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On a method for constructing a solution of integro-differential equations of fractional order

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Abstract. In this paper, we propose a new method for constructing a solution of the integro-differential equations of Volterra type. The particular solutions of the homogeneous and of the inhomogeneous equation will be constructed and the Cauchy type problems will be investigated. Note that this method is based on construction of normalized systems functions with respect to the differential operator's fractional order.

Keywords: integro-differential equation of Volterra type, Riemann–Liouville fractional integrals and derivatives, generalized Mittag-Leffler function, method normalized systems of functions.

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1 Introduction

Let $\alpha > 0$, $\nu \geq 0$, $\beta \in R$, $\lambda \neq 0$. On the domain $0 < t \leq d < \infty$ consider an integro-differential equation of the following form:

$$D^{\alpha} y(t) = \lambda t^{\beta} I^{\nu} y(t) + f(t), \qquad 0 < t \leq d < \infty, \tag{1.1}$$

where for any $\delta > 0$:

$$I^{\delta} y(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\tau)^{\delta-1} y(\tau) d\tau,$$

and D^{α} is the derivative of α order in the Riemann–Liouville sense, i.e.

$$D^{\alpha} y(t) = \frac{d^m}{dt^m} I^{m-\alpha} y(t), \qquad m = [\alpha] + 1.$$

We denote

$$C_{\delta}[0,d] = \left\{ f(t) : \exists \delta \in [0,1), t^{\delta} f(t) \in C[0,d] \right\}.$$

By a solution of equation (1.1) we mean a function $y(t)$ such that $\exists \delta \in [0, m-\alpha], y(t) \in C_{\delta}[0,d], D^{\alpha} y(t) \in C[0,d]$ and $y(t)$ satisfies equation (1.1) at all points $t \in (0,d)$.

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Questions related to theorems about existence and uniqueness of solutions of Cauchy type and Dirichlet type problems for linear and nonlinear fractional order differential equations have been developed in sufficient detail (see [15] for the main results, review of papers, and references). In [1], equation (1.1) was studied in the case $\beta = 0$. In a more general case, an equation of the type (1.1) and a Cauchy-type problem for them were studied in [3,5,7–9,16]. Theorems on existence and uniqueness of a solution of Cauchy-type problem have been proved. We note that explicit solutions have been constructed only for certain types of linear differential equations of fractional order. Solutions of certain elementary homogeneous and inhomogeneous equations, obtained by the selection method or by expansion of the desired solution into a quasi-power series, are known. Moreover, explicit solutions of a Cauchy-type problem for certain differential equations of fractional order were found in [12] by the method of reduction to an equivalent Volterra integral equation. Further, in [14], using the properties of Mittag-Leffler type functions:

$$E_{\alpha,m,l}(z) = \sum_{i=0}^{\infty} c_i z^i, \qquad c_0 = 1, \qquad c_i = \prod_{k=0}^{i-1} \frac{\Gamma[\alpha(km+l)+1]}{\Gamma[\alpha(km+l+1)+1]}, \qquad i \geq 1, \tag{1.2}$$

an algorithm for constructing a solution of the differential equation (1.1) in the case $\nu = 0$ was proposed. Moreover, the case, when $f(t) = 0$ and $f(t)$ is a quasi-polynomial, was considered. Further, in [13] an analogous algorithm was used to construct a solution of the equation (1.1) in the case $\alpha = \beta = 0$.

In this paper we propose a new method for constructing an explicit solution of integro-differential equations of fractional order. This method is based on construction of normalized systems with respect to a pair of operators $(D^\alpha, \lambda t^\beta)$ (see Section 2). Moreover, in contrast to [14], we construct particular solutions of the inhomogeneous equation for a more general class of functions $f(t)$. We also note that this method was used in [2,17] to construct solutions of certain linear differential equations of integer and fractional order with constant coefficients.

2 Normalized systems

In this section we give some information on normalized systems related to linear differential operators. Let L_1 and L_2 be linear operators, acting from the functional space X to X, $L_k X \subset X$, $k = 1, 2$. Let functions from X be defined in a domain $\Omega \subset R^n$. Let us give the definition of normalized systems [10].

Definition 2.1. A sequence of functions $\{f_i(x)\}_{i=0}^{\infty}$, $f_i(x) \in X$ is called f-normalized with respect to (L_1, L_2) on Ω, having the base $f_0(x)$, if on this domain the following equality holds:

$$L_1 f_0(x) = f(x), \qquad L_1 f_i(x) = L_2 f_{i-1}(x), \qquad i \geq 1.$$

If $L_2 = \mathrm{E}$ is a unit operator, then f-normalized with respect to (L_1, I) system of functions is called f-normalized with respect to L_1, i.e.

$$L_1 f_0(x) = f(x), \qquad L_1 f_i(x) = f_{i-1}(x), \qquad i \geq 1.$$

If $f(x) = 0$, then the system of functions $\{f_i(x)\}$ is called just normalized.

The main properties of the f-normalized systems of functions with respect to the operators (L_1, L_2) on Ω have been described in [11]. Let us consider the main property of the f-normalized systems.

Proposition 2.2. *If a system of functions $\{f_i(x)\}_{i=0}^{\infty}$ is f-normalized with respect to (L_1, L_2) on Ω, then the functional series $y(x) = \sum_{i=0}^{\infty} f_i(x), x \in \Omega$, is a formal solution of the equation:*

$$(L_1 - L_2)\, y(x) = f(x), \qquad x \in \Omega. \tag{2.1}$$

The next proposition allows to construct an f-normalized system with respect to a pair of operators (L_1, L_2).

Proposition 2.3. *If for L_1 there exists a right inverse operator L_1^{-1}, i.e. $L_1 \cdot L_1^{-1} = E$, where E is a unit operator and $L_1 f_0(x) = f(x)$, then a system of the functions*

$$f_i(x) = \left(L_1^{-1} \cdot L_2\right)^i f_0(x), i \geq 1,$$

is f-normalized with respect to a pair of the operators (L_1, L_2) on Ω.

