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Blow-up problems for quasilinear reaction diffusion equations with weighted nonlocal source

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Abstract. In this paper, we investigate the following quasilinear reaction diffusion equations

$$\begin{cases} (b(u))_t = \nabla \cdot \left(\rho\left(|\nabla u|^2\right)\nabla u\right) + c(x)f(u) & \text{in } \Omega \times (0,t^*), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}. \end{cases}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. Weighted nonlocal source satisfies

$$c(x)f(u(x,t)) \leq a_1 + a_2\,(u(x,t))^p \left(\int_\Omega (u(x,t))^\alpha \,\mathrm{d}x\right)^m,$$

where a_2, p, α are some positive constants and a_1, m are some nonnegative constants. We make use of a differential inequality technique and Sobolev inequality to obtain a lower bound for the blow-up time of the solution. In addition, an upper bound for the blow-up time is also derived.

Keywords: blow-up problems, quasilinear reaction equation, weighted nonlocal source.
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1 Introduction

The blow-up problems to reaction diffusion equations has been extensively investigated by many researchers. Much of the work prior to the turn of the century is referenced in [1, 9, 10]. More recent work, we refer readers to [13–18, 21]. In practical situations, one would like to know whether the solutions blows up and if so, at which time blow-up occurs. Hence, finding bounds for blow-up time has become the focus of the researchers, especially the search for lower bounds of blow-up time. Since Payne and Schaefer [20] introduced a first-order inequality technique and obtained a lower bound for blow-up time, many authors are devoted to the lower bounds of blow-up time for various reaction diffusion problems, (see, for instance,

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[3–7]). We note that above mentioned studies mainly aimed at seeking lower bounds for blow-up time of local reaction-diffusion equations. In this paper, we concern the reaction diffusion equations with weighted nonlocal source

$$\begin{cases} (b(u))_t = \nabla \cdot \left(\rho\left(|\nabla u|^2\right) \nabla u\right) + c(x)f(u) & \text{in } \Omega \times (0,t^*), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}. \end{cases} \tag{1.1}$$

In (1.1), Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$, ν represents the unit normal vector to $\partial\Omega$, $u_0(x) \in C^1(\overline{\Omega})$ is a nonnegative function satisfying the compatibility condition, t^* is the blow-up time if blow-up occurs, or else $t^* = \infty$. Weighted nonlocal source satisfies

$$c(x)f(u(x,t)) \leq a_1 + a_2(u(x,t))^p \left(\int_\Omega (u(x,t))^\alpha \mathrm{d}x\right)^m,$$

where a_2, p, α are some positive constants and a_1, m are some nonnegative constants. Set $\mathbb{R}_+ = (0,\infty)$. Throughout this paper, we assume that b is a $C^2(\overline{\mathbb{R}_+})$ function with $b'(s) > 0$ for $s > 0$, ρ is a positive $C^2(\overline{\mathbb{R}_+})$ function satisfying $\rho(s) + 2s\rho'(s) > 0$ for $s > 0$, c is a positive $C(\overline{\Omega})$ function, and f is a nonnegative $C(\overline{\mathbb{R}_+})$ function. By maximum principles [22], we know that the classical solution u of (1.1) is nonnegative in $\overline{\Omega} \times [0,t^*)$.

For the information about the nonlocal reaction diffusion equations, we refer readers to [2,11,12,19,23]. Fang and Ma [11] dealt with the following problems

$$\begin{cases} u_t = \displaystyle\sum_{i,j=1}^{n} (a^{ij}(x)u_{x_i})_{xj} - c(x)f(u) & \text{in } \Omega \times (0,t^*), \\ \displaystyle\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i}\nu_j = g(u) & \text{on } \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) & \text{in } \overline{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n(n \geq 2)$ is a bounded star-shaped domain with smooth boundary $\partial\Omega$, nonlocal source satisfies

$$f(u(x,t)) \geq a_2(u(x,t))^p \left(\int_\Omega (u(x,t))^\alpha \mathrm{d}x\right)^m,$$

and a_2, p, α, and m are positive constants. They derived conditions which imply the solution blows up in finite time or exists globally. Furthermore, upper and lower bounds for blow-up time are obtained.

As far as we know, there is little information on the bounds for blow-up time of problem (1.1). Motivated by the above work [11], we study the problem (1.1). Our results of this paper are based on some Sobolev type inequalities and differential inequality technique. In Section 2, when $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, we obtain a criterion for blow-up of the solution of (1.1) and get an upper bound for blow-up time. In Section 3, when $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$, we derive a lower bound for blow-up time. An example is presented in Section 4 to illustrate our abstract results derived in this paper.

2 Blow-up solution

In this section, we establish conditions on data to ensure that the solution blows up at t^* and obtain an upper bound for t^*. To accomplish these tasks, we introduce the following auxiliary functions

$$D(t) = \int_\Omega G(u(x,t))\mathrm{d}x, \qquad E(t) = -\int_\Omega P(|\nabla u|^2)\mathrm{d}x + 2\int_\Omega c(x)F(u)\mathrm{d}x, \qquad t \geq 0, \tag{2.1}$$

$$G(u) = 2\int_0^u sb'(s)\mathrm{d}s, \qquad P(|\nabla u|^2) = \int_0^{|\nabla u|^2} \rho(s)\mathrm{d}s, \qquad F(u) = \int_0^u f(s)\mathrm{d}s, \tag{2.2}$$

where u is the classical solution of (1.1). Our main result of this section is the following Theorem 2.1

