: Preliminary results for an auxiliary problem with a bounded nonlinearity are stated in Section 2. Auxiliary lemmas necessary for proofs of the existence of escape solutions of the auxiliary problem are given in Section 3. The existence of escape solutions of this problem is further discussed in Section 4. Namely, the first existence result in Section 4 is derived by an approach based on the Lipschitz property. The other case without the Lipschitz condition is studied by means of the lower and upper functions method. In Section 5, the criteria for escape solutions of the original problem to be unbounded are proved. The main results about the existence of unbounded solutions with examples are given in Section 6.

2 Preliminary

In order to derive the main existence results about unbounded solutions of problem (1.1), (1.2), we first introduce the auxiliary equation with a bounded nonlinearity

$$(p(t)\phi(u'(t)))' + p(t)\tilde{f}(\phi(u(t))) = 0, \quad t \in (0,\infty), \tag{2.1}$$

where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x < \phi(L_0), \quad x > \phi(L). \end{cases} \tag{2.2}$$

Since $\tilde{f}$ is bounded on $\mathbb{R}$, the maximal interval of existence for each solution of problem (2.1), (1.2) is $[0,\infty)$. In this section, we collect preliminary results for solutions of problem (2.1), (1.2) derived in [6]. Properties, asymptotic behaviour and a priori estimates of such solutions are specified in Lemmas 2.1–2.8. The existence and continuous dependence on initial values of solutions is provided in Theorem 2.9 and Theorem 2.10, respectively.

Lemma 2.1 (Lemma 2.1 b) in [6]). *Let (1.3)–(1.7) hold and let u be a solution of equation (2.1). Assume that there exists $a \geq 0$ such that $u(a) \in (0, L)$ and $u'(a) = 0$. Then $u'(t) < 0$ for $t \in (a, \theta]$, where θ is the first zero of u on (a,∞). If such θ does not exist, then $u'(t) < 0$ for $t \in (a,\infty)$.*

Lemma 2.2 (Lemma 2.2 in [6]). *Let (1.3)–(1.7) hold and let u be a solution of equation (2.1). Assume that there exists $a \geq 0$ such that $u(a) = L$ and $u'(a) = 0$.*

a) Let $\theta > a$ be the first zero of u on (a,∞). Then there exists $a_1 \in [a, \theta)$ such that

$$u(a_1) = L, \quad u'(a_1) = 0, \quad 0 \leq u(t) < L,\ u'(t) < 0,\ t \in (a_1, \theta].$$

b) Let $u > 0$ on $[a,\infty)$ and $u \not\equiv L$ on $[a,\infty)$. Then there exists $a_1 \in [a,\infty)$ such that

$$u(a_1) = L, \quad u'(a_1) = 0, \quad 0 < u(t) < L,\ u'(t) < 0,\ t \in (a_1, \infty).$$

In both cases, $u(t) = L$ for $t \in [a, a_1]$.

Lemma 2.3 (Lemma 2.6 in [6]). *Assume (1.3)–(1.7),*

$$\lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0, \tag{2.3}$$

and

$$\exists \bar{B} \in (L_0, 0) \colon \tilde{F}(\bar{B}) = \tilde{F}(L), \quad \text{where } \tilde{F}(x) = \int_0^x \tilde{f}(\phi(s))\, \mathrm{d}s, \quad x \in \mathbb{R}. \tag{2.4}$$

Let u be a solution of equation (2.1) and $b \geq 0$ and $\theta > b$ be such that

$$u(b) \in [\bar{B}, 0), \quad u'(b) = 0, \quad u(\theta) = 0, \quad u(t) < 0,\ t \in [b, \theta).$$

Then there exists $a \in (\theta,\infty)$ such that

$$u'(a) = 0, \quad u'(t) > 0,\ t \in (b, a), \quad u(a) \in (0, L).$$

Lemma 2.4 (Lemma 2.7 in [6]). *Assume that* (1.3)–(1.7), (2.3) *and* (2.4) *hold. Let u be a solution of equation* (2.1) *and $a \ge 0$ and $\theta > a$ be such that*

$$u(a) \in (0, L], \quad u'(a) = 0, \quad u(\theta) = 0, \quad u(t) > 0,\ t \in [a, \theta).$$

Then there exists $b \in (\theta, \infty)$ such that

$$u'(b) = 0, \quad u'(t) < 0, \quad t \in (a, b), \quad u(b) \in (\bar{B}, 0).$$

Lemma 2.5 (Lemma 2.8 in [6]). *Assume that* (1.3)–(1.7) *and* (2.3) *hold. Let u be a solution of equation* (2.1) *and $b \ge 0$ be such that*

$$u(b) \in (L_0, 0), \quad u'(b) = 0, \quad u(t) < 0,\ t \in [b, \infty).$$

Then

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0.$$

Lemma 2.6 (Lemma 3.1 in [6]). *Assume that* (1.3)–(1.7), (2.3) *and* (2.4) *hold. Let u be a solution of problem* (2.1), (1.2) *with $u_0 \in (L_0, \bar{B})$. Let $\theta > 0$, $a > \theta$ be such that*

$$u(\theta) = 0, \quad u(t) < 0, \quad t \in [0, \theta), \qquad u'(a) = 0, \quad u'(t) > 0, \quad t \in (\theta, a).$$

Then

$$u(a) \in (0, L], \quad u'(t) > 0, \quad t \in (0, a).$$

Lemma 2.7 (Lemma 3.2 in [6]). *Let assumptions* (1.3)–(1.7), (2.3) *and* (2.4) *hold. Let u be a solution of problem* (2.1), (1.2) *with $u_0 \in (L_0, 0) \cup (0, L)$. Then*

$$\begin{aligned} u_0 \in [\bar{B}, 0) \cup (0, L) \quad &\Rightarrow \quad \bar{B} < u(t) < L, \quad t \in (0, \infty), \\ u_0 \in (L_0, \bar{B}) \quad &\Rightarrow \quad u_0 < u(t), \quad t \in (0, \infty). \end{aligned}$$

For the following result, we introduce a function φ

$$\varphi(t) := \frac{1}{p(t)} \int_0^t p(s)\, \mathrm{d}s, \quad t \in (0, T], \quad \varphi(0) = 0. \tag{2.5}$$

This function is continuous on $[0, T]$ and satisfies

$$0 < \varphi(t) \le t, \quad t \in (0, T], \quad \lim_{t\to 0^+} \varphi(t) = 0. \tag{2.6}$$

Moreover, we point out that $\tilde{f}$ is bounded and there exists a constant $\tilde{M} > 0$ such that

$$|\tilde{f}(x)| \le \tilde{M}, \quad x \in \mathbb{R}. \tag{2.7}$$

Lemma 2.8 (Lemma 3.4 in [6]). *Assume* (1.3)–(1.7). *Let u be a solution of problem* (2.1), (1.2) *with $u_0 \in [L_0, L]$. The inequality*

$$\int_0^\beta \frac{p'(t)}{p(t)} \left|\phi(u'(t))\right| \mathrm{d}t \le \tilde{M}(\beta - \varphi(\beta))$$

is valid for every $\beta > 0$. If moreover (2.3) *and* (2.4) *hold, then there exists $\tilde{c} > 0$ such that*

$$|u'(t)| \le \tilde{c}, \quad t \in [0, \infty),$$

for every solution u of (2.1), (1.2) *with $u_0 \in (L_0, 0) \cup (0, L)$.*

The existence of solutions of the auxiliary problem (2.1), (1.2) is proved in [6] by means of the Schauder fixed point theorem. We state this existence result in the next theorem.

Theorem 2.9 (Theorem 4.1 in [6])**.** *Assume* (1.3)–(1.7)*. Then, for each $u_0 \in [L_0, L]$, there exists a solution u of problem* (2.1)*,* (1.2)*.*

The uniqueness of solutions of (2.1), (1.2) follows from the continuous dependence on initial values. This assertion is based on the Lipschitz property, see (2.8) and (2.9).

Theorem 2.10 (Theorem 4.3 in [6])**.** *Assume* (1.3)–(1.7) *and*

$$f \in \mathrm{Lip}\,[\phi(L_0), \phi(L)]\,, \tag{2.8}$$

$$\phi^{-1} \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}). \tag{2.9}$$

Let u_i be a solution of problem (2.1)*,* (1.2) *with $u_0 = B_i \in [L_0, L]$, $i = 1, 2$. Then, for each $\beta > 0$, there exists $K > 0$ such that*

$$\|u_1 - u_2\|_{C^1[0,\beta]} \le K|B_1 - B_2|.$$

Furthermore, any solution of problem (2.1)*,* (1.2) *with $u_0 \in [L_0, L]$ is unique.*

Remark 2.11. The above lemmas are proved in [6] under the weaker assumption

$$\limsup_{t\to\infty} \frac{p'(t)}{p(t)} < \infty$$

instead of condition (2.3). Similarly, no sign condition of $f(x)$, $x \notin [L_0, L]$ is needed in [6] while here we use (1.6). To keep the formulation as simple as possible, we decided to use these additional conditions in formulations of results in this section, whereas the results are proved in [6] without it.

3 Auxiliary results

In this section, we provide auxiliary lemmas, which are used in Section 4 for proofs of the existence of escape solutions of the auxiliary problem (2.1), (1.2).

Note that all solutions of problem (2.1), (1.2) with $u_0 \in [\bar{B}, L)$ are damped solutions, see Remark 1.3 and Lemma 2.7. Therefore, we consider only $u_0 \in [L_0, \bar{B})$ for investigation of escape solutions of problem (2.1), (1.2). Such solutions can be equivalently characterized as follows.

Lemma 3.1. *Let* (1.3)–(1.7)*,* (2.3) *and* (2.4) *hold and let u be a solution of problem* (2.1)*,* (1.2)*. Then u is an escape solution if and only if*

$$\sup\{u(t) \colon t \in [0, \infty)\} > L. \tag{3.1}$$

Proof. Let u fulfils (3.1). According to Definition 1.2, u is not a damped solution and hence, due to Lemma 2.7, $u(0) < \bar{B} < 0$. Consequently, there exists a maximal $c > 0$ such that $u(t) < L$ for $t \in [0, c)$ and

$$u(c) = L, \quad u'(c) \ge 0.$$

Assume that $u'(c) = 0$. Using Lemma 2.2 (and in the case of more roots of u also Lemma 2.3 and Lemma 2.4), we get that

$$\sup\{u(t) \colon t \in [0, \infty)\} = u(c) = L,$$

contrary to (3.1). Therefore, u fulfils (1.8). On the other hand, if u is an escape solution of problem (2.1), (1.2), then (3.1) follows immediately from Definition 1.4. □

The proofs of the existence of escape solutions are based on Lemma 3.2 and Lemma 3.5. These lemmas are denoted here as Basic lemmas because they are essential for the proof of existence of escape solutions. The Basic lemma I, Lemma 3.2, fully covers the case when the uniqueness of solutions of (2.1), (1.2) is guaranteed. In particular, $u \equiv L_0$ is the unique solution with $u_0 = L_0$. Therefore, $u_0 = L_0$ is not discussed in the context of escape solutions. The situation is different when (2.8) and (2.9) do not hold, see Basic lemma II, Lemma 3.5.