Proof. Since $L_1 \cdot L_1^{-1} = E$ is the unit operator, then for all $i = 1, 2, \ldots,$ we have

$$\begin{aligned} L_1 f_i(x) &= L_1 \left(L_1^{-1} \cdot L_2\right)^i f_0(x) = L_1 \left(L_1^{-1} \cdot L_2\right) \left(L_1^{-1} \cdot L_2\right)^{i-1} f_0(x) \\ &= L_2 \left(L_1^{-1} \cdot L_2\right)^{i-1} f_0(x) = L_2 f_{i-1}(x). \end{aligned}$$

Consequently, $L_1 f_i(x) = L_2 f_{i-1}(x)$ and by assumption of the theorem $L_1 f_0(x) = f(x)$ i.e. the system $f_i(x) = (L_1^{-1} \cdot L_2)^i f_0(x),\ i \geq 0$, is f-normalized with respect to the pair of operators (L_1, L_2). □

3 Properties of operators I^α and D^α

Consider some properties of the operators I^α and D^α.

The following statements are known [15].

Lemma 3.1. *Let $\alpha > 0$. Then for all $f(t) \in C_\delta[0, d]$ the equality*

$$D^\alpha \left[I^\alpha[f]\right](t) = f(t) \tag{3.1}$$

holds for all $t \in (0, d]$. If $f(t) \in C[0, d]$, then (3.2) holds for all $t \in [0, d]$.

Lemma 3.2. *Let $\alpha > 0, m = [\alpha] + 1$ and $s \in R$. Then the following equalities hold:*

$$I^\alpha t^s = \frac{\Gamma(s+1)}{\Gamma(s+1+\alpha)} t^{s+\alpha}, \qquad s > -1, \tag{3.2}$$

$$D^\alpha t^s = \frac{\Gamma(s+1)}{\Gamma(s+1-\alpha)} t^{s-\alpha}, \qquad s > \alpha - 1, \tag{3.3}$$

$$D^\alpha t^s = 0, \qquad s = \alpha - j, \qquad j = 1, 2, \ldots, m. \tag{3.4}$$

Corollary 3.3. *Let $\alpha > 0, m = [\alpha] + 1$. Then the equality $D^\alpha y(t) = 0$ holds if and only if*

$$y(t) = \sum_{j=1}^{m} c_j t^{\alpha - j},$$

where c_j are arbitrary constants.

Lemma 3.4. *Let $\alpha > 0, m = [\alpha] + 1, 0 \leq \delta < 1$ and $f(t) \in C_\delta[a,b]$. Then*

1) if $\alpha < \delta$, then $I^\alpha f(t) \in C_{\delta-\alpha}[a,b]$ and

$$\|I^\alpha f\|_{C_{\delta-\alpha}[a,b]} \leq M\|f\|_{C_\delta[a,b]}, \qquad M = \frac{\Gamma(1-\delta)}{\Gamma(1+\alpha-\delta)}; \tag{3.5}$$

2) if $\alpha \geq \delta$, then $I^\alpha f(t) \in C[a,b]$ and

$$\|I^\alpha f\|_{C[a,b]} \leq M\|f\|_{C_\delta[a,b]}, \qquad M = \frac{(b-a)^{\alpha-\delta}\Gamma(1-\delta)}{\Gamma(1+\alpha-\delta)}. \tag{3.6}$$

4 Construction of 0-normalized systems

In this section we construct 0 - normalized systems with respect to the pair of the operators $\left(D^\alpha, \lambda t^\beta I^\nu\right)$. To do it from Proposition 2.3 it follows that it is necessary to find all solutions of the equation $D^\alpha y(t) = 0$ and a right inverse for the operator D^α. According to statement of Lemma 3.1 the right inverse of the the operator D^α is the operator I^α, and due to (3.4) linear independent solutions of the equation $D^\alpha y(t) = 0$ are functions $t^{s_j}, s_j = \alpha - j, j = 1, 2, \ldots, m$. Hereinafter, we denote $L_1 = D^\alpha$ and $L_2 = \lambda t^\beta I^\nu$. Then the equation (1.1) is represented as (2.1). For real numbers $\alpha > 0$, $\nu \geq 0$, $\delta > 0$, $s \in R$ we introduce the following coefficients:

$$C_{\alpha,\nu}(\delta,s,i) = \prod_{k=0}^{i-1} \frac{\Gamma(\delta k + s + 1)}{\Gamma(\delta k + s + 1 + \nu)} \cdot \frac{\Gamma\left[\delta(k+1) + s + 1 - \alpha\right]}{\Gamma\left[\delta(k+1) + s + 1\right]}, \quad i \geq 1, \quad C_{\alpha,\nu}(\delta,s,0) = 1, \quad s \in R.$$

From the properties of the gamma function we have $C_{\alpha,\nu}(\delta,s,i) \neq 0$. It's obvious that

$$C_{\alpha,0}(\delta,s,i) = \prod_{k=0}^{i-1} \frac{\Gamma\left[\delta(k+1) + 1 + s - \alpha\right]}{\Gamma\left[\delta(k+1) + 1 + s\right])}.$$

Let $s_j = \alpha - j,\ j = 1, 2, \ldots, m$ and $f_{0,s_j}(t) = t^{s_j}$, then due to (3.4): $L_1 f_{0,s_j}(t) = 0,\ j = 1, 2, \ldots, m$. We consider the system of functions:

$$f_{i,j}(t) = \left(I^\alpha \cdot \lambda t^\beta I^\nu\right)^i f_{0.s_j}(t), \qquad i \geq 1. \tag{4.1}$$

Since $(D^\alpha)^{-1} = I^\alpha$ and $D^\alpha f_{0,s_j}(t) = 0$, then Proposition 2.3 implies that the system (4.1) is 0-normalized with respect to the pair of the operators $\left(D^\alpha, \lambda t^\beta I^\nu\right)$. We find explicit form of the system of functions $f_i(t)$. Hereinafter, everywhere we will assume that $\alpha > 0$, $m = [\alpha] + 1$, $\nu \geq 0$, $\beta > -\{\alpha\} - \nu$. The following proposition is valid.