Theorem 2.1. *Let u be a classical solution of (1.1). We suppose that functions b, c, ρ, and f satisfy*

$$\begin{gathered} b''(s) < 0, \qquad s\rho(s) \leq (1+\beta)P(s), \\ \int_\Omega c(x)s(x,t)f(s(x,t))\mathrm{d}x \geq 2(1+\beta)\int_\Omega c(x)F(s(x,t))\mathrm{d}x, \qquad s \geq 0, \end{gathered} \tag{2.3}$$

where β is a nonnegative constant. In addition, initial data are assumed to satisfy

$$E(0) = -\int_\Omega P(|\nabla u_0|^2)\mathrm{d}x + 2\int_\Omega c(x)F(u_0)\mathrm{d}x > 0. \tag{2.4}$$

Then u must blow up at $t^* \leq T$ in measure $D(t)$ with

$$T = \begin{cases} \dfrac{D(0)}{2\beta(1+\beta)E(0)}, & \beta > 0, \\ \infty, & \beta = 0. \end{cases}$$

Proof. It follows from Green's formula and (2.3) that

$$\begin{aligned} D'(t) &= \int_\Omega G'(u)u_t\mathrm{d}x = 2\int_\Omega ub'(u)u_t\mathrm{d}x \\ &= 2\int_\Omega u\left[\nabla\cdot\left(\rho(|\nabla u|^2)\nabla u\right) + c(x)f(u)\right]\mathrm{d}x \\ &= 2\int_\Omega \nabla\cdot\left(u\rho(|\nabla u|^2)\nabla u\right)\mathrm{d}x - 2\int_\Omega \rho(|\nabla u|^2)|\nabla u|^2\mathrm{d}x + 2\int_\Omega c(x)uf(u)\mathrm{d}x \\ &= 2\int_{\partial\Omega} u\rho(|\nabla u|^2)\frac{\partial u}{\partial\nu}\mathrm{d}S - 2\int_\Omega \rho(|\nabla u|^2)|\nabla u|^2\mathrm{d}x + 2\int_\Omega c(x)uf(u)\mathrm{d}x \\ &\geq -2(1+\beta)\int_\Omega P(|\nabla u|^2)\mathrm{d}x + 4(1+\beta)\int_\Omega c(x)F(u)\mathrm{d}x \\ &= 2(1+\beta)\left[-\int_\Omega P(|\nabla u|^2)\mathrm{d}x + 2\int_\Omega c(x)F(u)\mathrm{d}x\right] = 2(1+\beta)E(t). \end{aligned} \tag{2.5}$$

Differentiating $E(t)$, we get

$$\begin{aligned} E'(t) &= -2\int_\Omega \rho(|\nabla u|^2)\,(\nabla u\cdot\nabla u_t)\,\mathrm{d}x + 2\int_\Omega c(x)f(u)u_t\mathrm{d}x \\ &= 2\int_{\partial\Omega} \rho(|\nabla u|^2)u_t\frac{\partial u}{\partial \nu}\mathrm{d}S - 2\int_\Omega \rho(|\nabla u|^2)\,(\nabla u\cdot\nabla u_t)\,\mathrm{d}x + 2\int_\Omega c(x)f(u)u_t\mathrm{d}x \\ &= 2\int_\Omega \nabla\cdot\left(\rho(|\nabla u|^2)u_t\nabla u\right)\mathrm{d}x - 2\int_\Omega \rho\left(|\nabla u|^2\right)(\nabla u\cdot\nabla u_t)\,\mathrm{d}x + 2\int_\Omega c(x)f(u)u_t\mathrm{d}x \\ &= 2\int_\Omega u_t\left[\nabla\cdot\left(\rho(|\nabla u|^2)\nabla u\right) + c(x)f(u)\right]\mathrm{d}x \\ &= 2\int_\Omega b'(u)u_t^2\mathrm{d}x \ge 0, \end{aligned} \tag{2.6}$$

which with (2.4) imply $E(t) > 0$ and $D'(t) > 0$ for all $t \in (0, t^*)$. By the Hölder inequality, (2.5) and $b'(s) > 0$ for $s > 0$, we obtain

$$\begin{aligned} 2(1+\beta)E(t)D'(t) \le \left(D'(t)\right)^2 &= \left(2\int_\Omega b'(u)uu_t\mathrm{d}x\right)^2 \\ &\le 4\left(\int_\Omega b'(u)u^2\mathrm{d}x\right)\left(\int_\Omega b'(u)u_t^2\mathrm{d}x\right). \end{aligned} \tag{2.7}$$

Using (2.3) and integrating by part, we lead to

$$G(u) = 2\int_0^u sb'(s)\mathrm{d}s = \int_0^u b'(s)\mathrm{d}s^2 = b'(u)u^2 - \int_0^u s^2b''(s)\mathrm{d}s \ge b'(u)u^2. \tag{2.8}$$

We combine (2.7) and (2.8) to derive

$$(1+\beta)E(t)D'(t) \le 2\left(\int_\Omega G(u)\mathrm{d}x\right)\left(\int_\Omega b'(u)u_t^2\mathrm{d}x\right) = D(t)E'(t);$$

that is

$$\left(E(t)D^{-(1+\beta)}(t)\right)' \ge 0. \tag{2.9}$$

Integrating (2.9) over $[0, t]$, we have

$$E(t)D^{-(1+\beta)}(t) \ge E(0)D^{-(1+\beta)}(0).$$

By (2.5), we can deduce

$$D'(t)D^{-(1+\beta)}(t) \ge 2(1+\beta)E(0)D^{-(1+\beta)}(0). \tag{2.10}$$

If $\beta > 0$, integrating (2.10) over $[0, t]$, we derive

$$D^{-\beta}(t) \le D^{-\beta}(0) - 2\beta(1+\beta)E(0)D^{-(1+\beta)}(0)t. \tag{2.11}$$