Lemma 3.2 (Basic lemma I)**.** *Let (1.3)–(1.7), (2.3) and (2.4) hold. Choose $C \in (L_0, \bar{B})$ and a sequence $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. Let for each $n \in \mathbb{N}$, u_n be a solution of problem (2.1), (1.2) with $u_0 = B_n$ and let $(0, b_n)$ be the maximal interval such that*

$$u_n(t) < L, \quad u_n'(t) > 0, \quad t \in (0, b_n). \tag{3.2}$$

Finally, let $\gamma_n \in (0, b_n)$ be such that

$$u_n(\gamma_n) = C, \quad \forall\, n \in \mathbb{N}. \tag{3.3}$$

If the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded, then the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (2.1), (1.2).

Proof. Let the sequence $\{\gamma_n\}_{n=1}^{\infty}$ be unbounded, then there exists a subsequence going to infinity as $n \to \infty$. For simplicity, let us denote it by $\{\gamma_n\}_{n=1}^{\infty}$. Then we have

$$\lim_{n\to\infty} \gamma_n = \infty, \quad \gamma_n < b_n, \quad n \in \mathbb{N}.$$

Assume on the contrary that for any $n \in \mathbb{N}$, u_n is not an escape solution of problem (2.1), (1.2). By Lemma 3.1,

$$\sup\{u_n(t) : t \in [0, \infty)\} \le L,\ n \in \mathbb{N}. \tag{3.4}$$

STEP 1. Fix $n \in \mathbb{N}$ and consider a solution u_n of problem (2.1), (1.2) with $u_0 = B_n$. First assume that $u_n < 0$ on $[0, \infty)$. Then, by Lemma 2.1, we get $u_n' > 0$ on $(0, \infty)$, and for $b_n = \infty$, we obtain (3.2). In addition, we get by Lemma 2.5

$$\lim_{t\to\infty} u_n(t) = 0, \quad \lim_{t\to\infty} u_n'(t) = 0.$$

If we put

$$\lim_{t\to\infty} u_n(t) =: u_n(b_n), \quad \lim_{t\to\infty} u_n'(t) =: u_n'(b_n),$$

we get

$$u_n(b_n) = 0, \quad u_n'(b_n) = 0. \tag{3.5}$$

Now we assume that $\theta > 0$ is the first zero of u_n. By Lemma 2.1, $u_n' > 0$ on $(0, \theta]$.

(i) Let $u_n' > 0$ on (θ, ∞). Then according to (3.4), $0 < u_n < L$ on (θ, ∞) and (3.2) is valid for $b_n = \infty$. First we prove that

$$\lim_{t\to\infty} u_n(t) = L, \quad \lim_{t\to\infty} u_n'(t) = 0.$$

Since u_n is increasing on $(0, \infty)$, then according to (3.4), $0 < u_n < L$ on $(0, \infty)$. We denote

$$\lim_{t\to\infty} u_n(t) =: \ell \in (0, L].$$

Since u_n is a solution of equation (2.1), then

$$\phi'(u_n'(t))\, u_n''(t) + \frac{p'(t)}{p(t)}\, \phi(u_n'(t)) + \tilde{f}(\phi(u_n(t))) = 0, \quad t \in (0,\infty). \tag{3.6}$$

If we restrict the previous equation to the interval (θ,∞) then, by (1.3)–(1.7), we have that

$$\frac{p'(t)}{p(t)}\, \phi(u_n'(t)) > 0, \quad \tilde{f}(\phi(u_n(t))) > 0, \quad \phi'(u_n'(t)) > 0,$$

so we deduce that

$$u_n''(t) < 0, \quad t \in (\theta,\infty).$$

Consequently, u_n' is decreasing on (θ,∞) and so, there must exist $\lim_{t\to\infty} u_n'(t) \geq 0$. If $\lim_{t\to\infty} u_n'(t) = a > 0$, then $\lim_{t\to\infty} u_n(t) = \infty$, which is a contradiction. Therefore,

$$\lim_{t\to\infty} u_n'(t) = 0.$$

Finally, assume that $\ell \in (0,L)$. Letting $t \to \infty$ in (3.6), we get, by (1.4) and (2.3),

$$\phi'(0) \cdot \lim_{t\to\infty} u_n''(t) = -\tilde{f}(\phi(\ell)).$$

Since $\tilde{f}(\phi(\ell)) \in (0,\infty)$, we get $\lim_{t\to\infty} u_n''(t) < 0$, contrary to $\lim_{t\to\infty} u_n'(t) = 0$. Therefore, $\ell = L$. Then

$$u_n(b_n) = L, \quad u_n'(b_n) = 0. \tag{3.7}$$

(ii) Let $a > \theta$ be the first zero of u_n'. By (3.4) we have $u_n(a) \leq L$. For $b_n = a$ we get (3.2) and

$$u_n(b_n) \in (0,L], \quad u_n'(b_n) = 0. \tag{3.8}$$

To summarize (3.5), (3.7), (3.8), we see that u_n fulfils:

$$u_n(b_n) \in [0,L], \quad u_n'(b_n) = 0. \tag{3.9}$$

STEP 2. Let n be fixed. We define

$$E_n(t) := \int_0^{u_n'(t)} x\phi'(x)\,\mathrm{d}x + \tilde{F}(u_n(t)), \quad t \in (0,b_n),$$

and

$$K_n := \sup\left\{\frac{p'(t)}{p(t)} : t \in [\gamma_n, b_n)\right\}.$$

Due to (2.3), $\lim_{n\to\infty} K_n = 0$. In addition,

$$\exists \overline{\gamma}_n \in [\gamma_n, b_n) : u_n'(\overline{\gamma}_n) = \max\{u_n'(t) : t \in [\gamma_n, b_n)\}. \tag{3.10}$$

Then, by (3.6), the following holds

$$\begin{aligned}\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} &= u_n'(t)\,\phi'(u_n'(t))\,u_n''(t) + \tilde{f}(\phi(u_n(t)))\,u_n'(t) \\ &= -\frac{p'(t)}{p(t)}\quad \phi(u_n'(t))\,u_n'(t) < 0, \quad t \in (0,b_n).\end{aligned}$$

Integrating the above equality over (γ_n, b_n) and using (3.2), (3.10), we obtain

$$E_n(\gamma_n) - E_n(b_n) = \int_{\gamma_n}^{b_n} \frac{p'(t)}{p(t)} \phi(u_n'(t)) u_n'(t)\,\mathrm{d}t \le \phi(u_n'(\overline{\gamma}_n)) \int_{\gamma_n}^{b_n} \frac{p'(t)}{p(t)} u_n'(t)\,\mathrm{d}t$$
$$\le \phi(u_n'(\overline{\gamma}_n)) K_n \int_{\gamma_n}^{b_n} u_n'(t)\,\mathrm{d}t \le \phi(u_n'(\overline{\gamma}_n)) K_n (L - C).$$

Hence, we have

$$E_n(\gamma_n) \le E_n(b_n) + \phi(u_n'(\overline{\gamma}_n)) K_n (L - C).$$

Moreover, from (3.9), we have

$$E_n(\gamma_n) > F(u_n(\gamma_n)) = F(C), \quad E_n(b_n) = F(u_n(b_n)) \le F(L).$$

This leads to

$$F(C) < E_n(\gamma_n) \le F(L) + \phi(u_n'(\overline{\gamma}_n)) K_n (L - C).$$

Hence, we derive the estimate

$$\frac{F(C) - F(L)}{L - C} \frac{1}{K_n} < \phi(u_n'(\overline{\gamma}_n)). \tag{3.11}$$

STEP 3. We consider a sequence $\{u_n\}_{n=1}^{\infty}$. Since $\lim_{n\to\infty} K_n = 0$, we derive from (3.11) that

$$\lim_{n\to\infty} \phi(u_n'(\overline{\gamma}_n)) = \infty. \tag{3.12}$$

Using (1.4), we obtain

$$\lim_{n\to\infty} u_n'(\overline{\gamma}_n) = \lim_{n\to\infty} \phi^{-1}(\phi(u_n'(\overline{\gamma}_n))) = \infty.$$

Since $\tilde{F} \ge 0$ and E_n is decreasing on $(0, b_n)$,

$$\int_0^{u_n'(\overline{\gamma}_n)} x\phi'(x)\,\mathrm{d}x \le E_n(\overline{\gamma}_n) \le E_n(\gamma_n) \le \tilde{F}(L) + \phi(u_n'(\overline{\gamma}_n)) K_n (L - C), \quad n \in \mathbb{N}$$

therefore,

$$\lim_{n\to\infty} \left(\int_0^{u_n'(\overline{\gamma}_n)} x\phi'(x)\,\mathrm{d}x - \phi(u_n'(\overline{\gamma}_n)) K_n (L - C) \right) \le \tilde{F}(L) < \infty.$$

Since

$$\lim_{n\to\infty} u_n'(\overline{\gamma}_n) = \infty,$$

then there exists $n_0 \in \mathbb{N}$ such that

$$u_n'(\overline{\gamma}_n) > 1, \quad n \ge n_0.$$

Therefore,

$$\int_0^{u_n(\overline{\gamma}_n)} x\phi'(x)\,\mathrm{d}x > \int_1^{u_n'(\overline{\gamma}_n)} x\phi'(x)\,\mathrm{d}x > \int_1^{u_n'(\overline{\gamma}_n)} \phi'(x)\,\mathrm{d}x = \phi(u_n'(\overline{\gamma}_n)) - \phi(1), \quad n \ge n_0.$$

By (3.12) and $\lim_{n\to\infty} K_n = 0$ we derive

$$\lim_{n\to\infty} \left(\int_0^{u_n'(\overline{\gamma}_n)} x\phi'(x)\,\mathrm{d}x - \phi(u_n'(\overline{\gamma}_n)) K_n (L - C) \right) \ge \lim_{n\to\infty} \phi(u_n'(\overline{\gamma}_n)) \left(1 - K_n(L - C)\right) - \phi(1) = \infty.$$

This yields a contradiction. Therefore, the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (2.1), (1.2). $\square$

If ϕ^{-1} and f are not Lipschitz continuous, then problem (2.1), (1.2) with $u_0 \in [L_0, L] \setminus \{0\}$ can have more solutions. These solutions may be escape solutions. In particular, more solutions can start at L_0, not only the constant solution $u(t) \equiv L_0$. Therefore, we need to extend the assertions of Lemma 3.2 which deal with values greater than L_0 for $u_0 = L_0$. For this purpose next two lemmas are helpful.