Lemma 4.1. *Let $s \geq \alpha - m, g_i(t) = \left(I^\alpha t^\beta I^\nu\right)^i t^s, i \geq 1$. Then*

1) for the function $g_i(t)$ the following equality holds:

$$g_i(t) = C_{\alpha,\nu}(\alpha+\beta+\nu, s, i) t^{(\alpha+\beta+\nu)i+s}, \qquad i \geq 1; \tag{4.2}$$

2) the function $g_i(t)$ at least belongs to the class $C_{m-\alpha}[0,d]$.

Proof. Note that due to (3.2) for $s \geq \alpha - m$ the equality

$$I^\nu t^s = \frac{\Gamma(s+1)}{\Gamma(s+1+\nu)} t^{s+\nu}$$

holds. Let $i=1$. Then due to the inequality $s+\beta+\nu \geq \alpha-m+\beta+\nu=-1+\{\alpha\}+\beta+\nu>-1$ and properties of the operator I^{α}, for the function $g_1(t)$ we have

$$\begin{aligned} g_1(t) &= I^{\alpha}\left(t^{\beta} I^{\nu} t^{s}\right) = \frac{\Gamma(s+1)}{\Gamma(s+1+\nu)} I^{\alpha} t^{s+\nu+\beta} = \frac{\Gamma(s+1)}{\Gamma(s+1+\nu)} \frac{\Gamma(\beta+\nu+s+1)}{\Gamma(\alpha+\nu+\beta+s+1)} t^{s+\nu+\beta+\alpha} \\ &= C_{\alpha,\nu}(\alpha+\beta+\nu,s,1) t^{\alpha+\beta+\nu+s}. \end{aligned}$$

Due to the inequality $\alpha+\beta+\nu+s>\alpha-m$ it follows that at least $g_1(t) \in C_{m-\alpha}[0,d]$.

Further, in general case by the mathematical induction method it is possible to show validity of the equality (4.2). Indeed, let for some positive integer r the equality (4.2) holds. Then for $r+1$ we get:

$$\begin{aligned} g_{r+1}(t) &= \left(I^{\alpha} \cdot t^{\beta} I^{\nu}\right)^{r+1} t^{s} = \left(I^{\alpha} \cdot t^{\beta} I^{\nu}\right)\left(I^{\alpha} \cdot t^{\beta} I^{\nu}\right)^{r} t^{s} = I^{\alpha}\left[t^{\beta} I^{\nu} g_r(t)\right] \\ &= C_{\alpha,\nu}(\alpha+\beta+\nu,s,r) I^{\alpha}\left[t^{\beta} I^{\nu} t^{r(\alpha+\beta+\nu)+s}\right] \\ &= C_{\alpha,\nu}(\alpha+\beta+\nu,s,r) \frac{\Gamma(r(\alpha+\beta+\nu)+s+1)}{\Gamma(r(\alpha+\beta+\nu)+s+1+\nu)} I^{\alpha} t^{r(\alpha+\beta+\nu)+\beta+\nu+s} \\ &= C_{\alpha,\nu}(\alpha+\beta+\nu,s,r) \frac{\Gamma(r(\alpha+\beta+\nu)+s+1)}{\Gamma(r(\alpha+\beta+\nu)+s+1+\nu)} \\ &\quad \times \frac{\Gamma(r(\alpha+\beta+\nu)+\alpha+\beta+\nu+s+1-\alpha)}{\Gamma(r(\alpha+\beta+\nu)+\alpha+\beta+\nu+s+1)} t^{r(\alpha+\beta+\nu)+\alpha+\beta+\nu+s} \\ &= C_{\alpha,\nu}(\alpha+\beta+\nu,s,r+1) t^{(r+1)(\alpha+\beta+\nu)+s}. \end{aligned}$$

Therefore, (4.2) is true also for the case $r+1$. It is obvious that for any $r \geq 1$ at least $g_{r+1}(t) \in C_{m-\alpha}[0,d]$ and $D^{\alpha} g_{r+1}(t) \in C(0,d)$. □

From the lemma in the case $s_j=\alpha-j,\ j=1,2,\ldots,m$, we obtain

$$\begin{aligned} &C_{\alpha,\nu}(\delta,\alpha-j,i) \\ &\quad = \prod_{k=0}^{i-1} \frac{\Gamma\left[k\delta+1+\alpha-j\right]}{\Gamma\left[k\delta+1+\alpha-j+\nu\right]} \cdot \frac{\Gamma\left[(k+1)\delta+1-j\right]}{\Gamma\left[(k+1)\delta+1+\alpha-j\right]}, \quad i \geq 1,\ \delta=\alpha+\beta+\nu. \end{aligned} \tag{4.3}$$

Consider the function

$$u_j(z)=\sum_{i=0}^{\infty} C_{\alpha,\nu}(\alpha+\beta+\nu,s_j,i) z^{i}, \tag{4.4}$$

where z is a complex number. If in (4.3) $\beta=0$, then

$$C_{\alpha,\nu}(\alpha+\nu,\alpha-j,i)=\frac{\Gamma(1+\alpha-j)}{\Gamma(i(\alpha+\nu)+1+\alpha-j)}, \qquad i \geq 1,$$

and

$$u_j(z)=\sum_{i=0}^{\infty} \frac{\Gamma(1+\alpha-j)}{\Gamma(i(\alpha+\nu)+1+\alpha-j)} z^{i}=\Gamma(1+\alpha-j) E_{\alpha+\nu,1+\alpha-j}(z),$$

where $E_{\rho,\delta}(z)$ is a Mittag-Leffler type function [15].