This inequality can not hold for all $t > 0$. Hence, $u(x, t)$ must blow up at some finite time t^* in the measure $D(t)$. Furthermore, we conclude from (2.11)

$$t^* \le T = \frac{D(0)}{2\beta(1+\beta)E(0)}.$$

If $\beta = 0$, we integrate (2.10) to get

$$D(t) \ge D(0)e^{2E(0)D^{-1}(0)t},$$

which implies $T = \infty$. □

3 Lower bound for blow-up time

In this section, we restrict $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$. Our goal is to determine a lower bound for blow-up time t^*. Here we impose the following constraints on data

$$\begin{gathered}\rho(s) \geq b_1 + b_2 s^q, \qquad b'(s) \geq \gamma, \\ c(x)f(s(x,t)) \leq a_1 + a_2 (s(x,t))^p \left(\int_\Omega (s(x,t))^\alpha \mathrm{d}x \right)^m, \qquad s \geq 0,\end{gathered} \tag{3.1}$$

where a_2, b_2, p, q, γ are positive constants, a_1, b_1, m are nonnegative constants, $p > 2q+1$, $\alpha = 2r(q+1) - 2q$, and parameter r is restricted by the condition

$$r > \max\left\{1, \frac{n(p-2q-1)+4q}{4(q+1)}\right\}. \tag{3.2}$$

We introduce two auxiliary functions

$$A(t) = \int_\Omega B(u)\mathrm{d}x, \quad t \geq 0, \qquad B(u) = \alpha \int_0^u s^{\alpha-1} b'(s)\mathrm{d}s.$$

In this section, we also need to apply the following Sobolev inequality (see [8, Theorem 2, p. 265]) for $n \geq 3$,

$$\left(\int_\Omega (v^{q+1})^{\frac{2n}{n-2}} \mathrm{d}x \right)^{\frac{n-2}{2n}} \leq C \left(\int_\Omega v^{2(q+1)} \mathrm{d}x + \int_\Omega |\nabla v^{q+1}|^2 \mathrm{d}x \right)^{\frac{1}{2}}, \tag{3.3}$$

where $C = C(n, \Omega)$ is an embedding constant. The main result of this section is stated as follows.

Theorem 3.1. *Let u be a classical solution of (1.1). Assume that (3.1)–(3.2) hold. If u blows up at finite time t^* in measure $A(t)$, we then conclude that blow-up time t^* is bounded from blow by*

$$t^* \geq \int_{A(0)}^\infty \frac{d\tau}{K_1 \tau^{\frac{\alpha-1}{\alpha}} + K_2 \tau^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + K_3 \tau^{\frac{2r(q+1)}{2r(q+1)-2q}}},$$

where

$$K_1 = a_1 \alpha |\Omega|^{\frac{1}{\alpha}} \gamma^{\frac{1-\alpha}{\alpha}}, \tag{3.4}$$

$$\begin{aligned}K_2 = {} & \frac{a_2 \alpha [4r(q+1) + 2q(n-2) - n(p-1)]}{4r(q+1) + 2q(n-2)} \left(2C^2\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\ & \times \left(1 + \sigma_1^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \right) \gamma^{-\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}},\end{aligned} \tag{3.5}$$

$$\begin{aligned}K_3 = {} & \left(\frac{b_2 q \alpha(\alpha-1)}{r^{2(q+1)}} + \frac{a_2 n \alpha (p-1)}{4r(q+1)+2q(n-2)} \left(2C^2\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \right) \\ & \times \frac{4r(q+1)-4q}{2r(q+1)+q(n-2)} \left(2C^2\right)^{\frac{nq}{2r(q+1)-2q}} \gamma^{-\frac{2r(q+1)}{2r(q+1)-2q}} \\ & \times \left[\sigma_2^{-\frac{nq}{2r(q+1)-2q}} + \left(\frac{2r(q+1)+q(n-2)}{2nq} \right)^{-\frac{nq}{2r(q+1)-2q}} \right],\end{aligned} \tag{3.6}$$

$$\sigma_1 = \frac{b_2(\alpha-1)[2r(q+1)+q(n-2)]}{a_2 n(p-1)(q+1)r^{2(q+1)}} \left(2C^2\right)^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}, \tag{3.7}$$

$$\begin{aligned}\sigma_2 = &\frac{b_2(\alpha-1)[2r(q+1)+q(n-2)]}{2nq(q+1)} \\ &\times \left[2b_2q(\alpha-1) + \frac{a_2n(p-1)r^{2(q+1)}}{2r(q+1)+q(n-2)} \left(2C^2\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\right]^{-1}.\end{aligned} \tag{3.8}$$