Lemma 3.3. *Let (1.3)–(1.7) hold and let u be a solution of problem (2.1), (1.2) such that*

$$u_0 = L_0, \quad u \not\equiv L_0, \quad u(t) \geq L_0 \quad \text{for } t \in [0, \infty). \tag{3.13}$$

Then there exists $a \geq 0$ such that

$$u(t) = L_0 \quad \text{for } t \in [0, a] \tag{3.14}$$

and

$$u'(t) > 0 \quad \text{for } t \in (a, \theta],$$

where θ is the first zero of u on (a, ∞). If such θ does not exist, then $u'(t) > 0$ for $t \in (a, \infty)$. Let $\theta \in (a, \infty)$ and $a_1 > \theta$ be such that

$$u'(a_1) = 0, \quad u'(t) > 0, \ t \in (\theta, a_1). \tag{3.15}$$

Then $u(a_1) \in (0, L]$.

Proof. By (3.13), there exists $\tau > 0$ such that

$$L_0 < u(\tau) < 0. \tag{3.16}$$

Put $a := \inf\{\tau > 0; \text{(3.16) holds}\}$. Then u fulfils (3.14) and $u'(a) = 0$.

Put $\theta := \sup\{\tau > a; \text{(3.16) holds}\}$. Then

$$p(t) \tilde{f}(\phi(u(t))) < 0, \quad t \in (a, \theta). \tag{3.17}$$

Integrating equation (2.1) over $[a, t]$, we get, by (3.17),

$$p(t) \phi\left(u'(t)\right) = -\int_a^t p(s) \tilde{f}\left(\phi(u(s))\right) \, \mathrm{d}s > 0, \quad t \in (a, \theta) \tag{3.18}$$

and, since $p(t) > 0$, necessarily $u'(t) > 0$ for $t \in (a, \theta)$.

If $\theta = \infty$, then the proof is finished. On the other hand, if $\theta < \infty$, then θ is the first zero of u on (a, ∞) and (3.18) yields $u'(\theta) > 0$.

Let $\theta \in (a, \infty)$ and (3.15) hold. Then $u(a_1) > 0$. Assume that $u(a_1) > L$. Then there exists $a_0 \in (\theta, a_1)$ such that $u > L$ on $(a_0, a_1]$. Integrating equation (2.1) over (a_0, a_1) and using (2.2), we obtain

$$p(a_0)\phi(u'(a_0)) - p(a_1)\phi(u'(a_1)) = \int_{a_0}^{a_1} p(s)\tilde{f}(\phi(u(s)))\, \mathrm{d}s = 0$$

and so, $p(a_0)\phi(u'(a_0)) = 0$. Consequently, $u'(a_0) = 0$, contrary to $u' > 0$ on (a, a_1). We have proved that $u(a_1) \leq L$, which completes the proof. $\square$

Lemma 3.4. *Let (1.3)–(1.7) and (2.3) hold and let u be a solution of (2.1), (1.2) satisfying (3.13). Assume that*

$$u(t) < 0, \quad t \in [0, \infty).$$

Then

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0.$$

Proof. The proof is analogous to the proof of Lemma 2.5 but using Lemma 3.3 instead of Lemma 2.1. □

Lemma 3.5 (Basic lemma II)**.** *Let (1.3)–(1.7), (2.3) and (2.4) hold. Choose $C \in (L_0, \bar{B})$. Let for each $n \in \mathbb{N}$, u_n be a solution of problem (2.1), (1.2) with $u_0 = L_0$ and let (a_n, b_n) be the maximal interval such that*

$$L_0 < u_n(t) < L, \quad u_n'(t) > 0, \quad t \in (a_n, b_n).$$

Finally, let $\gamma_n \in (a_n, b_n)$ be such that

$$u_n(\gamma_n) = C, \ \forall n \in \mathbb{N}.$$

If the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded, then the sequence $\{u_n\}_{n=1}^{\infty}$ contains an escape solution of problem (2.1), (1.2) with $u_0 = L_0$.

Proof. The proof is held in an analogous way to the proof of Lemma 3.2 where in Step 1, Lemmas 3.3 and 3.4 are used instead of Lemmas 2.1 and 2.5, respectively. □

4 Existence of escape solutions

This section is devoted to the existence of escape solutions of problem (2.1), (1.2). First, we discuss the existence of escape solutions provided the Lipschitz continuity of ϕ^{-1} and f. For this purpose we choose a sequence of solutions which converges locally uniformly to the constant solution $u \equiv L_0$. In this manner we obtain an unbounded sequence $\{\gamma_n\}_{n=1}^{\infty}$ required in the Basic lemma I, Lemma 3.2 for the existence of an escape solution. This approach fails without the assumption on the Lipschitz condition. This situation is subject of investigation in the rest of this section.

Theorem 4.1 (Existence of escape solutions of problem (2.1), (1.2) I)**.** *Let (1.3)–(1.7), (2.3), (2.4), (2.8) and (2.9) hold. Then there exist infinitely many escape solutions of problem (2.1), (1.2) with different starting values in $(L_0, \bar{B})$.*

Proof. Choose $n \in \mathbb{N}$, $C \in (L_0, \bar{B})$ and $B_n \in (L_0, C)$. By Theorem 2.9 and Theorem 2.10, there exists a unique solution u_n of problem (2.1), (1.2) with $u_0 = B_n$. By Lemma 2.1, there exists a maximal $a_n > 0$ such that $u_n' > 0$ on $(0, a_n)$. Since $u_n(0) < 0$, there exists a maximal $\tilde{a}_n > 0$ such that $u_n < L$ on $[0, \tilde{a}_n)$. If we put $b_n = \min\{a_n, \tilde{a}_n\}$, then (3.2) holds. Further, either $\lim_{t\to\infty} u_n(t) = 0$ or u_n has a zero $\theta_n \in (0, b_n)$, due to Lemmas 2.1 and 2.5. Consequently, there exists $\gamma_n \in (0, b_n)$ satisfying $u_n(\gamma_n) = C$. We see that (3.3) is fulfilled.

Consider a sequence $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. Then we get the sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of problem (2.1), (1.2) with $u_0 = B_n$, and the corresponding sequence of $\{\gamma_n\}_{n=1}^{\infty}$. Assume that $\lim_{n\to\infty} B_n = L_0$. Then, by Theorem 2.10, the sequence $\{u_n\}_{n=1}^{\infty}$ converges locally uniformly on $[0, \infty)$ to the constant function $u \equiv L_0$. Therefore, $\lim_{n\to\infty} \gamma_n = \infty$ and the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded. By Lemma 3.2 there exists $n_0 \in \mathbb{N}$ such that u_{n_0} is an escape solution of problem (2.1), (1.2). We have $u_{n_0}(0) = B_{n_0} > L_0$. Now, consider the unbounded sequence $\{\gamma_n\}_{n=n_0+1}^{\infty}$. By Lemma 3.2 there exists $n_1 \in \mathbb{N}$ such that u_{n_1} is an escape solution of problem (2.1), (1.2) such that $u_{n_1}(0) = B_{n_1} > L_0$. Repeating this procedure, we obtain the sequence $\{u_{n_k}\}_{k=0}^{\infty}$ of escape solutions of problem (2.1), (1.2). □

Now, we investigate the existence of escape solutions in the case when ϕ^{-1} and f do not have to be Lipschitz continuous. In order to prove the existence result, we consider the lower and upper functions method for an auxiliary mixed problem on $[0,T]$. In particular, we use this method to find solutions of (2.1) which satisfy

$$u'(0)=0,\ u(T)=C,\quad C\in[L_0,L]. \tag{4.1}$$

Definition 4.2. A function $u\in\mathcal{C}^1[0,T]$ with $\phi(u')\in\mathcal{C}^1(0,T]$ is a *solution* of problem (2.1), (4.1) if u fulfils (2.1) for $t\in(0,T]$ and satisfies (4.1).

Definition 4.3. A function $\sigma_1\in\mathcal{C}[0,T]$ is a *lower function* of problem (2.1), (4.1) if there exists a finite (possibly empty) set $\Sigma_1\subset(0,T)$ such that $\sigma_1\in\mathcal{C}^2((0,T]\setminus\Sigma_1)$ and

$$\big(p(t)\,\phi(\sigma_1'(t))\big)'+p(t)\,\tilde f(\phi(\sigma_1(t)))\ge 0,\quad t\in(0,T]\setminus\Sigma_1, \tag{4.2}$$
$$-\infty<\sigma_1'(\tau^-)<\sigma_1'(\tau^+)<\infty,\quad \tau\in\Sigma_1, \tag{4.3}$$
$$\sigma_1'(0^+)\ge 0,\ \sigma_1(T)\le C. \tag{4.4}$$

Upper functions are defined analogously as follows.

Definition 4.4. A function $\sigma_2\in\mathcal{C}[0,T]$ is an *upper function* of problem (2.1), (4.1) if there exists a finite (possibly empty) set $\Sigma_2\subset(0,T)$ such that $\sigma_2\in\mathcal{C}^2((0,T]\setminus\Sigma_2)$ and

$$\big(p(t)\,\phi(\sigma_2'(t))\big)'+p(t)\,\tilde f(\phi(\sigma_2(t)))\le 0,\quad t\in(0,T]\setminus\Sigma_2, \tag{4.5}$$
$$-\infty<\sigma_2'(\tau^+)<\sigma_2'(\tau^-)<\infty,\quad \tau\in\Sigma_2, \tag{4.6}$$
$$\sigma_2'(0^+)\le 0,\ \sigma_2(T)\ge C. \tag{4.7}$$

Theorem 4.5 (Lower and upper functions method)**.** *Let* (1.3)–(1.7) *hold and let σ_1 and σ_2 be lower and upper functions of problem* (2.1), (4.1) *such that*

$$\sigma_1(t)\le\sigma_2(t),\quad t\in[0,T].$$

Then problem (2.1), (4.1) *has a solution u such that*

$$\sigma_1(t)\le u(t)\le\sigma_2(t),\quad t\in[0,T].$$

Proof. The proof is divided into two steps.
STEP 1. For $t\in[0,T]$ and $x\in\mathbb{R}$ we define the following auxiliary nonlinearity

$$f^*(t,x)=\begin{cases}\tilde f(\phi(\sigma_1(t)))+\frac{\sigma_1(t)-x}{\sigma_1(t)-x+1}, & x<\sigma_1(t),\\ \tilde f(\phi(x)), & \sigma_1(t)\le x\le\sigma_2(t),\\ \tilde f(\phi(\sigma_2(t)))-\frac{x-\sigma_2(t)}{x-\sigma_2(t)+1}, & x>\sigma_2(t).\end{cases}$$

