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Weak solutions to Dirichlet boundary value problem driven by $p(x)$-Laplacian-like operator

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Abstract. We prove the existence of weak solutions to the Dirichlet boundary value problem for equations involving the $p(x)$-Laplacian-like operator in the principal part, with reaction term satisfying a sub-critical growth condition. We establish the existence of at least one nontrivial weak solution and three weak solutions, by using variational methods and critical point theory.

Keywords: Dirichlet boundary value problem, $p(x)$-Laplacian-like operator, variable exponent Sobolev space.

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1 Introduction

In this article we consider the following Dirichlet boundary value problem:

$$\begin{cases} -\Delta^l_{p(x)} u(x) + |u(x)|^{p(x)-2} u(x) = \lambda\, g(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where

$$\Delta^l_{p(x)} u := \operatorname{div} \left(\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) |\nabla u|^{p(x)-2} \nabla u \right)$$

is the $p(x)$-Laplacian-like, $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary, $p \in C(\overline{\Omega})$ is a function with some regularity satisfying

$$1 < p^- := \inf_{x \in \Omega} p(x) \le p(x) \le p^+ := \sup_{x \in \Omega} p(x) < +\infty.$$

The function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory (that is, for all $z \in \mathbb{R}$, $x \to g(x,z)$ is measurable and for a.a. $x \in \Omega$, $z \to g(x,z)$ is continuous) and λ is a real positive parameter. In the sequel of this article, we assume that the reaction term $g(x,z)$ satisfies the hypothesis:

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(g_1) there exist $a_1, a_2 \in [0, +\infty[$ and $\alpha \in C(\overline{\Omega})$ with $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$, such that

$$|g(x,z)| \leq a_1 + a_2|z|^{\alpha(x)-1} \quad \text{for all } (x,z) \in \Omega \times \mathbb{R},$$

where $p^*(x) = \dfrac{np(x)}{n - p(x)}$ if $p(x) < n$ and $p^*(x) = +\infty$ if $p(x) \geq n$.

Now, let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in the generalized Lebesgue–Sobolev space $W^{1,p(x)}(\Omega)$ given in Section 2. For a weak solution of problem (P_λ), we mean a function $u \in W_0^{1,p(x)}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)dx + \int_\Omega \frac{|\nabla u(x)|^{2p(x)-2}\nabla u(x)}{\sqrt{1+|\nabla u(x)|^{2p(x)}}}\nabla v(x)dx$$
$$+ \int_\Omega |u(x)|^{p(x)-2}u(x)v(x)dx = \lambda \int_\Omega g(x,u(x))v(x)\,dx,$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

Existence and multiplicity results for problems involving the $p(x)$-Laplacian-like were obtained by Rodrigues [13] (Dirichlet boundary condition), Afrouzi–Kirane–Shokooh [1] (Neumann boundary condition). For other problems driven by the $p(x)$-Laplacian operator, there are the works of Fan–Zhang [9], Bonanno–Chinnì [3] (Dirichlet boundary condition), and Deng–Wang [7], Pan–Afrouzi–Li [12] (Neumann boundary condition). Also, we mention the comprehensive book on nonlinear boundary value problems by Motreanu–Motreanu–Papageorgiou [11].

Here, we prove the existence of weak solutions to the Dirichlet boundary value problem (P_λ), by using variational methods and critical point theory. Precisely, we apply a result of Bonanno [2] for functionals satisfying the Palais–Smale condition cut off upper at r (the $(PS)^{[r]}$-condition for short), to obtain the existence of at least one nontrivial weak solution. Then, we use a result of Bonanno–Marano [4] to obtain the existence of three weak solutions. The motivation of this study comes from the use of such problems to model the behaviour of electrorheological fluids in physics (as discussed in Diening–Harjulehto–Hästö–Růžička [8]) and, in particular, the phenomenon of capillarity which depends on solid and liquid interfacial properties such as surface tension, contact angle, and solid surface geometry.