Proof. By (3.2), we have $\alpha > 2$. It follows from Green's formula and (3.1) that

$$\begin{aligned}A'(t) = &\int_\Omega B'(u)u_t\mathrm{d}x = \alpha\int_\Omega u^{\alpha-1}b'(u)u_t\mathrm{d}x \\ &= \alpha\int_\Omega u^{\alpha-1}\left[\nabla\cdot\left(\rho(|\nabla u|^2)\nabla u\right) + c(x)f(u)\right]\mathrm{d}x \\ &= \alpha\int_\Omega \nabla\cdot\left(u^{\alpha-1}\rho(|\nabla u|^2)\nabla u\right)\mathrm{d}x - \alpha(\alpha-1)\int_\Omega \rho(|\nabla u|^2)u^{\alpha-2}|\nabla u|^2\mathrm{d}x \\ &\quad + \alpha\int_\Omega u^{\alpha-1}c(x)f(u)\mathrm{d}x \\ &\le \alpha\int_{\partial\Omega} u^{\alpha-1}\rho(|\nabla u|^2)\frac{\partial u}{\partial \nu}\mathrm{d}S - \alpha(\alpha-1)\int_\Omega u^{\alpha-2}\left(b_1 + b_2|\nabla u|^{2q}\right)|\nabla u|^2\mathrm{d}x \\ &\quad + a_1\alpha\int_\Omega u^{\alpha-1}\mathrm{d}x + a_2\alpha\int_\Omega u^{\alpha+p-1}\mathrm{d}x\left(\int_\Omega u^\alpha\mathrm{d}x\right)^m \\ &\le -b_2\alpha(\alpha-1)\int_\Omega u^{\alpha-2}|\nabla u|^{2(q+1)}\mathrm{d}x + a_1\alpha\int_\Omega u^{\alpha-1}\mathrm{d}x \\ &\quad + a_2\alpha\int_\Omega u^{\alpha+p-1}\mathrm{d}x\left(\int_\Omega u^\alpha\mathrm{d}x\right)^m.\end{aligned} \tag{3.9}$$

We apply the Hölder inequality to get

$$\int_\Omega u^{\alpha-1}\mathrm{d}x \le |\Omega|^{\frac{1}{\alpha}}\left(\int_\Omega u^\alpha\mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}}. \tag{3.10}$$

For brevity, we denote $v = u^r$ and

$$|\nabla u^r|^{2(q+1)} = r^{2(q+1)}u^{2(r-1)(q+1)}|\nabla u|^{2(q+1)}. \tag{3.11}$$

Hence, by (3.10)–(3.11), (3.9) can be rewritten as

$$\begin{aligned}A'(t) \le &-\frac{b_2\alpha(\alpha-1)}{r^{2(q+1)}}\int_\Omega |\nabla u^r|^{2(q+1)}\mathrm{d}x + a_1\alpha|\Omega|^{\frac{1}{\alpha}}\left(\int_\Omega u^\alpha\mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} \\ &+ a_2\alpha\int_\Omega u^{\alpha+p-1}\mathrm{d}x\left(\int_\Omega u^\alpha\mathrm{d}x\right)^m \\ = &-\frac{b_2\alpha(\alpha-1)}{r^{2(p+1)}}\int_\Omega |\nabla v|^{2(q+1)}\mathrm{d}x + a_1\alpha|\Omega|^{\frac{1}{\alpha}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} \\ &+ a_2\alpha\int_\Omega v^{2(q+1)+\frac{p-2q-1}{r}}\mathrm{d}x\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^m.\end{aligned} \tag{3.12}$$

Using the Hölder inequality and the Young inequality, we have

$$\begin{aligned}\int_\Omega |\nabla v^{q+1}|^2 \mathrm{d}x &= (q+1)^2 \int_\Omega v^{2q}|\nabla v|^2 \mathrm{d}x \\ &\le (q+1)^2 \left(\int_\Omega v^{2(q+1)}\mathrm{d}x\right)^{\frac{q}{q+1}} \left(\int_\Omega |\nabla v|^{2(q+1)}\mathrm{d}x\right)^{\frac{1}{q+1}} \\ &\le q(q+1)\int_\Omega v^{2(q+1)}\mathrm{d}x + (q+1)\int_\Omega |\nabla v|^{2(q+1)}\mathrm{d}x;\end{aligned}$$

that is

$$\int_\Omega |\nabla v|^{2(q+1)}\mathrm{d}x \ge \frac{1}{q+1}\int_\Omega \left|\nabla v^{q+1}\right|^2 \mathrm{d}x - q\int_\Omega v^{2(q+1)}\mathrm{d}x. \tag{3.13}$$

Substituting (3.13) into (3.12), we get

$$\begin{aligned}A'(t) \le{}& \frac{b_2 q\alpha(\alpha-1)}{r^{2(q+1)}}\int_\Omega v^{2(q+1)}\mathrm{d}x - \frac{b_2\alpha(\alpha-1)}{(q+1)r^{2(q+1)}}\int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x + a_1\alpha|\Omega|^{\frac{1}{\alpha}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} \\ &+ a_2\alpha\int_\Omega v^{2(q+1)+\frac{p-2q-1}{r}}\mathrm{d}x\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^m.\end{aligned} \tag{3.14}$$

Now, we deal with the last term of (3.14). Applying the Hölder inequality and (3.3), we derive

$$\begin{aligned}&\int_\Omega v^{2(q+1)+\frac{p-2q-1}{r}}\mathrm{d}x\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^m \\ &\le \left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)}+m}\left(\int_\Omega (v^{q+1})^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{(n-2)(p-1)}{4r(q+1)+2q(n-2)}} \\ &\le \left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)}+m}\left[C^{\frac{2n}{n-2}}\left(\int_\Omega v^{2(q+1)}\mathrm{d}x + \int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x\right)^{\frac{n}{n-2}}\right]^{\frac{(n-2)(p-1)}{4r(q+1)+2q(n-2)}} \\ &= \left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)}+m}\left(C^2\int_\Omega v^{2(q+1)}\mathrm{d}x + C^2\int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}},\end{aligned} \tag{3.15}$$

where $0 < \frac{(n-2)(p-1)}{4r(q+1)+2q(n-2)} < 1$ in view of (3.2). Using in (3.15) the basic inequality

$$(k_1+k_2)^j \le 2^j(k_1^j + k_2^j),\ k_1,\ k_2,\ j > 0, \tag{3.16}$$

we have

$$\begin{aligned}&\int_\Omega v^{2(q+1)+\frac{p-2q-1}{r}}\mathrm{d}x\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^m \\ &\le (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)}+m}\left(\int_\Omega v^{2(q+1)}\mathrm{d}x\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}} \\ &\quad + (2C^2)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)}+m} \\ &\quad \times\left(\int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}}.\end{aligned} \tag{3.17}$$