Note that f^* is bounded, that is there exists $M^*>0$ such that

$$|f^*(t,x)|\le M^*,\quad (t,x)\in[0,T]\times\mathbb{R}. \tag{4.8}$$

Consider the auxiliary equation

$$\big(p(t)\,\phi(u'(t))\big)'+p(t)\,f^*(t,u(t))=0,\quad t\in(0,T]. \tag{4.9}$$

Integrating (4.9), we get the equivalent form of problem (4.9), (4.1):

$$u(t)=C-\int_t^T\phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\,f^*(\tau,u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s,\quad t\in[0,T].$$

Now, consider the Banach space $C[0,T]$ with the maximum norm and define an operator $\mathcal{F}\colon C[0,T] \to C[0,T]$,

$$(\mathcal{F}u)(t) := C - \int_t^T \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\, f^*(\tau,u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Put $\Lambda := \max\{|L_0|, L\}$ and consider the ball $\mathcal{B}(0,R) = \{u \in C[0,T] : \|u\|_{C[0,T]} \le R\}$, where $R := \Lambda + T\phi^{-1}(M^*T)$ and M^* is from (4.8). Since ϕ is increasing on $\mathbb{R}$, ϕ^{-1} is also increasing on $\mathbb{R}$ and, by (2.6), $\phi^{-1}(M^*\varphi(t)) \le \phi^{-1}(M^*T)$, $t \in [0,T]$, where φ is defined in (2.5). The norm of $\mathcal{F}u$ can be estimated as follows

$$\begin{aligned}\|\mathcal{F}u\|_{C[0,T]} &= \max_{t\in[0,T]}\left|C - \int_t^T \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\, f^*(\tau,u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\right| \\ &\le \Lambda + \int_t^T \left|\phi^{-1}(M^*\varphi(s))\right|\mathrm{d}s \le \Lambda + \int_t^T \phi^{-1}(M^*T)\,\mathrm{d}s \le \Lambda + T\phi^{-1}(M^*T) = R,\end{aligned}$$

which yields that $\mathcal{F}$ maps $\mathcal{B}(0,R)$ to itself.

Let us prove that $\mathcal{F}$ is compact on $\mathcal{B}(0,R)$. Choose a sequence $\{u_n\} \subset C[0,T]$ such that $\lim_{n\to\infty}\|u_n - u\|_{C[0,T]} = 0$. We have

$$\begin{aligned}(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t) = -\int_t^T \Bigg(&\phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)f^*(\tau,u_n(\tau))\,\mathrm{d}\tau\right) \\ &+\phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)f^*(\tau,u(\tau))\,\mathrm{d}\tau\right)\Bigg)\,\mathrm{d}s.\end{aligned}$$

Since f^* is continuous on $[0,T]\times\mathbb{R}$, we get

$$\lim_{n\to\infty}\|f^*(\cdot,u_n(\cdot))) - f^*(\cdot,u(\cdot))\|_{C[0,T]} = 0.$$

Put

$$\begin{aligned}A_n(t) &:= -\frac{1}{p(t)}\int_0^t p(\tau)f^*(\tau,u_n(\tau))\,\mathrm{d}\tau, \\ A(t) &:= -\frac{1}{p(t)}\int_0^t p(\tau)f^*(\tau,u(\tau))\,\mathrm{d}\tau, \quad t\in(0,T], \quad A_n(0) = A(0) = 0, \quad n\in\mathbb{N}.\end{aligned}$$

Then, for a fixed $n \in \mathbb{N}$,

$$|A_n(t) - A(t)| = \left|\frac{1}{p(t)}\int_0^t p(\tau)\left(f^*(\tau,u(\tau)) - f^*(\tau,u_n(\tau))\right)\mathrm{d}\tau\right|, \quad t\in(0,T]$$

and, by (2.6) and (4.8), $\lim_{t\to0^+}|A_n(t) - A(t)| = 0$. Therefore, $A_n - A \in C[0,T]$ and

$$\|A_n - A\|_{C[0,T]} \le \|f^*(\cdot,u_n(\cdot)) - f^*(\cdot,u(\cdot))\|_{C[0,T]}\,T, \quad n\in\mathbb{N}.$$

This implies that $\lim_{n\to\infty}\|A_n - A\|_{C[0,T]} = 0$. Using the continuity of ϕ^{-1} on $\mathbb{R}$, we have

$$\lim_{n\to\infty}\left\|\phi^{-1}(A_n) - \phi^{-1}(A)\right\|_{C[0,T]} = 0.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\|\mathcal{F}u_n - \mathcal{F}u\|_{C[0,T]} &= \lim_{n\to\infty}\left\|\int_t^T\left(\phi^{-1}(A_n(s)) - \phi^{-1}(A(s))\right)\mathrm{d}s\right\|_{C[0,T]} \\ &\le T\lim_{n\to\infty}\left\|\phi^{-1}(A_n) - \phi^{-1}(A)\right\|_{C[0,T]} = 0,\end{aligned}$$

that is the operator $\mathcal{F}$ is continuous.

Choose an arbitrary $\varepsilon > 0$ and put $\delta := \frac{\varepsilon}{\phi^{-1}(M^*T)}$. Then, for $t_1, t_2 \in [0,T]$ and $u \in \mathcal{B}(0,R)$,

$$|t_1 - t_2| < \delta \Rightarrow |(\mathcal{F}u)(t_1) - (\mathcal{F}u)(t_2)| = \left| \int_{t_2}^{t_1} \phi^{-1}\left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u(\tau))\, \mathrm{d}\tau \right) \mathrm{d}s \right|$$
$$\leq \left| \int_{t_2}^{t_1} \phi^{-1}(M^* \varphi(s))\, \mathrm{d}s \right| \leq \left| \int_{t_2}^{t_1} \phi^{-1}(M^*T)\, \mathrm{d}s \right| = \phi^{-1}(M^*T)\,|t_1 - t_2| < \phi^{-1}(M^*T)\,\delta = \varepsilon.$$

Hence, functions in $\mathcal{F}(\mathcal{B}(0,R))$ are equicontinuous, and, by the Arzelà–Ascoli theorem, the set $\mathcal{F}(\mathcal{B}(0,R))$ is relatively compact. Consequently, the operator $\mathcal{F}$ is compact on $\mathcal{B}(0,R)$. The Schauder fixed point theorem yields the existence of a fixed point $u^\star$ of $\mathcal{F}$ in $\mathcal{B}(0,R)$. Therefore,

$$u^\star(t) = C - \int_t^T \phi^{-1}\left(-\frac{1}{p(s)} \int_0^s p(\tau) f^*(\tau, u^\star(\tau))\, \mathrm{d}\tau \right) \mathrm{d}s$$

is a solution of (4.9), (4.1).

STEP 2. Now we prove that any solution u of problem (4.9), (4.1) satisfies that

$$\sigma_1(t) \leq u(t) \leq \sigma_2(t), \quad t \in [0,T],$$

and, therefore, it is a solution of problem (2.1), (4.1). Put $v(t) = u(t) - \sigma_2(t)$ for $t \in [0,T]$ and assume that

$$\max\{v(t) :\ t \in [0,T]\} = v(t_0) > 0. \tag{4.10}$$

By (4.6), $v'(\tau^-) < v'(\tau^+)$ for each $\tau \in \Sigma_2$, so $t_0 \notin \Sigma_2$. Moreover, $\sigma_2(T) \geq C$ and $u(T) = C$, so $v(T) \leq 0$ and, consequently, $t_0 \neq T$. Therefore, $t_0 \in [0,T) \setminus \Sigma_2$. We distinguish two cases

(i) If $t_0 = 0$, then (4.1) and (4.7) yield $v'(0^+) = u'(0^+) - \sigma_2'(0^+) = -\sigma_2'(0^+) \geq 0$. If $v'(0^+) > 0$, we get a contradiction with (4.10); hence, $v'(0^+) = 0$.

(ii) If $t_0 \in (0,T) \setminus \Sigma_2$, (4.10) also implies that $v'(t_0) = 0$.

Since $t_0 \in [0,T) \setminus \Sigma_2$, there exists $\delta > 0$ such that $(t_0, t_0 + \delta) \subset (0,T) \setminus \Sigma_2$ and $v(t) > 0$ for $t \in (t_0, t_0 + \delta)$. Moreover, for $t \in (t_0, t_0 + \delta)$, we have that

$$(p(t)\,\phi(u'(t)))' - (p(t)\,\phi(\sigma_2'(t)))' \geq p(t)\left(-f^*(t, u(t)) + \tilde{f}(\phi(\sigma_2(t))) \right) = p(t) \frac{v(t)}{v(t)+1} > 0$$

and integrating the previous expression, we obtain that

$$\int_{t_0}^t \left((p(s)\,\phi(u'(s)))' - (p(s)\,\phi(\sigma_2'(s)))' \right) \mathrm{d}s = p(t)\left(\phi(u'(t)) - \phi(\sigma_2'(t)) \right) > 0, \quad t \in (t_0, t_0 + \delta).$$

Therefore, since ϕ is increasing, we have that $v'(t) > 0$ on $(t_0, t_0 + \delta)$, which is a contradiction with (4.10). Consequently, we have proved that

$$u(t) \leq \sigma_2(t), \quad t \in [0,T].$$

Analogously, it can be proved that

$$u(t) \geq \sigma_1(t), \quad t \in [0,T].$$

We conclude that the solution u of problem (4.9), (4.1) is a solution of (2.1), (4.1). $\square$

The main result of this section is contained in Theorem 4.7. Its proof is based on Lemmas 3.2 and 3.5, where a suitable sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of problem (2.1), (1.2) is used. In order to get such sequence with the starting values equal to L_0 (see part (ii) in the proof of Theorem 4.7), we need the next lemma.