2 Mathematical background

Let X be a real Banach space and X^* its topological dual. In developing our study, we consider both the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and the generalized Lebesgue–Sobolev space $W^{1,p(x)}(\Omega)$. Indeed, these spaces, in respect to the norms defined below, are separable, reflexive and uniformly convex Banach spaces (see Fan–Zhang [9]). So, we have the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ given as

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ : \ \text{u is measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty \right\},$$

where we consider the following norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\} \quad \text{(i.e., Luxemburg norm).}$$

On the other hand, the generalized Lebesgue–Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Also, we take the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \| \, |\nabla u| \, \|_{L^{p(x)}(\Omega)},$$

which is equivalent to the norm

$$\|u\| := \inf \left\{ \lambda > 0 : \int_\Omega \left(\left| \frac{u(x)}{\lambda} \right|^{p(x)} + \left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} \right) dx \leq 1 \right\},$$

(see D'Aguì–Sciammetta [6]). In the following, we will use the norm $\|u\|$ instead of $\|u\|_{W^{1,p(x)}(\Omega)}$ on $W_0^{1,p(x)}(\Omega)$. In the proofs of our theorems, we use a Sobolev embedding result; precisely we refer to the following proposition due to Fan–Zhao [10].

Proposition 2.1. *Let $p \in C(\overline{\Omega})$ with $p(x) > 1$ for each $x \in \overline{\Omega}$. Then, there exists a continuous and compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$, provided that $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$.*

Another useful theorem, which links $\|u\|_{L^{p(x)}(\Omega)}$ to $\int_\Omega |u(x)|^{p(x)} dx$ (respectively, $\|u\|$ to $\int_\Omega \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) dx$), can be stated as follows (Fan–Zhao [10, Theorem 1.3] and Cammaroto–Chinnì–Di Bella [5, Proposition 2.1]).

Theorem 2.2. *Let $u \in L^{p(x)}(\Omega)$ (resp., $u \in W_0^{1,p(x)}(\Omega)$) and put $\|u\|_* = \|u\|_{L^{p(x)}(\Omega)}$ (resp., $\|u\|_* = \|u\|$) and $\rho_*(u) = \int_\Omega |u(x)|^{p(x)} dx$ (resp., $\rho_*(u) = \int_\Omega \left(|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)} \right) dx$). Then, we have:*

(i) $\|u\|_* < 1 \ (= 1, \ > 1) \Leftrightarrow \rho_*(u) < 1 \ (= 1, \ > 1)$;

(ii) *if* $\|u\|_* > 1$, *then* $\|u\|_*^{p^-} \leq \rho_*(u) \leq \|u\|_*^{p^+}$;

(iii) *if* $\|u\|_* < 1$, *then* $\|u\|_*^{p^+} \leq \rho_*(u) \leq \|u\|_*^{p^-}$.

Next, let $G : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$G(x,t) = \int_0^t g(x,z) dz \quad \text{for all } t \in \mathbb{R}, \ x \in \Omega,$$

and consider the functional $\Psi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\Psi(u) = \int_\Omega G(x, u(x)) \, dx, \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

By using (g_1), we get $\Psi \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$. Also, by Proposition 2.1 we deduce that Ψ has a compact derivative given as

$$\Psi'(u)(v) = \int_\Omega g(x, u(x)) v(x) \, dx, \quad \text{for all } u, v \in W_0^{1,p(x)}(\Omega).$$

Moreover, let $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the functional defined by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx + \int_\Omega \frac{1}{p(x)} \left[\sqrt{1 + |\nabla u(x)|^{2p(x)}} - 1 \right] dx + \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} dx$$

for all $u \in W_0^{1,p(x)}(\Omega)$, so that Φ is in $C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$. We recall that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative $\Phi' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ is

$$\Phi'(u)(v) = \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) dx + \int_\Omega \frac{|\nabla u(x)|^{2p(x)-2} \nabla u(x)}{\sqrt{1 + |\nabla u(x)|^{2p(x)}}} \nabla v(x) dx$$
$$+ \int_\Omega |u(x)|^{p(x)-2} u(x) v(x) dx$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$. From Rodrigues [13], we recall the following proposition.