It is easy to show that at $\nu=0$ the equality

$$C_{\alpha,0}(\alpha+\beta,\alpha-j,i)=c_i$$

holds, i.e. these coefficients coincide with coefficients of expansion of the function (1.2), with indexes $m = 1 + \frac{\beta}{\alpha}$, $\ell = 1 + \frac{\beta - j}{\alpha}$. In [6] it is shown that for the coefficients of the function (1.2) the following asymptotic estimate holds:

$$\frac{c_i}{c_{i+1}} = \frac{\Gamma[\alpha(im+l+1)+1]}{\Gamma[\alpha(im+l)+1]} \sim (\alpha m i)^{\alpha} \qquad (i \to \infty).$$

Thus, the function (1.2) is entire. Denote $\delta = \alpha + \beta + \nu$ and rewrite the coefficients $C_{\alpha,\nu}(\delta, \alpha - j, i)$ as follows:

$$C_{\alpha,\nu}(\delta,\alpha-j,i) = \prod_{k=0}^{i-1} \frac{\Gamma\left[\nu\left(k\frac{\delta}{\nu} + \frac{\alpha-j}{\nu}\right)+1\right]}{\Gamma\left[\nu\left(k\frac{\delta}{\nu} + \frac{\alpha-j}{\nu}+1\right)+1\right]} \cdot \frac{\Gamma\left[\alpha\left(k\frac{\delta}{\alpha} + \frac{\delta-j}{\alpha}\right)+1\right]}{\Gamma\left[\alpha\left(k\frac{\delta}{\alpha} + \frac{\delta-j}{\alpha}+1\right)+1\right]}, \quad i \geq 1, \ \nu > 0.$$

Further, the asymptotic estimate

$$\frac{C_{\alpha,\nu}(\delta,\alpha-j,i)}{C_{\alpha,\nu}(\delta,\alpha-j,i+1)} \sim (\delta i)^{\nu+\alpha} \to \infty \qquad (i \to \infty)$$

yields that $u_j(z), j = 1, 2, \ldots, m$, from (4.4) are also entire functions. Lemma 4.1 and Proposition 2.3 implies the following lemma.

Lemma 4.2. *Let $s_j = \alpha - j$, $j = 1, 2, \ldots, m$. Then at all values $j = 1, 2, \ldots, m$ the system of functions*

$$f_{i,j}(t) = \lambda^i C_{\alpha,\nu}(\alpha+\beta+\nu, s_j, i) t^{(\alpha+\beta+\nu)i+s_j}, \qquad i = 0, 1, \ldots$$

is 0-normalized with respect to the pair of operators $\left(D^{\alpha}, \lambda t^{\beta} I^{\nu}\right)$ on the domain $t > 0$.

Using the main property of normalized systems we get the following theorem.

Theorem 4.3. *Let $s_j = \alpha - j$, $j = 1, 2, \ldots, m$. Then at all values $j = 1, 2, \ldots, m$ the functions*

$$y_j(t) = \sum_{i=0}^{\infty} f_{i,j}(t) = t^{s_j} \sum_{i=0}^{\infty} \lambda^i C_{\alpha,\nu}(\alpha+\beta+\nu, s_j, i) t^{(\alpha+\beta+\nu)i} \tag{4.5}$$

are linearly independent solutions of the homogeneous equation (1.1).

Moreover, for all $j = 1, 2, \ldots, m-1$, $y_j(t) \in C[0,d]$ and $y_m(t) \in C_{m-\alpha}[0,d]$.

Proof. Consider the function

$$u_j(t) = \sum_{i=0}^{\infty} \lambda^i C_{\alpha,\nu}(\alpha+\beta+\nu, s_j, i) t^{(\alpha+\beta+\nu)i}.$$

Since the function (4.4) is entire, then it is obvious that $y_j(t) = t^{\alpha-j} u_j(t) \in C[0,d]$ at $j = 1, 2, \ldots, m-1$ and $y_m(t) = t^{\alpha-m} u_m(t) \in C_{m-\alpha}[0,d]$. Moreover, for all $j = 1, 2, \ldots, m$:

$$\begin{aligned} D^{\alpha} f_{0,j}(t) = 0, D^{\alpha} f_{i,j}(t) &= \lambda I^{\nu} f_{i-1,j}(t) = \lambda^i C_{\alpha,\nu}(\alpha+\beta+\nu, s_j, i-1) I^{\nu} t^{(\alpha+\beta+\nu)i+s_j} \\ &= \lambda^i C_{\alpha,\nu}(\alpha+\beta+\nu, s_j, i-1) \frac{\Gamma\left[(\alpha+\beta+\nu)i+s_j+1\right]}{\Gamma\left[(\alpha+\beta+\nu)i+s_j+1+\nu\right]} t^{(\alpha+\beta+\nu)i+\nu+s_j}, \qquad i \geq 1. \end{aligned}$$

Consequently, the series $\sum_{i=0}^{\infty} D^{\alpha} f_{i,j}(t), \sum_{i=0}^{\infty} I^{\nu} f_{i,j}(t)$ uniformly converge on any closed domain $[\varepsilon, d]$, $0 < \varepsilon < d$ and, therefore, termwise use of the operators D^{α} and I^{ν} to the series (4.5) is rightful. Then functions $y_j(t)$ from (4.5) are solutions of the homogeneous equation (1.1). Proof of linearly independence of the solutions (4.5) we will show below in Theorem 6.2 of Section 6. □

Remark 4.4. In the case $\nu = 0$ the functions $y_j(t)$ are represented in the form:

$$y_j(t) = t^{\alpha-j} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-j}{\alpha}}\left(\lambda t^{\alpha+\beta}\right), \qquad j = 1,2,\ldots,m.$$

This representation of solution of the equation (1.1) coincides with the result of [14] (see Theorem 1, formulas (19) and (21)).

5 Construction of f-normalized systems

Now we turn to construction of a solution of inhomogeneous equation. Let $f(t) \in C[0,d]$. Then by the statement of Lemma 3.1 for the function $f_0(t) = I^\alpha f(t)$ the following equality is true:

$$L_1 f_0(t) = D^\alpha I^\alpha f(t) = f(t).$$

Consider the system

$$f_i(t) = \left(I^\alpha \lambda t^\beta I^\nu\right)^i f_0(t) \equiv \lambda^i \left(I^\alpha t^\beta I^\nu\right)^i f_0(t), \qquad i = 1,2,\ldots \tag{5.1}$$

Lemma 5.1. *Let $f(t) \in C[0,d]$, $d < \infty$. Then the system of functions (5.1) is $f(t)$-normalized with respect to the pair of operators $\left(D^\alpha, \lambda t^\beta I^\nu\right)$ on the domain $t > 0$.*