Lemma 4.6. *Let (1.3)–(1.7), (2.3) and (2.4) hold. Choose $C \in (L_0, \bar{B})$ and assume that there exists at least one solution u of problem (2.1), (1.2) satisfying (3.13), that is*

$$u_0 = L_0, \quad u \not\equiv L_0, \quad u(t) \geq L_0 \quad \text{for } t \in [0, \infty).$$

Then there exists $\gamma > 0$ such that for each $T > \gamma$, problem (2.1), (1.2) with $u_0 = L_0$ has a solution u_T satisfying

$$u_T(T) = C, \quad u_T(t) \geq L_0, \ t \in [0, \infty). \tag{4.11}$$

Proof. As a consequence of Lemmas 3.3 and 3.4, we know that either it exists $\theta > 0$ such that $u(\theta) = 0$ or $\lim_{t\to\infty} u(t) = 0$. Because of this we can take

$$\gamma = \min\{t \in [0, \infty); \ u(t) = C\} > 0. \tag{4.12}$$

Now, we fix $T > \gamma$ and prove the assertion in three steps.
STEP 1. Construction of a lower function of problem (2.1), (4.1):
We prove that $\sigma_1 \equiv L_0$ satisfies conditions (4.2)–(4.4). First,

$$(p(t)\phi(\sigma_1'(t)))' + p(t)\tilde{f}(\phi(\sigma_1(t))) = (p(t)\phi(0))' + p(t)\tilde{f}(\phi(L_0)) = 0 \geq 0, \ t \in [0, T].$$

Moreover, in this case, $\sigma_1 \in C^2[0, T]$, so $\Sigma_1 = \emptyset$. Finally,

$$\sigma_1'(0^+) = 0 \geq 0 \text{ and } \sigma_1(T) = L_0 < C.$$

Therefore, σ_1 is a lower function of (2.1), (4.1).
STEP 2. Construction of an upper function of problem (2.1), (4.1):
We distinguish two different cases.

(i) If $u < 0$ on $[0, \infty)$, we choose $\sigma_2 = u$. First,

$$(p(t)\phi(\sigma_2'(t)))' + p(t)\tilde{f}(\phi(\sigma_2(t))) = 0 \leq 0, \ t \in (0, T].$$

Moreover, in this case, $\sigma_2 \in C^2(0, T]$, so $\Sigma_2 = \emptyset$. Finally,

$$\sigma_2'(0^+) = 0 \leq 0 \quad \text{and} \quad \sigma_2(T) > \sigma_2(\gamma) = C.$$

The last inequality $\sigma_2(T) > C$ is a consequence of the fact that from Lemma 3.3 we know that σ_2 is increasing on $[a, \infty)$ for some $a \in [0, \gamma)$. Hence, σ_2 satisfies conditions (4.5)–(4.7).

(ii) If there exists $\theta > 0$ such that $u(\theta) = 0$ then $\gamma \in (0, \theta)$ and we choose

$$\sigma_2(t) = \begin{cases} u(t), & t \in [0, \theta], \\ 0, & t \in (\theta, \infty). \end{cases}$$

First,

$$(p(t)\phi(\sigma_2'(t)))' + p(t)\tilde{f}(\phi(\sigma_2(t))) = 0 \leq 0, \quad t \in (0, T] \setminus \{\theta\}.$$

In this case, $\Sigma_2 = \{\theta\}$. From Lemma 3.3, we know that $u' > 0$ on $(a, \theta]$ for some $a \in [0, \gamma)$ and hence, $\sigma_2'(\theta^-) > 0$. It is clear that $\sigma_2'(\theta^+) = 0$, so $\sigma_2'(\theta^+) < \sigma_2'(\theta^-)$.

Finally, analogously to case (i),

$$\sigma_2'(0^+) = 0 \leq 0 \text{ and } \sigma_2(T) > \sigma_2(\gamma) = u(\gamma) = C.$$

Therefore, σ_2 satisfies conditions (4.5)–(4.7) and so, σ_2 is an upper function of (2.1), (4.1).
STEP 3. Existence of a solution u_T:
We have found a pair of lower and upper functions which clearly satisfy that

$$\sigma_1(t) \le \sigma_2(t), \quad t \in [0,T] \text{ for each } T > \gamma.$$

As a consequence, Theorem 4.5 ensures the existence of a solution u_T of problem (2.1), (4.1) such that

$$L_0 \le u_T(t) \le \sigma_2(t), \quad t \in [0,T].$$

Since $\sigma_2(0) = u_T(0) = L_0$, u satisfies (1.2) with $u_0 = L_0$.

Finally, since $\tilde{f}(\phi)$ is bounded on $\mathbb{R}$, u_T can be extended to interval $[0,\infty)$ as a solution of equation (2.1). This classical extension result follows from more general Theorem 11.5 in [14]. The estimate $u_T > L_0$ on $[0,\infty)$ can be proved in the same way as in the proof of Lemma 3.2 in [6] using Lemma 3.3 instead of Lemmas 2.1 and 2.6.

Therefore, u_T is a solution of problem (2.1), (1.2) with $u_0 = L_0$ and satisfies (4.11). □

Theorem 4.7 (Existence of escape solutions of problem (2.1), (1.2) II)**.** *Let* (1.3)–(1.7), (2.3) *and* (2.4) *hold. Then there exist infinitely many escape solutions of problem* (2.1), (1.2) *with not necessarily different starting values in* $[L_0, \bar{B})$*.*

Proof. Choose $n \in \mathbb{N}$, $C \in (L_0, \bar{B})$ and $B_n \in (L_0, C)$. By Theorem 2.9, there exists a solution u_n of problem (2.1), (1.2) with $u_0 = B_n$. By Lemma 2.1, there exists a maximal $a_n > 0$ such that $u_n' > 0$ on $(0, a_n)$. Since $u_n(0) < 0$, there exists a maximal $\tilde{a}_n > 0$ such that $u_n < L$ on $[0, \tilde{a}_n)$. If we put $b_n = \min\{a_n, \tilde{a}_n\}$, then (3.2) holds. Due to Lemmas 2.1 and 2.5, there exists $\gamma_n \in (0, b_n)$ such that $u_n(\gamma_n) = C$.

Consider a sequence $\{B_n\}_{n=1}^{\infty} \subset (L_0, C)$. Then we get a sequence $\{u_n\}_{n=1}^{\infty}$ of solutions of problem (2.1), (1.2) with $u_0 = B_n$, and the corresponding sequence of $\{\gamma_n\}_{n=1}^{\infty}$. Assume that $\lim_{n\to\infty} B_n = L_0$.

Now, integrating equation (2.1) we get the equivalent form of problem (2.1), (1.2) for u_n

$$u_n(t) = B_n + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\,\tilde{f}(\phi(u_n(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [0,\infty). \tag{4.13}$$

We prove that the sequence $\{u_n\}_{n=1}^{\infty}$ is uniformly bounded on $[0,\beta]$ for all $\beta > 0$. Indeed, for $t \in [0,\beta]$,

$$|u_n(t)| \le |L_0| + \int_0^t \left|\phi^{-1}(\tilde{M}\,\varphi(s))\right|\mathrm{d}s \le |L_0| + \int_0^t \phi^{-1}(\tilde{M}\,\beta)\,\mathrm{d}s \le |L_0| + \beta\,\phi^{-1}(\tilde{M}\,\beta) =: K_\beta,$$

where φ is defined in (2.5) and $\tilde{M}$ is from (2.7). Moreover, as a consequence of Lemma 2.8, we know that the sequence of derivatives $\{u_n'\}_{n=1}^{\infty}$ is uniformly bounded. Therefore, the sequence $\{u_n\}_{n=1}^{\infty}$ is equicontinuous. Therefore, by Arzelà–Ascoli theorem, there exists a subsequence of $\{u_n\}_{n=1}^{\infty}$ which converges locally uniformly on $[0,\infty)$ to a continuous function u. To the sake of simplicity we denote this subsequence also as $\{u_n\}_{n=1}^{\infty}$. In particular, if we take the limit when t goes to infinity on equation (4.13), since the convergence is locally uniform, we obtain that u satisfies the following

$$u(t) = L_0 + \int_0^t \phi^{-1}\left(-\frac{1}{p(s)}\int_0^s p(\tau)\,\tilde{f}(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [0,\infty),$$

and therefore, u is a solution of problem (2.1), (1.2) for $u_0 = L_0$.

Now, we distinguish three different cases.

(i) $u \equiv L_0$:

In this case, $\lim_{n\to\infty} \gamma_n = \infty$ and the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is unbounded.

By Lemma 3.2 there exists $n_0 \in \mathbb{N}$ such that u_{n_0} is an escape solution of problem (2.1), (1.2). We have $u_{n_0}(0) = B_{n_0} > L_0$. Now consider the unbounded sequence $\{\gamma_n\}_{n=n_0+1}^{\infty}$. By Lemma 3.2 there exists $n_1 \in \mathbb{N}$ such that u_{n_1} is an escape solution of problem (2.1), (1.2) with $u_{n_1}(0) = B_{n_1} > L_0$. We repeat this procedure and we obtain the sequence $\{u_{n_k}\}_{k=0}^{\infty}$ of escape solutions of problem (2.1), (1.2) with starting values in $(L_0, \bar{B})$.

(ii) $u \not\equiv L_0$ is not an escape solution:

In this case, we define $\tilde{B}_n = L_0$ for all $n \in \mathbb{N}$ and consider γ defined in (4.12). Now, we can take an unbounded sequence $\{\tilde{\gamma}_n\}_{n=1}^{\infty}$ such that $\tilde{\gamma}_n > \gamma$ for all $n \in \mathbb{N}$. By Lemma 4.6, for all $n \in \mathbb{N}$ there exists a solution $\tilde{u}_n$ of problem (2.1), (1.2) with $u_0 = \tilde{B}_n$ such that

$$\tilde{u}_n(\tilde{\gamma}_n) = C, \quad \tilde{u}_n(t) \geq L_0, \quad t \in [0, \infty).$$

Therefore, we have a sequence of solutions $\{\tilde{u}_n\}_{n=1}^{\infty}$ in the conditions of Lemma 3.5 and so, this sequence contains an escape solution $\tilde{u}_{n_0}$ of (2.1), (1.2) with $u_0 = L_0$. As in the previous case, we could consider now the unbounded sequence $\{\tilde{\gamma}_n\}_{n=n_0+1}^{\infty}$ and repeat the procedure from (i). This way we obtain a sequence $\{\tilde{u}_{n_k}\}_{k=0}^{\infty}$ of escape solutions of problem (2.1), (1.2) with $u_0 = L_0$.

(iii) $u \not\equiv L_0$ is an escape solution:

In this case, we can argue as in (ii) and we also obtain a sequence $\{\tilde{u}_{n_k}\}_{k=0}^{\infty}$ of escape solutions of problem (2.1), (1.2) with $u_0 = L_0$.

Moreover, in this case, since the sequence $\{u_n\}_{n=0}^{\infty}$ converges locally uniformly to an escape solution of (2.1), (1.2), there must exist some n_0 such that u_n is also an escape solution for all $n \geq n_0$. As a consequence we also obtain a sequence $\{u_n\}_{n=n_0}^{\infty}$ of escape solutions of problem (2.1), (1.2) with starting values in $(L_0, \bar{B})$. □

5 Unbounded solutions

In this section, we discuss the original problem (1.1), (1.2) and provide conditions which guarantee that an escape solution of (1.1), (1.2) is unbounded.