Proposition 2.3. *The functional $\Phi' : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)^*$ is a strictly monotone and bounded homeomorphism.*

Finally, consider the functional $I_\lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by $I_\lambda(u) = \Phi(u) - \lambda \Psi(u)$ for all $u \in W_0^{1,p(x)}(\Omega)$. We have

$$\inf_{u \in W_0^{1,p(x)}(\Omega)} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

We conclude this section with the following notion.

Definition 2.4. Let X be a real Banach space and X^* its topological dual. Then, $I_\lambda : X \to \mathbb{R}$ satisfies the Palais–Smale condition cut off upper at r, with fixed $r \in]-\infty, +\infty]$, if any sequence $\{u_n\}$ such that

(i) $\{I_\lambda(u_n)\}$ is bounded;

(ii) $\lim_{n \to +\infty} \|I'_\lambda(u_n)\|_{X^*} = 0$;

(iii) $\Phi(u_n) < r$,

has a convergent subsequence.

3 Existence of one weak solution

In this section we establish an existence theorem producing at least one nontrivial weak solution of (P_λ). To this aim, we apply a theorem proved by Bonanno [2, Theorem 2.3], which reads as follows.

Theorem 3.1. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_{u \in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < r$, such that*

(i) $\sigma = \frac{1}{r} \sup_{\Phi(u) \le r} \Psi(u) < \frac{\Psi(\bar{u})}{\Phi(\bar{u})} = \rho$;

(ii) *for each* $\lambda \in \left]\frac{1}{\rho}, \frac{1}{\sigma}\right[$ *the functional* $I_\lambda := \Phi - \lambda\Psi$ *satisfies the* $(P.S.)^{[r]}$*-condition.*

Then, for each $\lambda \in \Lambda_r := \left]\frac{1}{\rho}, \frac{1}{\sigma}\right[$, *there is* $u_{0,\lambda} \in \Phi^{-1}(]0,r[)$ *such that* $I'_\lambda(u_{0,\lambda}) \equiv \vartheta_{X^*}$ *and* $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]0,r[)$.

Here, we need the function $\delta : \overline{\Omega} \to \mathbb{R}$ given as $\delta(x) = d(x, \partial\overline{\Omega})$, with d to denote the Euclidean distance. Let $x_0 \in \Omega$ be a point of maximum for δ and let $D = \delta(x_0)$, then $B(x_0, D) = \{x \in \mathbb{R}^n : d(x_0, x) < D\} \subset \Omega$. Now, we fix $s \in]1, +\infty[$ and put $s_D = \frac{1}{s}$ and $\kappa_D = \frac{s}{(s-1)D}$. Clearly $(1 - s_D)D\kappa_D = 1$. Then, for $\beta > 0$ and $h \in C(\overline{\Omega})$ with $1 < h^-$, we put

$$[\beta]^h := \max\left\{\beta^{h^-}, \beta^{h^+}\right\}.$$

The hypothesis on the function $G : \Omega \times \mathbb{R} \to \mathbb{R}$ is as follows:

(g_2) $\inf_{x\in\Omega} G(x,t) \geq 0$ for all $t \in [0,1]$ and $\limsup_{t\to 0^+} \frac{\inf_{x\in\Omega} G(x,t)}{t^{p^-}} = +\infty$.

Let $\lambda^* := \left(a_1 k_1 (p^+)^{1/p^-} + \frac{a_2}{\alpha^-}[k_\alpha]^\alpha (p^+)^{\alpha^+/p^-}\right)^{-1}$, where k_1 and k_α are the best constants for the compact embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$, respectively. We establish the following result.