Proof. Since $f(t) \in C[0,d]$, then due to statement of Lemma 3.4 $f_0(t) = I^\alpha f(t) \in C[0,d]$. Moreover,

$$|f_0(t)| = |I^\alpha f(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} |f(\tau)| d\tau \le \|f\|_{C[0,d]} \frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Thus

$$|f_0(t)| \le \frac{\|f\|_{C[0,d]}}{\Gamma(\alpha+1)} t^\alpha,$$
$$\|f_0\|_{C[0,d]} \le \frac{d^\alpha}{\Gamma(\alpha+1)} \|f\|_{C[0,d]}.$$

Hereinafter, we denote $M = \frac{\|f\|_{C[0,d]}}{\Gamma(\alpha+1)}$. Then $|f_0(t)| \le M t^\alpha$. Since $|f_0(t)| \le M t^\alpha$, then

$$\left|\lambda \left(I^\alpha t^\beta I^\nu\right) f_0(t)\right| \le M|\lambda| \left(I^\alpha t^\beta I^\nu\right) t^\alpha.$$

Therefore, for any $i \ge 1$ the following estimate holds:

$$|f_i(t)| = \left|\lambda^i \left(I^\alpha t^\beta I^\nu\right)^i f_0(t)\right| \le M|\lambda|^i \left(I^\alpha t^\beta I^\nu\right)^i t^\alpha.$$

Further, due to (4.2) the function $\left(I^\alpha t^\beta I^\nu\right)^i t^\alpha$ is represented as follows:

$$\left(I^\alpha t^\beta I^\nu\right)^i t^\alpha = C_{\alpha,\nu}(\alpha+\beta+\nu,\alpha,i) t^{(\alpha+\beta+\nu)i+\alpha},$$

thus,

$$|f_i(t)| \le M|\lambda|^i C_{\alpha,\nu}(\alpha+\beta+\nu,\alpha,i) t^{i(\alpha+\beta+\nu)+\alpha}. \tag{5.2}$$

Consequently, at any $i \geq 1$ we have $f_i(t) \in C[0,d]$ as well as (5.2).
Moreover,

$$L_1 f_i(t) = D^\alpha \left(I^\alpha \cdot \lambda t^\beta\right)^i f_{0,s_j}(t) = D^\alpha I^\alpha \cdot \lambda t^\beta \left(I^\alpha \cdot \lambda t^\beta\right)^{i-1} f_{0,s_j}(t)$$
$$= \lambda t^\beta f_{i-1}(t) = L_2 f_{i-1}(t), \qquad i \geq 1.$$

Thus, in the class of functions $X = C[0,d]$ we get:

$$L_1 f_0(t) = f(t), \qquad L_1 f_i(t) = L_2 f_{i-1}(t), \qquad i \geq 1,$$

i.e. the system (5.1) is f-normalized with respect to the pair of operators $\left(D^\alpha, \lambda t^\beta\right)$. □

Theorem 5.2. *Let $f(t) \in C[0,d]$ and functions $f_i(t)$, $i \geq 0$ be defined by* (5.1)*. Then the function*

$$y_f(x) = \sum_{i=0}^{\infty} f_i(t) \tag{5.3}$$

is a particular solution of the equation (1.1) *from the class $C[0,d]$.*

Proof. Estimate the series (5.3). Due to (5.2), we have

$$|y_f| \leq \sum_{i=0}^{\infty} |f_i(t)| \leq \frac{\|f\|_{C[0,d]} t^\alpha}{\Gamma(\alpha+1)} \left[1 + \sum_{i=1}^{\infty} |\lambda|^i C_{\alpha,\nu}\left(\alpha+\beta+\nu,\alpha,i\right) t^{i(\alpha+\beta)}\right].$$

Since the last series is uniformly convergent on the domain $0 \leq t \leq d$, then sum of the series, and hence the function $y_f(t)$ belong to the class $C[0,d]$. □

Now we study representation of the functions (5.1) for certain particular cases of functions $f(t)$.

Lemma 5.3. *Let $f(t) = t^\mu$, $\mu > -1$. Then a particular solution of the equation* (1.1) *has the following form:*

$$y_f(t) = \frac{\Gamma(\mu+1)t^{\alpha+\mu}}{\Gamma(\mu+1+\alpha)} \sum_{k=0}^{\infty} \lambda^k C_{\alpha,\nu}(\alpha+\beta+\nu,\mu+\alpha,k) t^{k(\alpha+\beta+\nu)}.$$

Proof of the lemma follows from (4.2).

Theorem 5.4. *Let $f(t) = \sum\limits_{j=1}^{p} \lambda_j t^{\mu_j}$, $\mu_j > -1$. Then a particular solution of the equation* (1.1) *has the following form:*

$$y_f(t) = \sum_{j=1}^{p} \frac{\lambda_j \Gamma(\mu_j+1)t^{\alpha+\mu_j}}{\Gamma(\mu_j+1+\alpha)} \sum_{k=0}^{\infty} \lambda^k C_{\alpha,\nu}(\alpha+\beta+\nu,\mu_j+\alpha,k) t^{k(\alpha+\beta+\nu)}. \tag{5.4}$$

In the case $\nu = 0$ the representation (5.4) of a particular solution of (1.1) coincides with the result of [14] (see Theorem 2, formula (27)).

Now we give an algorithm for constructing particular solutions of the inhomogeneous equation (1.1) in the case when $f(t)$ is an analytic function.