An application of the Young inequality to the first term of (3.17) yields

$$\begin{aligned}
&\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)}+m}\left(\int_\Omega v^{2(q+1)}\mathrm{d}x\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}}\\
&=\left[\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\right]^{\frac{4r(q+1)+2q(n-2)-n(p-1)}{4r(q+1)+2q(n-2)}}\left(\int_\Omega v^{2(q+1)}\mathrm{d}x\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}}\\
&\leq\frac{4r(q+1)+2q(n-2)-n(p-1)}{4r(q+1)+2q(n-2)}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\\
&\quad+\frac{n(p-1)}{4r(q+1)+2q(n-2)}\int_\Omega v^{2(q+1)}\mathrm{d}x, \qquad (3.18)
\end{aligned}$$

where we use the fact that $0<\frac{n(p-1)}{4r(q+1)+2q(n-2)}<1$ due to (3.2). Similarly, for the second term of (3.17), we apply the Young inequality to obtain

$$\begin{aligned}
&\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{4r(q+1)+2q(n-2)-(n-2)(p-1)}{4r(q+1)+2q(n-2)}+m}\left(\int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}}\\
&=\left[\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\right]^{\frac{4r(q+1)+2q(n-2)-n(p-1)}{4r(q+1)+2q(n-2)}}\left(\int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)}}\\
&\leq\frac{4r(q+1)+2q(n-2)-n(p-1)}{4r(q+1)+2q(n-2)}\sigma_1^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\\
&\quad+\frac{n(p-1)\sigma_1}{4r(q+1)+2q(n-2)}\int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x, \qquad (3.19)
\end{aligned}$$

where σ_1 is given in (3.7). Substituting (3.18)–(3.19) into (3.17), we have

$$\begin{aligned}
&\int_\Omega v^{2(q+1)+\frac{p-2q-1}{r}}\mathrm{d}x\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^m\\
&\leq\frac{4r(q+1)+2q(n-2)-n(p-1)}{4r(q+1)+2q(n-2)}\left(2C^2\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\\
&\quad\times\left(1+\sigma_1^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\right)\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\\
&\quad+\frac{n(p-1)}{4r(q+1)+2q(n-2)}\left(2C^2\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\\
&\quad\times\left(\sigma_1\int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x+\int_\Omega v^{2(q+1)}\mathrm{d}x\right). \qquad (3.20)
\end{aligned}$$

Inserting (3.20) into (3.14), we deduce

$$
\begin{aligned}
A'(t) \leq & \, a_1\alpha|\Omega|^{\frac{1}{\alpha}} \left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} + \frac{a_2\alpha[4r(q+1)+2q(n-2)-n(p-1)]}{4r(q+1)+2q(n-2)} \left(2C^2\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\
& \times \left(1+\sigma_1^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\right) \left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \\
& + \left(\frac{b_2q\alpha(\alpha-1)}{r^{2(q+1)}} + \frac{a_2n\alpha(p-1)}{4r(q+1)+2q(n-2)} \left(2C^2\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\right) \int_\Omega v^{2(q+1)}\mathrm{d}x \\
& + \left(\frac{a_2n\alpha(p-1)\sigma_1}{4r(q+1)+2q(n-2)} \left(2C^2\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} - \frac{b_2\alpha(\alpha-1)}{(q+1)r^{2(q+1)}}\right) \\
& \times \int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x.
\end{aligned}
\tag{3.21}
$$

Next, we pay our attention to the third term of (3.21). By the Hölder inequality and (3.3), we obtain

$$
\begin{aligned}
& \int_\Omega v^{2(q+1)}\mathrm{d}x \\
& \quad \leq \left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left(\int_\Omega \left(v^{q+1}\right)^{\frac{2n}{n-2}}\mathrm{d}x\right)^{\frac{q(n-2)}{2r(q+1)+q(n-2)}} \\
& \quad \leq \left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left[C^{\frac{2n}{n-2}} \left(\int_\Omega v^{2(q+1)}\mathrm{d}x + \int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x\right)^{\frac{n}{n-2}}\right]^{\frac{q(n-2)}{2r(q+1)+q(n-2)}} \\
& \quad = \left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left(C^2\int_\Omega v^{2(q+1)}\mathrm{d}x + C^2\int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x\right)^{\frac{nq}{2r(q+1)+q(n-2)}},
\end{aligned}
\tag{3.22}
$$

where $0 < \frac{q(n-2)}{2r(q+1)+q(n-2)} < 1$ in view of (3.2). Using (3.16) in (3.22), we have

$$
\begin{aligned}
& \int_\Omega v^{2(q+1)}\mathrm{d}x \\
& \quad \leq \left(2C^2\right)^{\frac{nq}{2r(q+1)+q(n-2)}} \left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left(\int_\Omega v^{2(q+1)}\mathrm{d}x\right)^{\frac{nq}{2r(q+1)+q(n-2)}} \\
& \qquad + \left(2C^2\right)^{\frac{nq}{2r(q+1)+q(n-2)}} \left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}} \left(\int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x\right)^{\frac{nq}{2r(q+1)+q(n-2)}}.
\end{aligned}
\tag{3.23}
$$