Theorem 3.2. *If hypotheses* (g_1), (g_2) *hold, then problem* (P_λ) *admits at least one nontrivial weak solution, for each* $\lambda \in]0, \lambda^*[$.

Proof. We consider the functionals Φ and Ψ given in Section 2 on the Banach space $W_0^{1,p(x)}(\Omega)$, and prove that all the hypotheses of Theorem 3.1 hold true with $r = 1$. Since $\Phi, \Psi \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and Ψ' is compact, the functional I_λ satisfies the $(P.S.)^{[r]}$-condition for all $r > 0$ (see, Afrouzi–Kirane–Shokooh [1, Theorem 3.1]). We deduce that Theorem 3.1 (ii) holds true. Then, fixed $\lambda \in]0, \lambda^*[$, by (g_2) we get

$$0 < \delta_\lambda < \min\left\{1, \left(\frac{p^-}{mD^n(2[\kappa_D]^p(1 - s_D^n) + 1)}\right)^{1/p^-}\right\}$$

so that

$$\frac{p^- s_D^n \inf_{x\in\Omega} G(x, \delta_\lambda)}{(2[\kappa_D]^p(1 - s_D^n) + 1)(\delta_\lambda)^{p^-}} > \frac{1}{\lambda}.$$

Now, we consider the function $u_\lambda : \Omega \to \mathbb{R}$ given as

$$u_\lambda(x) = \begin{cases} 0, & x \in \Omega \setminus B(x_0, D), \\ \delta_\lambda, & x \in B(x_0, s_D D), \\ \delta_\lambda \kappa_D (D - |x - x_0|), & x \in B(x_0, D) \setminus B(x_0, s_D D), \end{cases}$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. We obtain

$$\begin{aligned} p^-\Phi(u_\lambda) &\leq \int_\Omega |\nabla u_\lambda(x)|^{p(x)} dx + \int_\Omega \left[\sqrt{1 + |\nabla u_\lambda(x)|^{2p(x)}} - 1\right] dx + \int_{B(x_0,D)} |u_\lambda(x)|^{p(x)} dx \\ &\leq \int_\Omega 2|\nabla u_\lambda(x)|^{p(x)} dx + \int_{B(x_0,D)} (\delta_\lambda)^{p(x)} dx \\ &\leq mD^n \left(2[\kappa_D]^p(1 - s_D^n) + 1\right)(\delta_\lambda)^{p^-} \\ \Rightarrow\quad & \Phi(u_\lambda) < 1, \end{aligned}$$

where $m := \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ denotes the measure of unit ball of $\mathbb{R}^n$ and Γ is the Gamma function. We get

$$\begin{aligned} &\Psi(u_\lambda) \geq \int_{B(x_0,s_D D)} G(x,u_\lambda)dx \geq \inf_{x\in\Omega} G(x,\delta_\lambda) m s_D^n D^n \quad \text{(by left part of } (g_2)) \\ \Rightarrow \quad &\frac{\Psi(u_\lambda)}{\Phi(u_\lambda)} \geq \frac{p^- m s_D^n D^n \inf_{x\in\Omega} G(x,\delta_\lambda)}{mD^n \left(2[\kappa_D]^p(1-s_D^n)+1\right)(\delta_\lambda)^{p^-}} = \frac{p^- s_D^n \inf_{x\in\Omega} G(x,\delta_\lambda)}{\left(2[\kappa_D]^p(1-s_D^n)+1\right)(\delta_\lambda)^{p^-}} > \frac{1}{\lambda}. \end{aligned}$$

Let $r = 1$. For each $u \in \Phi^{-1}(]-\infty,1])$, we can use Theorem 2.2 and conclude that

$$\begin{aligned} &\|u\| \leq \left[\int_\Omega \left(|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)}\right) dx\right]^{1/p} \leq \left[p^+\Phi(u)\right]^{1/p} \leq (p^+)^{1/p^-}, \\ \Rightarrow \quad &\|u\| \leq (p^+)^{1/p^-}. \end{aligned} \tag{3.1}$$