Theorem 5.5. *Let $f(t)$ be an analytic function. Then a particular solution of the equation (1.1) has the form*

$$y_f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)t^{\alpha+k}}{\Gamma(\alpha+k+1)} y_{k+\alpha}(t), \tag{5.5}$$

where $y_{k+\alpha}(t)$ is defined by the equality:

$$y_{k+\alpha}(t) = \sum_{i=0}^{\infty} \lambda^i C_{\alpha,\nu}\left(\alpha+\beta, k+\alpha, i\right) t^{i(\alpha+\beta)}.$$

Proof. Let $f(t)$ be an analytical function. Then it can be represented in the form

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k,$$

and assuming that $f_0(t) = I^\alpha f(t)$, we have

$$\begin{aligned} f_i(t) &= \left(I^\alpha \cdot \lambda t^\beta I^\nu\right)^i f_0(t) = \left(I^\alpha \cdot \lambda t^\beta I^\nu\right)^i \left(\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} I^\alpha t^k\right) \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} \left(I^\alpha \cdot \lambda t^\beta I^\nu\right)^i t^{k+\alpha}. \end{aligned}$$

Due to (4.2):

$$\left(I^\alpha \cdot \lambda t^\beta I^\nu\right)^i t^{k+\alpha} = \lambda^i C_{\alpha,\nu}(\alpha+\beta+\nu, k+\alpha, i) t^{(\alpha+\beta+\nu)i+k+\alpha}.$$

Then

$$f_i(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} \lambda^i C_{\alpha,\nu}(\alpha+\beta+\nu, k+\alpha, i) t^{(\alpha+\beta+\nu)i+k+\alpha}.$$

Hence for the function $y_f(t)$ we get (5.5):

$$y_f(t) = \sum_{i=0}^{\infty} f_i(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)t^{k+\alpha}}{\Gamma(\alpha+k+1)} \sum_{i=0}^{\infty} \lambda^i C_{\alpha,\nu}\left(\alpha+\beta, k+\alpha, i\right) t^{i(\alpha+\beta)} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)t^{k+\alpha}}{\Gamma(\alpha+k+1)} y_{k+\alpha}(t).$$

□

Theorem 5.6. *Let $\beta = n,\ n = 0, 1, \dots, f(t) \in C[0, d]$. Then a particular solution of the equation (1.1) has the form:*

$$y_f(t) = \int_0^t G_{n,\alpha+\nu}(t-\tau, \tau, \lambda) f(\tau) d\tau, \tag{5.6}$$

where $G_{n,\alpha+\nu}(u, w, \lambda)$ is defined by the equality:

$$\begin{aligned} G_{n,\alpha+\nu}(u, w, \lambda) &= \sum_{i=0}^{\infty} G_{n,\alpha+\nu,i}(u, w, \lambda), \\ G_{n,\alpha,\nu,i}(u, w, \lambda) &= \frac{\lambda^i}{\Gamma(\alpha)} \sum_{j_1=0}^{n} \dots \sum_{j_i=0}^{n} C_n^{j_1} \dots C_n^{j_1} C_{\alpha,\nu}(\alpha+\nu, j_1+j_2+\dots+j_i+\alpha-1, i) \\ &\quad \times u^{i(\alpha+\nu)+j_1+\dots+j_i+\alpha-1} w^{in-j_1-\dots-j_i}, C_n^{j_i} = \frac{n!}{j_i!(n-j_i)!}. \end{aligned}$$

$$C_{\alpha,\nu}(\alpha+\nu, j_1+j_2+\cdots+j_i+\alpha-1, i)$$
$$=\prod_{p=0}^{i-1}\frac{\Gamma(p(\alpha+\nu)+j_0+\cdots+j_p+\alpha)}{\Gamma(p(\alpha+\nu)+j_0+\cdots+j_p+\alpha+\nu)}\cdot\frac{\Gamma\left[(p+1)(\alpha+\nu)+j_1+\cdots+j_{p+1}\right]}{\Gamma\left[(p+1)(\alpha+\nu)+j_1+\cdots+j_{p+1}+\alpha\right]}, \quad (5.7)$$

where $j_0=0$.

Proof. Let $i=1,\ \beta=n,\ n=0,1,\ldots,\ f_0(t)=I^\alpha f(t)$. Then

$$f_1(t)=(I^\alpha\cdot\lambda t^n I^\nu)\,f_0(t)=(I^\alpha\cdot\lambda t^n)\,I^{\nu+\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\lambda\tau^n I^{\nu+\alpha}f(\tau)d\tau$$
$$=\frac{\lambda}{\Gamma(\alpha)\Gamma(\nu+\alpha)}\int_0^t f(z)\int_z^t(t-\tau)^{\alpha-1}(\tau-z)^{\alpha+\nu-1}\tau^n d\tau dz.$$

Investigate the inner integral:

$$I_{n,1}=\int_z^t(t-\tau)^{\alpha-1}(\tau-z)^{\alpha+\nu-1}\tau^n d\tau.$$

After the change of variables $\tau=z+(t-z)\xi$ we have:

$$I_{n,1}=\int_z^t(t-\tau)^{\alpha-1}(\tau-z)^{\alpha+\nu-1}\tau^n d\tau=(t-z)^{2\alpha+\nu-1}\int_0^1(1-\xi)^{\alpha-1}\xi^{\alpha+\nu-1}((t-z)\xi+z)^n d\xi.$$

Consequently,

$$f_1(t)=\frac{\lambda}{\Gamma(\alpha+\nu)}\sum_{j_1=0}^{n}C_n^{j_1}\frac{\Gamma(\alpha+\nu+j_1)}{\Gamma(\alpha+\nu+j_1+\alpha)}\int_0^t(t-z)^{j_1+2\alpha+\nu-1}z^{n-j_1}f(z)dz.$$

Further, for the function $I^\nu f_1(t)$ we obtain:

$$I^\nu f_1(t)=\frac{\lambda}{\Gamma(\nu)\Gamma(\alpha+\nu)}\sum_{j_1=0}^{n}C_n^{j_1}\frac{\Gamma(\alpha+\nu+j_1)}{\Gamma(\alpha+\nu+j_1+\alpha)}\int_0^t(t-\tau)^{\nu-1}\int_0^\tau(\tau-z)^{j_1+2\alpha+\nu-1}z^{n-j_1}f(z)dzd\tau$$
$$=\frac{\lambda}{\Gamma(\nu)\Gamma(\alpha+\nu)}\sum_{j_1=0}^{n}C_n^{j_1}\frac{\Gamma(\alpha+\nu+j_1)}{\Gamma(\alpha+\nu+j_1+\alpha)}\int_0^t z^{n-j_1}f(z)\int_z^t(t-\tau)^{\nu-1}(\tau-z)^{j_1+2\alpha+\nu-1}d\tau dz$$
$$=\frac{\lambda}{\Gamma(\nu)\Gamma(\alpha+\nu)}\sum_{j_1=0}^{n}\frac{C_n^{j_1}\Gamma(\alpha+\nu+j_1)}{\Gamma(\alpha+\nu+j_1+\alpha)}\int_0^t(t-\tau)^{j_1+2\alpha+2\nu-1}z^{n-j_1}f(z)\int_0^1(1-\xi)^{j_1+2\alpha+\nu-1}\xi^{\nu-1}d\xi dz$$
$$=\frac{\lambda}{\Gamma(\nu)\Gamma(\alpha+\nu)}\sum_{j_1=0}^{n}\frac{C_n^{j_1}\Gamma(\alpha+\nu+j_1)}{\Gamma(\alpha+\nu+j_1+\alpha)}\frac{\Gamma(\nu)\Gamma(j_1+2\alpha+\nu)}{\Gamma(j_1+2\alpha+2\nu)}\int_0^t(t-\tau)^{j_1+2\alpha+2\nu-1}z^{n-j_1}f(z)dz$$
$$=\frac{\lambda}{\Gamma(\alpha+\nu)}\sum_{j_1=0}^{n}C_n^{j_1}\frac{\Gamma(\alpha+\nu+j_1)}{\Gamma(\alpha+\nu+j_1+\alpha)}\frac{\Gamma(2(\alpha+\nu)+j_1-\nu)}{\Gamma(2(\alpha+\nu)+j_1)}\int_0^t(t-\tau)^{j_1+2\alpha+2\nu-1}z^{n-j_1}f(z)dz.$$

Then for $f_2(t)$ we get:

$$f_2(t)=(I^\alpha\cdot\lambda t^n I^\nu)\,f_1(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\lambda\tau^n I^\nu f_1(\tau)d\tau$$
$$=\frac{\lambda^2}{\Gamma(\alpha)\Gamma(\alpha+\nu)}\sum_{j_1=0}^{n}C_n^{j_1}\frac{\Gamma(\alpha+\nu+j_1)}{\Gamma(\alpha+\nu+j_1+\alpha)}\frac{\Gamma(2(\alpha+\nu)+j_1-\nu)}{\Gamma(2(\alpha+\nu)+j_1)}$$
$$\times\int_0^t z^{n-j_1}f(z)\int_z^t(t-\tau)^{\alpha-1}(\tau-z)^{j_1+2\alpha+2\nu-1}\tau^n d\tau dz.$$

Consequently,

$$f_2(t) = \frac{\lambda^2}{\Gamma(\alpha)} \sum_{j_1=0}^{n} \sum_{j_2=0}^{n} C_n^{j_1} C_n^{j_2} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\nu)} \frac{\Gamma(\alpha+\nu+j_1+\alpha)}{\Gamma(\alpha+\nu+j_1+\alpha+\nu)} \frac{\Gamma(\alpha+\nu+j_1)}{\Gamma(\alpha+\nu+j_1+\alpha)}$$
$$\times \frac{\Gamma(2(\alpha+\nu)+j_1+j_2)}{\Gamma(2(\alpha+\nu)+j_1+j_2+\alpha)} \int_0^t (t-z)^{2(\alpha+\nu)+\alpha+j_1+j_2-1} z^{2n-j_1-j_2} f(z)dz.$$

Put $j_0 = 0$. Then for $f_2(t)$ we obtain the following representation:

$$f_2(t) = \frac{\lambda^2}{\Gamma(\alpha)} \sum_{j_1=0}^{n} \sum_{j_2=0}^{n} C_n^{j_1} C_n^{j_2} \prod_{p=0}^{1} \frac{\Gamma(p(\alpha+\nu)+j_0+\cdots+j_p+\alpha)}{\Gamma(p(\alpha+\nu)+j_0+\cdots+j_p+\alpha+\nu)}$$
$$\times \frac{\Gamma\left[(p+1)(\alpha+\nu)+j_0+\cdots+j_{p+1}\right]}{\Gamma\left[(p+1)(\alpha+\nu)+j_0+\cdots+j_{p+1}+\alpha\right]} \int\limits_0^t (t-z)^{2(\alpha+\nu)+\alpha+j_1+j_2-1} z^{2n-j_1-j_2} f(z)dz.$$

In general case, using the representation of coefficients $C_{\alpha,\nu}(\delta, s, i)$ for $f_i(t)$, we get:

$$f_i(t) = \frac{\lambda^i}{\Gamma(\alpha)} \sum_{j_1=0}^{n} \cdots \sum_{j_i=0}^{n} C_n^{j_1} \ldots C_n^{j_i} C_{\alpha,\nu}(\alpha+\nu, j_1+j_2+\cdots+j_i+\alpha-1, i)$$
$$\times \int\limits_0^t (t-z)^{k(\alpha+\nu)+\alpha+j_1+j_2+\cdots+j_i-1} z^{kn-j_1-j_2-\cdots-j_i} f(z)dz,$$

where coefficients $C_{\alpha,\nu}(\alpha+\nu, j_1+j_2+\cdots+j_i+\alpha-1, i)$ are defined by the equality (5.7). Then a particular solution of the equation (1.1) is represented in the form (5.6). □

Example 5.7. Let $\beta \equiv n = 0$. Then $j_1 = j_2 = \cdots = j_i = 0$,

$$C_{\alpha,\nu}(\alpha+\nu, 0, i) = \prod_{p=0}^{i-1} \frac{\Gamma(p(\alpha+\nu)+\alpha)}{\Gamma(p(\alpha+\nu)+\alpha+\nu)} \cdot \frac{\Gamma\left[(p+1)(\alpha+\nu)\right]}{\Gamma\left[(p+1)(\alpha+\nu)+\alpha\right]} = \frac{\Gamma(\alpha)}{\Gamma\left[i(\alpha+\nu)+\alpha\right]},$$
$$G_{0,\alpha,i}(u, w, \lambda) = \frac{\lambda^i}{\Gamma(i(\alpha+\nu)+\alpha)} u^{i(\alpha+\nu)+\alpha-1},$$
$$G_{0,\alpha}(u, v, \lambda) = \sum_{i=0}^{\infty} \frac{\lambda^i u^{i(\alpha+\nu)+\alpha-1}}{\Gamma(i(\alpha+\nu)+\alpha)} = u^{\alpha-1} E_{\alpha+\nu,\alpha}(\lambda u^{\alpha+\nu}).$$

In this case

$$y_f(t) = \int_0^t G_{0,\alpha}(t-\tau, \tau, \lambda) f(\tau)d\tau = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha+\nu,\alpha}(\lambda(t-\tau)^{\alpha+\nu}) f(\tau)d\tau.$$

This formula has been obtained in [1] (see formula (15)).