For the first term of (3.23), we use the Young inequality to get

$$
\begin{aligned}
&(2C^2)^{\frac{nq}{2r(q+1)+q(n-2)}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}}\left(\int_\Omega v^{2(q+1)}\mathrm{d}x\right)^{\frac{nq}{2r(q+1)+q(n-2)}}\\
&\quad=\left[(2C^2)^{\frac{nq}{2r(q+1)-2q}}\left(\frac{2r(q+1)+q(n-2)}{2nq}\right)^{-\frac{nq}{2r(q+1)-2q}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)-2q}}\right]^{\frac{2r(q+1)-2q}{2r(q+1)+q(n-2)}}\\
&\qquad\times\left(\frac{2r(q+1)+q(n-2)}{2nq}\int_\Omega v^{2(q+1)}\mathrm{d}x\right)^{\frac{nq}{2r(q+1)+q(n-2)}}\\
&\quad\le\frac{2r(q+1)-2q}{2r(q+1)+q(n-2)}(2C^2)^{\frac{nq}{2r(q+1)-2q}}\left(\frac{2r(q+1)+q(n-2)}{2nq}\right)^{-\frac{nq}{2r(q+1)-2q}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)-2q}}\\
&\qquad+\frac{1}{2}\int_\Omega v^{2(q+1)}\mathrm{d}x,
\end{aligned}\tag{3.24}
$$

where $0<\frac{nq}{2r(q+1)+q(n-2)}<1$ in consideration of (3.2). We again use the Young inequality to second term of (3.23) to obtain

$$
\begin{aligned}
&(2C^2)^{\frac{nq}{2r(q+1)+q(n-2)}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)+q(n-2)}}\left(\int_\Omega |\nabla v^{q+1}|^2\mathrm{d}x\right)^{\frac{nq}{2r(q+1)+q(n-2)}}\\
&\quad\le\left[(2C^2)^{\frac{nq}{2r(q+1)-2q}}\left(\int_\Omega v^{\frac{\alpha}{r}}\right)^{\frac{2r(q+1)}{2r(q+1)-2q}}\sigma_2^{-\frac{nq}{2r(q+1)-2q}}\right]^{\frac{2r(q+1)-2q}{2r(q+1)+q(n-2)}}\left(\sigma_2\int_\Omega|\nabla v^{q+1}|^2\mathrm{d}x\right)^{\frac{nq}{2r(q+1)+q(n-2)}}\\
&\quad\le\frac{2r(q+1)-2q}{2r(q+1)+q(n-2)}(2C^2)^{\frac{nq}{2r(q+1)-2q}}\sigma_2^{-\frac{nq}{2r(q+1)-2q}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)-2q}}\\
&\qquad+\frac{nq}{2r(q+1)+q(n-2)}\sigma_2\int_\Omega|\nabla v^{q+1}|^2\mathrm{d}x,
\end{aligned}\tag{3.25}
$$

where σ_2 is defined in (3.8). Substituting (3.24)–(3.25) into (3.23), we get

$$
\begin{aligned}
&\int_\Omega v^{2(q+1)}\mathrm{d}x\\
&\quad\le\frac{4r(q+1)-4q}{2r(q+1)+q(n-2)}(2C^2)^{\frac{nq}{2r(q+1)-2q}}\left[\left(\frac{2r(q+1)+q(n-2)}{2nq}\right)^{-\frac{nq}{2r(q+1)-2q}}+\sigma_2^{-\frac{nq}{2r(q+1)-2q}}\right]\\
&\qquad\times\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)-2q}}+\frac{2nq\sigma_2}{2r(q+1)+q(n-2)}\int_\Omega|\nabla v^{q+1}|^2\mathrm{d}x.
\end{aligned}\tag{3.26}
$$

Inserting (3.26) into (3.21), we have

$$
\begin{aligned}
A'(t)\le{}& a_1\alpha|\Omega|^{\frac{1}{\alpha}}\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}}+J_1\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\\
&+J_2\left(\int_\Omega v^{\frac{\alpha}{r}}\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)-2q}},
\end{aligned}\tag{3.27}
$$

where

$$J_1 = \frac{a_2\alpha[4r(q+1)+2q(n-2)-n(p-1)]}{4r(q+1)+2q(n-2)} \left(2C^2\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} \left(1+\sigma_1^{-\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\right),$$

$$J_2 = \left(\frac{b_2 q\alpha(\alpha-1)}{r^{2(q+1)}} + \frac{a_2 n\alpha(p-1)}{4r(q+1)+2q(n-2)} \left(2C^2\right)^{\frac{n(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}\right)$$
$$\times \frac{4r(q+1)-4q}{2r(q+1)+q(n-2)} \left(2C^2\right)^{\frac{nq}{2r(q+1)-2q}} \left[\left(\frac{2r(q+1)+q(n-2)}{2nq}\right)^{-\frac{nq}{2r(q+1)-2q}} + \sigma_2^{-\frac{nq}{2r(q+1)-2q}}\right].$$