Note that solutions of the original problem (1.1), (1.2) and solutions of the auxiliary problem (2.1), (1.2) are related in the following way (when (1.3)–(1.7), (2.3) and (2.4) are assumed): Each solution of (2.1), (1.2) which is not an escape solution, is a bounded solution of the original problem (1.1), (1.2) in $[0, \infty)$. This results from Lemma 2.7 and Lemma 3.1, where such solutions of (2.1), (1.2) satisfy

$$L_0 \leq u(t) \leq L, \quad t \in [0, \infty)$$

and, due to (2.2),

$$\tilde{f}(\phi(u(t))) = f(\phi(u(t))), \quad t \in [0, \infty).$$

If u is an escape solution of the auxiliary problem (2.1), (1.2), i.e.

$$\exists\, c \in (0, \infty) \colon u(t) \in [L_0, L),\ t \in [0, c), \quad u(c) = L, \quad u'(c) > 0, \tag{5.1}$$

then u fulfils at once the auxiliary equation (2.1) and the original equation (1.1) on $[0,c]$. The restriction of u on $[0,c]$ can be extended as an escape solution of problem (1.1), (1.2) on some maximal interval $[0,b)$. Therefore, we search for unbounded solutions of (1.1), (1.2) in the set of escape solutions of (1.1), (1.2) on $[0,b)$.

Since in general, an escape solution u of (1.1), (1.2) on $[0,b)$ need not to be unbounded, we derive criteria for u to tend to infinity.

Lemma 5.1. *Assume that* (1.3)–(1.7) *hold. Let u be an escape solution of problem* (1.1), (1.2) *on* $[0,b)$. *Then*

$$u(t) > L, \quad u'(t) > 0, \quad t \in (c,b), \tag{5.2}$$

where c is from (5.1). *If $b < \infty$, then*

$$\lim_{t\to b^-} u(t) = \infty.$$

Proof. Let u be an escape solution of problem (1.1), (1.2) on $[0,b)$. Then (5.1) holds. Assume that there exists $c_1 > c$ such that $u'(c_1) = 0$, $u(t) > L$, $u'(t) > 0$ for $t \in (c,c_1)$. Integrating equation (1.1) over $[c,c_1]$, dividing by $p(t)$ and using (1.3), (1.4), (1.6), (1.7), we get

$$\phi(u'(t)) = \frac{p(c)\phi(u'(c))}{p(t)} - \frac{1}{p(t)}\int_c^t p(s)f(\phi(u(s)))\,\mathrm{d}s > 0, \quad t \in [c,c_1],$$

contrary to $u'(c_1) = 0$. Hence, $u(t) > L$ and $u'(t) > 0$ for $t \in (c,b)$ which yields (5.2).

Let $b < \infty$. Since $[0,b)$ is the maximal interval, where the solution u is defined, u cannot be extended behind b. Therefore, (5.2) gives $\lim_{t\to b^-} u(t) = \infty$ and thus, the solution u is unbounded. □

Since all escape solutions of (2.1), (1.2) on $[0,b)$ which cannot be extended on the halfline $[0,\infty)$ are naturally unbounded, we continue our investigation about unboundedness of escape solutions defined on $[0,\infty)$.

Theorem 5.2. *Assume* (1.3)–(1.7) *hold and let*

$$\lim_{t\to\infty} p(t) < \infty. \tag{5.3}$$

Let u be an escape solution of problem (1.1), (1.2). *Then*

$$\lim_{t\to\infty} u(t) = \infty. \tag{5.4}$$

Proof. Let u be an escape solution of problem (1.1), (1.2). Lemma 5.1 gives (5.2) with $b = \infty$ and so, there exists $\lim_{t\to\infty} u(t) \in (L,\infty]$. Due to (1.3), (1.4), (1.7) and (5.1), $p(c)\phi(u'(c)) =: c_0 \in (0,\infty)$. Integrate equation (1.1) from c to $t > c$ and get, by (1.6) and (1.7),

$$u(t) = L + \int_c^t \phi^{-1}\left(\frac{c_0}{p(s)} - \frac{1}{p(s)}\int_c^s p(\tau)f(\phi(u(\tau)))\,\mathrm{d}\tau\right)\mathrm{d}s > \int_c^t \phi^{-1}\left(\frac{c_0}{p(s)}\right)\mathrm{d}s, \quad t \in (c,\infty).$$

Conditions (1.7) and (5.3) give $\lim_{s\to\infty} \frac{c_0}{p(s)} \in (0,\infty)$ and, by (1.3) and (1.4),

$$\int_1^\infty \phi^{-1}\left(\frac{c_0}{p(s)}\right)\mathrm{d}s = \infty.$$

Therefore,

$$\lim_{t\to\infty} u(t) \ge \int_c^\infty \phi^{-1}\left(\frac{c_0}{p(s)}\right)\mathrm{d}s = \infty,$$

which gives (5.4). □

Theorem 5.3. *Assume* (1.3)–(1.7), (2.3) *and*

$$f(x) < 0 \quad \textit{for } x > \phi(L). \tag{5.5}$$

Let u be an escape solution of problem (1.1), (1.2). *Then* (5.4) *holds.*

Proof. Let u be an escape solution of problem (1.1), (1.2). According to Lemma 5.1, $u' > 0$ on (c, ∞) and hence, there exists $\lim_{t\to\infty} u(t) \in (L, \infty]$. Assume on the contrary that

$$\lim_{t\to\infty} u(t) =: A \in (L, \infty). \tag{5.6}$$

STEP 1. We prove that u' is bounded. Assume that u' is unbounded. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} u'(t_n) = \infty$. The next approach is similar to the proof of Lemma 2.8 in [6]. Equation (1.1) has an equivalent form

$$\phi'(u'(t))u''(t) + \frac{p'(t)}{p(t)}\phi(u'(t)) + f(\phi(u(t))) = 0, \quad t \in (0, \infty). \tag{5.7}$$

Choose $n \in \mathbb{N}$. Multiplying this equation by u' and integrating it from c to $t > c$, we obtain for $t = t_n$ that

$$\psi_1(t_n) + \psi_2(t_n) + \psi_3(t_n) = 0, \quad t_n \in [c, \infty), \tag{5.8}$$

where

$$\psi_1(t_n) = \int_{u'(c)}^{u'(t_n)} x\phi'(x)\,\mathrm{d}x, \quad \psi_2(t_n) = \int_c^{t_n} \frac{p'(s)}{p(s)}\phi(u'(s))u'(s)\,\mathrm{d}s, \quad \psi_3(t_n) = \int_L^{u(t_n)} f(\phi(x))\,\mathrm{d}x.$$

Then $\psi_3(t_n) = F(u(t_n)) - F(L)$, where $F(x) := \int_0^x f(\phi(s))\,\mathrm{d}s$, $x \in \mathbb{R}$. Due to (1.3), (1.4) and (5.5), $F(x)$ is decreasing for $x > \phi(L)$. Since u is increasing on (c, ∞), $F(u(t_n))$ is decreasing for $t_n \in (c, \infty)$ and $\lim_{n\to\infty} F(u(t_n)) = F(A)$. According to (5.6),

$$\lim_{n\to\infty} \psi_3(t_n) \in (-\infty, 0)\,.$$

By (1.3), (1.4) and (1.7),

$$\lim_{n\to\infty} \psi_1(t_n) = \infty, \quad \lim_{n\to\infty} \psi_2(t_n) > 0.$$

Hence, letting $n \to \infty$ in (5.8), we obtain

$$0 = \lim_{n\to\infty} (\psi_1(t_n) + \psi_2(t_n) + \psi_3(t_n)) = \infty,$$

a contradiction. So, u' is bounded.

STEP 2. We prove (5.4). Since u' is bounded, letting $t \to \infty$ in (5.7) and using (2.3), (5.5) and (5.6), we get

$$\lim_{t\to\infty} \phi'(u'(t))u''(t) = -f(\phi(A)) > 0.$$

Since $\phi'(u'(t)) > 0$ for $t > c$, there exists $\tau > c$ such that $u''(t) > 0$ for $t \geq \tau$. Therefore, u' is increasing on $[\tau, \infty)$ and there exists $\lim_{t\to\infty} u'(t) > 0$, which contradicts $\lim_{t\to\infty} u(t) = A < \infty$. Thus, (5.4) is valid. □

Remark 5.4. The proof of Theorem 5.3 yields that if a solution u of problem (1.1), (1.2) satisfies $\lim_{t\to\infty} u(t) =: A \in (L, \infty)$, then $f(\phi(A)) = 0$, which is equivalent with the fact that $u(t) \equiv A$ is a solution of equation (1.1).

For $f \equiv 0$ on $(\phi(L), \infty)$, we are able to find necessary and sufficient condition for the unboundedness of escape solutions of problem (1.1), (1.2).

Theorem 5.5. *Assume* (1.3)–(1.7),

$$f(x) \equiv 0 \quad \text{for } x > \phi(L), \tag{5.9}$$

and

$$\phi(x) = x^{\alpha}, \quad x \in (0, \infty),\ \alpha \geq 1. \tag{5.10}$$

Let u be an escape solution of problem (1.1), (1.2). *Then*

$$\lim_{t\to\infty} u(t) = \infty \quad \iff \quad \int_1^{\infty} \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s = \infty. \tag{5.11}$$

If we replace condition (5.10) *by*

$$\phi(ab) \leq \phi(a)\phi(b), \quad a, b \in (0, \infty), \tag{5.12}$$

then (5.4) *holds if*

$$\int_1^{\infty} \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s = \infty. \tag{5.13}$$

Proof. Let u be an escape solution of problem (1.1), (1.2). According to Lemma 5.1, $u' > 0$ on (c, ∞). Then there exists $t_0 > c$ such that $u(t_0) > L$, $u'(t) > 0$ for $t \in [t_0, \infty)$. Therefore, there exists $\lim_{t\to\infty} u(t) \in (L, \infty]$. By (5.10), $\phi^{-1}(ab) = \phi^{-1}(a)\phi^{-1}(b)$ for $a, b \in (0, \infty)$. Due to (1.3), (1.4), (1.7) and (5.9),

$$p(t_0)\phi(u'(t_0)) =: c_0 \in (0, \infty), \qquad f(\phi(u(t))) = 0 \ \text{ for } t \in [t_0, \infty).$$

Thus, integrating equation (1.1) from t_0 to $t > t_0$, we get

$$\begin{aligned} u(t) =& u(t_0) + \int_{t_0}^{t} \phi^{-1}\left(\frac{c_0}{p(s)}\right) \mathrm{d}s = u(t_0) \\ & + \phi^{-1}(c_0)\left(\int_1^{t} \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s - \int_1^{t_0} \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s\right), \quad t \in (t_0, \infty). \end{aligned}$$

Letting $t \to \infty$ here, we get (5.11).