6 Solution of Cauchy type problem

Consider the following Cauchy type equation:

$$D^\alpha y(t) = \lambda t^\beta I^\nu y(t) + \sum_{j=1}^{p} \lambda_j t^{\mu_j}, \qquad 0 < t \le d < \infty, \tag{6.1}$$

$$D^{\alpha-k} y(t)\Big|_{t=0} = b_k, \qquad k = 1, 2, \ldots, m-1, \tag{6.2}$$

where $D^{\alpha-m} = I^{m-\alpha}$, b_k are real numbers. First we study the homogeneous equation (6.1)–(6.2).

Theorem 6.1. *Let $\lambda_j = 0, j = 1, 2, \ldots, p$. Then solution of the Cauchy equation (6.1)–(6.2) exists, unique and can be represented in the form:*

$$y(t) = \sum_{j=1}^{m} \frac{b_j}{\Gamma(\alpha - j + 1)} t^{\alpha-j} \sum_{i=0}^{\infty} \lambda^i C_{\alpha,\nu}(\alpha + \beta + \nu, \alpha - j, i) t^{(\alpha+\beta+\nu)i}. \tag{6.3}$$

Proof. Let $\lambda_j = 0, j = 1, 2, \ldots, p$. According to Theorem 4.3 the function $y(t)$ in (6.3) is solution of the equation (6.1). Let us show that $y(t)$ satisfies initial conditions (6.2). Due to (3.2)–(3.4), we have:

$$I^{m-\alpha} t^{(\alpha+\beta+\nu)i+\alpha-j} = \frac{\Gamma[(\alpha+\beta+\nu)i + \alpha - j + 1]}{\Gamma[(\alpha+\beta+\nu)i + m - j + 1]} t^{(\alpha+\beta+\nu)i+m-j},$$

$$D^{\alpha-k} t^{\alpha-j} = 0, \qquad j > k,$$

$$D^{\alpha-k} t^{(\alpha+\beta+\nu)i+\alpha-j} = \frac{\Gamma[(\alpha+\beta+\nu)i + \alpha - j + 1]}{\Gamma[(\alpha+\beta+\nu)i + k - j + 1]} t^{(\alpha+\beta+\nu)i+k-j}, \qquad j \le k.$$

Thus, for the functions $f_{i,j}(t), j = 1, 2, \ldots, m$, we get:

$$D^{\alpha-k} f_{i,j}(t)\Big|_{t=0} = \lim_{t\to 0} D^{\alpha-k} f_{i,j}(t) = \begin{cases} \Gamma(\alpha - j + 1), & i = 0,\ k = j, \\ 0, & k \neq j,\ i \geq 0. \end{cases}$$

Then

$$D^{\alpha-j} y_j(0) = \lim_{t\to 0} D^{\alpha-j} y_j(t) = \Gamma(\alpha - j + 1), \quad D^{\alpha-k} y_j(0) = \lim_{t\to 0} D^{\alpha-k} y_j(t) = 0, \quad k \neq j. \tag{6.4}$$

Consequently,

$$D^{\alpha-k} y(t)\Big|_{t=0} = \lim_{t\to 0} D^{\alpha-k} y(t) = \frac{b_k}{\Gamma(\alpha - k + 1)} \Gamma(\alpha - k + 1) = b_k.$$ □

Theorem 6.1 implies also the following result.

Theorem 6.2. *If $\beta > -\{\alpha\} - \nu$, then solutions $y_j(t)$ in (4.5) of the homogeneous equation (6.1) are linearly independent.*

Proof. For solutions $y_1(t), y_2(t), \ldots, y_m(t)$ we introduce analogue of Wronskian [4, p. 225]: $W_\alpha(t) = \det\left(D^{\alpha-k} y_j(t)\right)_{k,j}^{m}$, $0 \le t \le d$. We have the following statement, which is proved similarly to the corresponding theorem for linear differential equations of order m.

Lemma 6.3. *Solutions $y_1(t), y_2(t), \ldots, y_m(t)$ of the equation (6.1) are linearly independent if and only if at some point $t_0 \in [0, d]$: $W_\alpha(t_0) \neq 0$.*

According to (6.4) we get $W_\alpha(0) = (-1)^n \Gamma(\alpha)\Gamma(\alpha - 1) \cdot \ldots \cdot \Gamma(\alpha - m + 1) \neq 0$ and, consequently, due to the lemma, solutions $y_1(t), y_2(t), \ldots, y_m(t)$ of the equation (6.1) are linearly independent. □

From the Theorems 5.4 and 6.1 we get the following statement.

Theorem 6.4. *If $\mu_j > -1$, $j = 1, 2, \ldots, p$, then solution of the Cauchy problem* (6.1)–(6.2) *exists, unique and can be represented in the form:*

$$y(t) = \sum_{j=1}^{m} \frac{b_j}{\Gamma(\alpha - j + 1)} t^{\alpha - j} \sum_{i=0}^{\infty} \lambda^i C_{\alpha,\nu}(\alpha + \beta + \nu, \alpha - j, i) t^{(\alpha+\beta+\nu)i}$$
$$+ \sum_{j=1}^{p} \frac{\lambda_j \Gamma(\mu_j + 1) t^{\alpha+\mu_j}}{\Gamma(\mu_j + 1 + \alpha)} \sum_{k=0}^{\infty} \lambda^k C_{\alpha,\nu}(\alpha + \beta + \nu, \mu_j + \alpha, k) t^{k(\alpha+\beta+\nu)}.$$

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