From (3.1), it follows that

$$B(u) = \alpha \int_0^u b'(s)s^{\alpha-1}\mathrm{d}s \geq \alpha\gamma \int_0^u s^{\alpha-1}\mathrm{d}s = \gamma u^\alpha;$$

that is

$$v^{\frac{\alpha}{r}} = u^\alpha \leq \frac{1}{\gamma}B(u). \tag{3.28}$$

Combining (3.27) and (3.28), we get

$$A'(t) \leq a_1\alpha|\Omega|^{\frac{1}{\alpha}}\gamma^{\frac{\alpha-1}{\alpha}} \left(\int_\Omega B(u)\mathrm{d}x\right)^{\frac{\alpha-1}{\alpha}} + J_1\gamma^{-\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}}$$
$$\times \left(\int_\Omega B(u)\mathrm{d}x\right)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + J_2\gamma^{-\frac{2r(q+1)}{2r(q+1)-2q}} \left(\int_\Omega B(u)\mathrm{d}x\right)^{\frac{2r(q+1)}{2r(q+1)-2q}}$$
$$= K_1A(t)^{\frac{\alpha-1}{\alpha}} + K_2A(t)^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + K_3A(t)^{\frac{2r(q+1)}{2r(q+1)-2q}}, \tag{3.29}$$

where K_1, K_2 and K_3 are defined in (3.4), (3.5) and (3.6), respectively. We integrate (3.29) from 0 to t to obtain

$$\int_{A(0)}^{A(t)} \frac{d\tau}{K_1\tau^{\frac{\alpha-1}{\alpha}} + K_2\tau^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + K_3\tau^{\frac{2r(q+1)}{2r(q+1)-2q}}} \leq t.$$

Letting $t \to t^*$, a lower bound for t^* is given by

$$t^* \geq \int_{A(0)}^{\infty} \frac{d\tau}{K_1\tau^{\frac{\alpha-1}{\alpha}} + K_1\tau^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + K_2\tau^{\frac{2r(q+1)}{2r(q+1)-2q}}}.$$ □

4 Application

In this section, an example is presented to illustrate the applications of Theorems 2.1 and 3.1.

Example 4.1. Let u be a classical solution of the following problem:

$$\begin{cases} (u+\ln(1+u))_t = \nabla\cdot\left(\frac{1}{10}\left(1+|\nabla u|\right)\nabla u\right) + \left(3+|x|^2\right)u^{\frac{5}{2}}\left(\int_\Omega u^3\mathrm{d}x\right)^{\frac{1}{4}} & \text{in } \Omega\times(0,t^*), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,t^*), \\ u(x,0) = 1+\left(1-|x|^2\right)^2 & \text{in } \overline{\Omega}, \end{cases}$$

where $\Omega = \left\{x = (x_1, x_2, x_3) \mid |x|^2 = x_1^2 + x_2^2 + x_3^2 < 1\right\}$ is a ball of $\mathbb{R}^3$. We then have

$$b(u) = u + \ln(1+u), \qquad \rho(|\nabla u|^2) = \frac{1}{10}(1 + |\nabla u|), \qquad c(x) = 3 + |x|^2, \tag{4.1}$$

$$f(u) = u^{\frac{5}{2}} \left(\int_\Omega u^3 \mathrm{d}x\right)^{\frac{1}{4}}, \qquad u_0(x) = 1 + (1 - |x|^2)^2. \tag{4.2}$$

It follows from (2.1)–(2.2) and (4.1)–(4.2) that

$$\begin{aligned}
F(u) &= \int_0^u f(s)\mathrm{d}s = \int_0^u s^{\frac{5}{2}} \left(\int_\Omega s^3 \mathrm{d}x\right)^{\frac{1}{4}} \mathrm{d}s, \\
G(u) &= 2\int_0^u s b'(s)\mathrm{d}s = 2\int_0^u s\left(1 + \frac{1}{1+s}\right)\mathrm{d}s = u^2 + 2u - 2\ln(1+u), \\
P(|\nabla u|^2) &= \int_0^{|\nabla u|^2} \rho(s)\mathrm{d}s = \frac{1}{10}\int_0^{|\nabla u|^2}\left(1 + s^{\frac{1}{2}}\right)\mathrm{d}s = \frac{1}{10}|\nabla u|^2 + \frac{1}{15}|\nabla u|^3, \\
D(t) &= \int_\Omega G(u)\mathrm{d}x = \int_\Omega \left(u^2 + 2u - 2\ln(1+u)\right)\mathrm{d}x, \\
E(t) &= -\int_\Omega P(|\nabla u|^2)\mathrm{d}x + 2\int_\Omega c(x)F(u)\mathrm{d}x \\
&= \int_\Omega \left(-\frac{1}{10}|\nabla u|^2 - \frac{1}{15}|\nabla u|^3\right)\mathrm{d}x + 2\int_\Omega (3 + |x|^2)\left[\int_0^u s^{\frac{5}{2}}\left(\int_\Omega s^3\mathrm{d}x\right)^{\frac{1}{4}}\mathrm{d}s\right]\mathrm{d}x.
\end{aligned}$$

Selecting $\beta = \frac{1}{2}$, it is easy to check that (2.3)–(2.4) hold. Moreover, we compute

$$\begin{aligned}
D(0) &= \int_\Omega \left(u_0^2 + 2u_0 - 2\ln(1+u_0)\right)\mathrm{d}x \\
&= \int_\Omega \left((|x|^4 - 2|x|^2 + 2)^2 + 2(|x|^4 - 2|x|^2 + 2) - 2\ln\left(|x|^4 - 2|x|^2 + 3\right)\right)\mathrm{d}x \\
&= 10.1931.
\end{aligned}$$