Let us consider (5.12) instead of (5.10) and assume (5.13). Then we continue analogously and obtain

$$\phi^{-1}(a)\phi^{-1}(b) = \phi^{-1}(\phi(\phi^{-1}(a)\phi^{-1}(b))) \leq \phi^{-1}(\phi(\phi^{-1}(a))\phi(\phi^{-1}(b))) = \phi^{-1}(ab), \ \ a, b \in (0, \infty),$$

$$\begin{aligned} u(t) = u(t_0) &+ \int_{t_0}^{t} \phi^{-1}\left(\frac{c_0}{p(s)}\right) \mathrm{d}s \geq u(t_0) \\ & + \phi^{-1}(c_0)\left(\int_1^{t} \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s - \int_1^{t_0} \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s\right), \quad t \in (t_0, \infty). \end{aligned}$$

We let $t \to \infty$ here and obtain, by (5.13), that (5.4) holds. □

6 Main results and examples

In this section, we first present the existence results about unbounded solutions of the original problem (1.1), (1.2) in the case when ϕ^{-1} and f are Lipschitz continuous, see Theorems 6.1, 6.3 and 6.5. Each of these theorems is afterwards illustrated by an example which is chosen in such a way that only this theorem is applicable, while none of the remaining two theorems can be used for this example.

Then, in Theorems 6.7, 6.9 and 6.11, we present the main existence results about unbounded solutions of the original problem (1.1), (1.2) provided ϕ^{-1} and f do not need to be Lipschitz continuous. The illustration by examples is done as in the previous case and shows that none of these theorems is included in any of two remaining ones.

In the whole section, we assume that (due to Definition 1.1) for each $n \in \mathbb{N}$, $[0, b_n) \subset [0, \infty)$ is a maximal interval such that a function u_n satisfies equation (1.1) for every $t \in (0, b_n)$.

Theorem 6.1. *Assume that* (1.3)–(1.7), (2.3), (2.4), (2.8), (2.9) *and* (5.3) *hold. Then there exist infinitely many unbounded solutions u_n of problem* (1.1), (1.2) *on $[0, b_n)$ with different starting values in $(L_0, \bar{B})$, $n \in \mathbb{N}$.*

Proof. By Theorem 4.1, there exist infinitely many escape solutions u_n of problem (2.1), (1.2) with starting values in $(L_0, \bar{B})$. Let us choose $n \in \mathbb{N}$. Then

$$\exists\, c_n \in (0, \infty) \colon u_n(t) \in (L_0, L),\ t \in [0, c_n), \quad u_n(c_n) = L, \quad u_n'(c_n) > 0.$$

Consider restriction of u_n to $[0, c_n]$. Then there exists $b_n > c_n$ such that u_n can be extended as a solution of problem (1.1), (1.2) on $[0, b_n)$. If $b_n < \infty$, then, due to Lemma 5.1,

$$\lim_{t \to b_n^-} u_n(t) = \infty,$$

so u_n is unbounded. If $b_n = \infty$, then Theorem 5.2 yields

$$\lim_{t \to \infty} u_n(t) = \infty,$$

that is u_n is unbounded, as well. □

Example 6.2. Consider problem (1.1), (1.2) with

$$\phi(x) = \sinh x = \frac{e^x - e^{-x}}{2}, \quad x \in \mathbb{R},$$
$$f(x) = \begin{cases} x(x + \sinh 4)(\sinh 1 - x) & \text{for } x \in [-\sinh 4, \sinh 1], \\ \cos(x - \sinh 1) - 1 & \text{for } x > \sinh 1, \end{cases}$$
$$p(t) = \arctan t \qquad \text{or} \qquad p(t) = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \qquad t \in [0, \infty).$$

Here $L_0 = -4$, $L = 1$, $\phi^{-1}(x) = \operatorname{arg\,sinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$. These functions p satisfy (1.7), (5.3) and

$$\lim_{t \to \infty} \frac{(\arctan t)'}{\arctan t} = \lim_{t \to \infty} \frac{\frac{1}{t^2+1}}{\arctan t} = 0, \qquad \lim_{t \to \infty} \frac{(\tanh t)'}{\tanh t} = \lim_{t \to \infty} \frac{\frac{1}{\cosh^2 t}}{\tanh t} = 0,$$

that is (2.3) holds, as well. Functions ϕ and f fulfil (1.3)–(1.6). Moreover, $0 < L < -L_0$, ϕ is odd and

$$\begin{aligned}\tilde{F}(L_0) &= \int_0^{-4} \phi(s)\,(\phi(s)+\sinh 4)\,(\sinh 1-\phi(s))\,\mathrm{d}s\\ &= \int_0^{4} \phi(s)\,(\sinh 4-\phi(s))\,(\sinh 1+\phi(s))\,\mathrm{d}s > \int_0^{1} \phi(s)\,(\sinh 4-\phi(s))\,(\sinh 1+\phi(s))\,\mathrm{d}s\\ &> \int_0^{1} \phi(s)\,(\phi(s)+\sinh 4)\,(\sinh 1-\phi(s))\,\mathrm{d}s = \tilde{F}(L),\end{aligned}$$

thus, (2.4) holds. Since f and ϕ^{-1} are Lipschitz continuous, conditions (2.8) and (2.9) are valid, too.

We have fulfilled all assumptions of Theorem 6.1. Since f has isolated zeros on $(\sinh 1, \infty)$, we cannot use Theorem 6.3 and 6.5 here.

In the same way as in the proof of Theorem 6.1, we can prove the following Theorems 6.3 or 6.5, if we use in the proof Theorems 5.3 or 5.5, respectively, instead of Theorem 5.2.

Theorem 6.3. *Let (1.3)–(1.7), (2.3), (2.4), (2.8), (2.9) and (5.5) hold. Then there exist infinitely many unbounded solutions u_n of problem (1.1), (1.2) on $[0, b_n)$ with different starting values in $(L_0, \bar{B})$, $n \in \mathbb{N}$.*

Example 6.4. Let us consider problem (1.1), (1.2) with

$$\begin{aligned}&\phi(x) = \ln(|x|+1)\operatorname{sgn} x, \quad x \in \mathbb{R},\\ &f(x) = x(x+\ln 4)(\ln 2 - x), \quad x \in [-\ln 4, \infty),\\ &p(t) = t^{\beta}, \quad \beta > 0,\ t \in [0,\infty).\end{aligned}$$

Here $L_0 = -3$, $L = 1$, $\phi^{-1}(x) = \left(e^{|x|}-1\right)\operatorname{sgn} x$. We can easily check that ϕ, f and p satisfy (1.3)–(1.7), (2.3) and (5.5). In addition, $0 < L < -L_0$, ϕ is odd and we can show similarly as in Example 6.2 that (2.4) holds. The Lipschitz continuity of f and ϕ^{-1} yields (2.8) and (2.9). Thus, we can apply Theorem 6.3 here. Since $\lim_{t\to\infty} t^{\beta} = \infty$ and $f(x) < 0$ for $x > \ln 2$, we cannot use either Theorem 6.1 or Theorem 6.5.

Theorem 6.5. *Assume that (1.3)–(1.7), (2.3), (2.4), (2.8), (2.9), (5.9), (5.12) and (5.13) hold. Then there exist infinitely many unbounded solutions u_n of problem (1.1), (1.2) on $[0, b_n)$ with different starting values in $(L_0, \bar{B})$, $n \in \mathbb{N}$.*

Example 6.6. Consider problem (1.1), (1.2) with

$$\begin{aligned}&\phi(x) = x, \quad x \in \mathbb{R},\\ &p(t) = \sqrt{t}, \quad t \in [0,\infty),\\ &f(x) = \begin{cases} x^3(x-\phi(L_0))(\phi(L)-x) & \text{for } x \in [\phi(L_0), \phi(L)],\\ 0 & \text{for } x > \phi(L),\end{cases} \qquad 0 < L < -L_0\,.\end{aligned}$$

Functions ϕ, f, p and $\phi^{-1}(x) = x$ satisfy (1.3)–(1.7), (2.3), (2.8), (2.9), (5.9), (5.10) and consequently, (5.12). Since $f(\phi(x)) = f(x)$ and $L < -L_0$, we have $\tilde{F}(L) < \tilde{F}(L_0)$ and (2.4) holds. In addition,

$$\int_1^{\infty} \phi^{-1}\left(\frac{1}{p(s)}\right)\mathrm{d}s = \int_1^{\infty} \frac{1}{\sqrt{s}}\,\mathrm{d}s = \infty,$$

which yields (5.13). We have verified all assumptions of Theorem 6.5. Since $\lim_{t\to\infty}\sqrt{t} = \infty$ and $f(x) = 0$ for $x > \phi(L)$, we cannot use either Theorem 6.1 or Theorem 6.3.

Now, applying Theorem 4.7 instead of Theorem 4.1, we get as before the existence results about unbounded solutions in the case when ϕ^{-1} and f do not have to be Lipschitz continuous.

Theorem 6.7. *Let* (1.3)–(1.7), (2.3), (2.4) *and* (5.3) *hold. Then there exist infinitely many unbounded solutions u_n of problem* (1.1), (1.2) *on $[0, b_n)$ with not necessarily different starting values in $[L_0, \bar{B})$, $n \in \mathbb{N}$.*

Example 6.8. Let us consider problem (1.1), (1.2) with

$$\phi(x) = |x|^\alpha \operatorname{sgn} x, \quad \alpha > 1, \ x \in \mathbb{R},$$

$$f(x) = \begin{cases} \sqrt{|x|} \operatorname{sgn} x (x - \phi(L_0))(\phi(L) - x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ (\phi(L) - x)(\phi(2L) - x) & \text{for } x \in (\phi(L), \phi(2L)), \\ 0 & \text{for } x \geq \phi(2L), \end{cases} \qquad 0 < L < -L_0,$$

$$p(t) = \arctan t \qquad \text{or} \qquad p(t) = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \qquad t \in [0, \infty).$$

According to Example 6.2, functions p satisfy (1.7), (2.3) and (5.3). Functions ϕ and f fulfil (1.3)–(1.6). Since f is continuous, $0 < L < -L_0$ and ϕ is a continuous and odd function, (2.4) holds, too.

We have verified all assumptions of Theorem 6.7. The form of f implies that neither Theorem 6.9 nor Theorem 6.11 can be applied.