Since $1 \le u_0 \le 2$, we have

$$\begin{aligned}
F(u_0) &= \int_0^{u_0} s^{\frac{5}{2}}\left(\int_\Omega s^3\mathrm{d}x\right)^{\frac{1}{4}}\mathrm{d}s \ge \int_{\frac{1}{2}}^{u_0} s^{\frac{5}{2}}\left(\int_\Omega s^3\mathrm{d}x\right)^{\frac{1}{4}}\mathrm{d}s \ge \int_{\frac{1}{2}}^{u_0} s^{\frac{5}{2}}\left(\int_\Omega \left(\frac{1}{2}\right)^3\mathrm{d}x\right)^{\frac{1}{4}}\mathrm{d}s \\
&= \frac{2}{7}\times\left(\frac{1}{2}\right)^{\frac{3}{4}}|\Omega|^{\frac{1}{4}}u_0^{\frac{7}{2}} - \frac{2}{7}\times\left(\frac{1}{2}\right)^{\frac{17}{4}}|\Omega|^{\frac{1}{4}} = 0.2430u_0^{\frac{7}{2}} - 0.0215.
\end{aligned} \tag{4.3}$$

In view of $E(t)$ and (4.3), we have

$$\begin{aligned}
E(0) &= -\int_\Omega P(|\nabla u_0|^2)\mathrm{d}x + 2\int_\Omega c(x)F(u_0)\mathrm{d}x \\
&\ge \int_\Omega\left(-\frac{1}{10}|\nabla u_0|^2 - \frac{1}{15}|\nabla u_0|^3\right)\mathrm{d}x + 2\int_\Omega (3 + |x|^2)\left(0.2430u_0^{\frac{7}{2}} - 0.0215\right)\mathrm{d}x \\
&= \int_\Omega \Big(-1.6\left(1 - |x|^2\right)^2|x|^2 - \frac{64}{15}\left(1 - |x|^2\right)^3|x|^3 \\
&\qquad + (6 + 2|x|^2)[0.2430(|x|^4 - 2|x|^2 + 2)^{\frac{7}{2}} - 0.0215]\Big)\mathrm{d}x \\
&= 15.3826.
\end{aligned}$$

Consequently, by Theorem 2.1, we know that the solution u blows up at $t^* \leq T$ in the measure $D(t)$ and

$$T = \frac{D(0)}{2\beta(1+\beta)E(0)} \leq 0.4418. \tag{4.4}$$

Next, we apply Theorem 3.1 to obtain a lower bound for t^*. Here we have $n = 3$ and $|\Omega| = \frac{4}{3}\pi$. Choosing $a_1 = 0$, $a_2 = 4$, $b_1 = \frac{1}{10}$, $b_2 = \frac{1}{10}$, $\gamma = 1$, $m = \frac{1}{4}$, $p = \frac{5}{2}$, $q = \frac{1}{2}$, $r = \frac{4}{3}$, and $\alpha = 3$, we can check that (3.1)–(3.2) hold. The Sobolev embedding constant $C = 4^{\frac{1}{3}} 3^{-\frac{1}{2}} \pi^{-\frac{2}{3}}$ is given in [11]. Inserting above constants into (3.4)–(3.8), we obtain $\sigma_1 = 0.0385, \sigma_2 = 0.0546, K_1 = 0, K_2 = 59.1007, K_2 = 9.5161$. Moreover, we compute

$$B(u) = \alpha \int_0^u b'(s) s^{\alpha-1} \mathrm{d}s = 3\int_0^u s^2 \left(1 + \frac{1}{1+s}\right) \mathrm{d}s = u^3 + \frac{3}{2}u^2 + 3\ln(u+1) - 3u,$$
$$A(t) = \int_\Omega B(u)\mathrm{d}x = \int_\Omega \left(u^3 + \frac{3}{2}u^2 + 3\ln(u+1) - 3u\right) \mathrm{d}x,$$

and

$$\begin{aligned} A(0) &= \int_\Omega \left(u_0^3 + \frac{3}{2}u_0^2 + 3\ln(u_0+1) - 3u_0\right) \mathrm{d}x \\ &= \int_\Omega \Bigg(\left(|x|^4 - 2|x|^2 + 2\right)^3 + \frac{3}{2}\left(|x|^4 - 2|x|^2 + 2\right)^2 \\ &\qquad + 3\ln\left(|x|^4 - 2|x|^2 + 3\right) - 3(|x|^4 - 2|x|^2 + 2)\Bigg)\mathrm{d}x \\ &= 13.1535. \end{aligned}$$

Since u blows up at t^* in measure $D(t)$, we know that u blows up at a finite time t^*. Therefore, u blows up at t^* in measure $A(t)$. By Theorem 3.1, we obtain a lower bound for the blow-up time

$$\begin{aligned} t^* &\geq \int_{A(0)}^\infty \frac{d\tau}{K_1 \tau^{\frac{\alpha-1}{\alpha}} + K_1 \tau^{\frac{[4r(q+1)+2q(n-2)](1+m)-(n-2)(p-1)}{4r(q+1)+2q(n-2)-n(p-1)}} + K_2 \tau^{\frac{2r(q+1)}{2r(q+1)-2q}}} \\ &= \int_{13.1535}^\infty \frac{d\tau}{59.1007\tau^{\frac{13}{6}} + 9.5161\tau^{\frac{4}{3}}} = 7.0988 \times 10^{-4}. \end{aligned} \tag{4.5}$$

It follows from (4.4)–(4.5) that

$$7.0988 \times 10^{-4} \leq t^* \leq 0.4418.$$

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