Theorem 6.9. *Assume that* (1.3)–(1.7), (2.3), (2.4) *and* (5.5) *hold. Then there exist infinitely many unbounded solutions u_n of problem* (1.1), (1.2) *on $[0, b_n)$ with not necessarily different starting values in $[L_0, \bar{B})$, $n \in \mathbb{N}$.*

Example 6.10. Consider problem (1.1), (1.2) with

$$\begin{aligned} \phi(x) &= x^3, \quad x \in \mathbb{R}, \\ f(x) &= \sqrt[3]{x}\,(x+8)(1-x), \quad x \in [-8, \infty), \\ p(t) &= t^\beta, \quad \beta > 0, \ t \in [0, \infty). \end{aligned}$$

Here $L_0 = -2$, $L = 1$, $\phi^{-1}(x) = \sqrt[3]{x}$. It is easy to see that ϕ, f and p fulfil (1.3)–(1.7), (2.3) and (5.5). Further,

$$\tilde{F}(L_0) = \int_0^{-2} s\left(s^3+8\right)\left(1-s^3\right)\,\mathrm{d}s = \frac{144}{5}, \qquad \tilde{F}(L) = \int_0^{1} s\left(s^3+8\right)\left(1-s^3\right)\,\mathrm{d}s = \frac{99}{40}.$$

So, $\tilde{F}(L_0) > \tilde{F}(L)$ which yields (2.4). Therefore, we can apply Theorem 6.9 here. Since $\lim_{t\to\infty} t^\beta = \infty$ and $f(x) < 0$ for $x > 1$, we cannot use either Theorem 6.7 or Theorem 6.11.

Theorem 6.11. *Let* (1.3)–(1.7), (2.3), (2.4), (5.9), (5.12) *and* (5.13) *hold. Then there exist infinitely many unbounded solutions u_n of problem* (1.1), (1.2) *on $[0, b_n)$ with not necessarily different starting values in $[L_0, \bar{B})$, $n \in \mathbb{N}$.*

Example 6.12. Let us consider problem (1.1), (1.2) with

$$\phi(x) = |x|^\alpha \operatorname{sgn} x, \ \alpha > 1, \ x \in \mathbb{R},$$

$$p(t) = t^\beta, \ \beta \in (0, \alpha], \ t \in [0, \infty),$$

$$f(x) = \begin{cases} \sqrt[3]{x}\,(x - \phi(L_0))(\phi(L) - x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x > \phi(L), \end{cases} \qquad 0 < L < -L_0.$$

Functions ϕ, f and p satisfy (1.3)–(1.7), (2.3), (5.9), (5.10) and consequently, (5.12). Moreover, $0 < L < -L_0$ and ϕ is odd function which yields (2.4). Further,

$$\phi^{-1}(x) = x^{\frac{1}{\alpha}} \text{ for } x > 0, \quad \int_1^\infty \phi^{-1}\left(\frac{1}{p(s)}\right) \mathrm{d}s = \int_1^\infty s^{-\frac{\beta}{\alpha}} \mathrm{d}s = \infty,$$

that is (5.13) holds and we have verified all assumptions of Theorem 6.11. Since $\lim_{t\to\infty} t^\beta = \infty$ and $f(x) = 0$ for $x > \phi(L)$, neither Theorem 6.7 nor Theorem 6.9 is applicable.

It si clear that every unbounded solution of problem (1.1), (1.2) is an escape solution. According to the proofs of above theorems, we can formulate also the reverse assertion.

Corollary 6.13. *Assume all assumptions of Theorem 6.1 or 6.3 or 6.5 or 6.7 or 6.9 or 6.11. Then each escape solution of problem* (1.1), (1.2) *is unbounded.*

In this paper we discuss the existence of unbounded solutions of the singular nonlinear initial value problem (1.1), (1.2) with a ϕ-Laplacian. In the case when functions f and ϕ^{-1} are Lipschitz continuous, a sequence of escape solutions with different initial values in $(L_0, \bar{B})$ is obtained. The basis of the proof is a sequence of solutions which converge locally uniformly to a solution u with $u_0 = L_0$. By virtue of uniqueness $u \equiv L_0$. This is not guaranteed in the other case when such sequence converging to the constant solution $u \equiv L_0$ might not exist. Therefore we would like to point out the approach without assuming the Lipschitz property of data functions. In this situation, the investigation is not straightforward and requires some efficient idea about how to deal with difficulties caused by the lack of uniqueness. In contrast to the case with Lipschitz data functions, the set of escape solutions with $u_0 \in (L_0, \bar{B})$ might be empty and all escape solutions could start at $u_0 = L_0$. Therefore we cannot just follow the method used in the first case. The technique used here is the method of lower and upper functions which is applied to a sequence of related boundary value problems. The sequence of obtained solutions contains an escape solution, in particular infinitely many escape solutions. These solutions are under suitable conditions unbounded. In this manner, we prove the existence of unbounded solutions of the investigated problem.

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References

[1] F. F. Abraham, *Homogeneous nucleation theory*, Academic Press, New York, 1974.

[2] R. P. Agarwal, D. O'Regan, *Infinite interval problems for differential, difference and integral equations*, Kluwer, Dordrecht 2001. MR1845855; https://doi.org/10.1007/978-94-010-0718-4

[3] H. Berestycki, P. L. Lions, L. A. Peletier, An ODE approach to the existence of positive solutions for semilinear problems in R^n, *Indiana Math. Univ. J.* **30**(1981), 141–157. MR0600039; https://doi.org/10.1512/iumj.1981.30.30012

[4] D. Bonheure, J. M. Gomes, L. Sanchez, Positive solutions of a second-order singular ordinary differential equation, *Nonlinear Anal.* **61**(2005), 1383–1399. MR2135816; https://doi.org/10.1016/j.na.2005.02.029

[5] J. Burkotová, M. Hubner, I. Rachůnková, E. B. Weinmüller, Asymptotic properties of Kneser solutions to nonlinear second order ODEs with regularly varying coefficients, *J. Appl. Math. Comp.* **274**(2015), 65–82. MR3433115; https://doi.org/10.1016/j.amc.2015.10.074

[6] J. Burkotová, I. Rachůnková M. Rohleder, J. Stryja, Existence and uniqueness of damped solutions of singular IVPs with ϕ-Laplacian, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 121, 1–28 MR3592201; https://doi.org/10.14232/ejqtde.2016.1.121

[7] J. Burkotová, M. Rohleder, J. Stryja, On the existence and properties of three types of solutions of singular IVPs, *Electron. J. Qual. Theory Differ. Equ.* **2015**, No. 29, 1–25. MR3353200; https://doi.org/10.14232/ejqtde.2015.1.29

[8] G. H. Derrick, Comments on nonlinear wave equations as models for elementary particles, *J. Math. Physics* **5**(1965), 1252–1254. MR0174304; https://doi.org/10.1063/1.1704233

[9] Z. Došlá, M. Marini, S. Matucci, A boundary value problem on a half-line for differential equations with indefinite weight, *Commun. Appl. Anal.* **15**(2011), 341–352. MR2867356

[10] P. C. Fife, *Mathematical aspects of reacting and diffusing systems*, Lecture Notes in Biomathematics, Vol. 28, Springer-Verlag, Berlin–New York, 1979. MR0527914; https://doi.org/10.1007/978-3-642-93111-6

[11] H. Gouin, G. Rotoli, An analytical approximation of density profile and surface tension of microscopic bubbles for Van der Waals fluids, *Mech. Research Communic.* **24**(1997), 255–260. https://doi.org/10.1016/S0093-6413(97)00022-0

[12] J. Jaroš, T. Kusano, J. V. Manojlović, Asymptotic analysis of positive solutions of generalized Emden–Fowler differential equations in the framework of regular variation, *Cent. Eur. J. Mathc.* **11**(2013), No. 12, 2215–2233. MR3111718; https://doi.org/10.2478/s11533-013-0306-9

[13] I. T. Kiguradze, *Some singular boundary value problems for ordinary differential equations*, ITU, Tbilisi, 1975. MR0499402

[14] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluver Academic, Dordrecht, 1993. MR1220223; https://doi.org/10.1007/978-94-011-1808-8

[15] I. T. Kiguradze, B. Shekhter, *Singular boundary value problems for ordinary differential equations of the second order*, Viniti, Moscow, 1987.

[16] T. Kusano, J. V. Manojlović, J. Milošević, Intermediate solutions of second order quasi-linear ordinary differential equations in the framework of regular variation, *Appl. Math. Comput.* **219**(2013), 8178–8191. MR3037526; https://doi.org/10.1016/j.amc.2013.02.007

[17] A. P. Linde, *Particle physics and inflationary cosmology*, Harwood Academic, Chur, Switzerland, 1990.

[18] M. Marini, S. Matucci, A boundary value problem on the half-line for superlinear differential equations with changing sign weight, *Rend. Istit. Mat. Univ. Trieste* **44**(2012), 117–132. MR3019556

[19] S. Matucci, P. Řehák, Asymptotics of decreasing solutions of coupled p-Laplacian systems in the framework of regular variation, *Ann. Mat. Pura Appl.* **193**(2014), 837–858. MR3210171; https://doi.org/10.1007/s10231-012-0303-9

[20] I. Rachůnková, L. Rachůnek, Asymptotic formula for oscillatory solutions of some singular nonlinear differential equation, *Abstr. Appl. Anal.* **2011**, Art. ID 981401, 9 pp. MR2802847; https://doi.org/10.1155/2011/981401

[21] I. Rachůnková, L. Rachůnek, J. Tomeček, Existence of oscillatory solutions of singular nonlinear differential equations, *Abstr. Appl. Anal.* **2011**, Art. ID 408525, 20 pp. MR2795071; https://doi.org/10.1155/2011/408525

[22] I. Rachůnková, J. Tomeček, Homoclinic solutions of singular nonautonomous second order differential equations, *Bound. Value Probl.* **2009**, Art. ID 959636, 21 pp. MR2552066; https://doi.org/10.1155/2009/959636

[23] I. Rachůnková, J. Tomeček, Bubble-type solutions of nonlinear singular problem, *Math. Comp. Modelling* **51**(2010), 658–669. MR2594716; https://doi.org/10.1016/j.mcm.2009.10.042

[24] I. Rachůnková, J. Tomeček, Strictly increasing solutions of a nonlinear singular differential equation arising in hydrodynamics, *Nonlinear Anal.* **72**(2010), 2114–2118. MR2577608; https://doi.org/10.1016/j.na.2009.10.011

[25] I. Rachůnková, J. Tomeček, J. Stryja, Oscillatory solutions of singular equations arising in hydrodynamics, *Adv. Difference Equ.* **2010**, Art. ID 872160, 13 pp. MR2652448; https://doi.org/10.1155/2010/872160

[26] M. Rohleder, On the existence of oscillatory solutions of the second order nonlinear ODE, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **51**(2012), No. 2, 107–127. MR3058877

[27] J. Vampolová, On existence and asymptotic properties of Kneser solutions to singular second order ODE, *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **52**(2013), No. 1, 135–152. MR3202755