Next, Proposition 2.1 and Theorem 2.2 imply that

$$\int_\Omega |u(x)|^{\alpha(x)}dx = \rho_\alpha(u) \leq \left[\|u\|_{L^{\alpha(x)}(\Omega)}\right]^\alpha \leq [k_\alpha\|u\|]^\alpha \tag{3.2}$$

for all $u \in W_0^{1,p(x)}(\Omega)$, where k_α is the best constant for the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$. Moreover, the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$ (with best constant k_1), (g_1), (3.1) and (3.2) imply that, for each $u \in \Phi^{-1}(]-\infty,1])$, we have

$$\begin{aligned} \Psi(u) &\leq a_1 \int_\Omega |u(x)|dx + \frac{a_2}{\alpha^-}\int_\Omega |u(x)|^{\alpha(x)}dx \leq a_1k_1\|u\| + \frac{a_2}{\alpha^-}[k_\alpha]^\alpha[\|u\|]^\alpha \\ &\leq a_1k_1(p^+)^{1/p^-} + \frac{a_2}{\alpha^-}[k_\alpha]^\alpha(p^+)^{\alpha^+/p^-} \\ \Rightarrow \quad &\sup_{\Phi(u)\leq 1} \Psi(u) \leq a_1k_1(p^+)^{1/p^-} + \frac{a_2}{\alpha^-}[k_\alpha]^\alpha(p^+)^{\alpha^+/p^-} = \frac{1}{\lambda^*} < \frac{1}{\lambda} \\ \Rightarrow \quad &\sup_{\Phi(u)\leq 1} \Psi(u) < \frac{1}{\lambda} < \frac{\Psi(u_\lambda)}{\Phi(u_\lambda)}. \end{aligned}$$

It follows that Theorem 3.1 (i) holds true. Since $\lambda \in \left]\frac{\Phi(u_\lambda)}{\Psi(u_\lambda)}, \frac{1}{\sup_{\Phi(u)\leq r}\Psi(u)}\right[$, by an application of Theorem 3.1 with $\overline{u} = u_\lambda$ and $r = 1$, we obtain the existence of a local minimum point v_λ of the functional I_λ such that $0 < \Phi(v_\lambda) < 1$. This means that v_λ is a nontrivial weak solution of problem (P_λ). □

4 Existence of three weak solutions

In this section we prove a theorem producing at least three weak solutions of (P_λ). To this aim, we apply a theorem proved by Bonanno–Marano [4, Theorem 3.6], which run as follows.

Theorem 4.1. *Let X be a reflexive real Banach space and let $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\inf_X \Phi(x) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{u} \in X$, with $0 < r < \Phi(\bar{u})$, such that*

(*i*) $\sigma = \frac{1}{r}\sup_{\phi(u)\leq r}\Psi(u) < \frac{\Psi(\bar{u})}{\Phi(\bar{u})} = \rho$;

(ii) *for each* $\lambda \in \left]\frac{1}{\rho}, \frac{1}{\sigma}\right[$ *the functional* $I_\lambda := \Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r := \left]\frac{1}{\rho}, \frac{1}{\sigma}\right[$, *the functional* $I_\lambda := \Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

The hypotheses on the function $G : \Omega \times \mathbb{R} \to \mathbb{R}$ are as follows:

(g_3) there exist $c \in [0, +\infty[$ and $\gamma \in C(\overline{\Omega})$ with $1 < \gamma^- \leq \gamma^+ < p^-$ such that

$$G(x,t) \leq c\left(1 + |t|^{\gamma(x)}\right) \qquad \text{for all } (x,t) \in \Omega \times \mathbb{R};$$

(g_4) $G(x,t) \geq 0$ for all $(x,t) \in \Omega \times [0, +\infty[$;

(g_5) there exist $r > 0$ and $\delta > 0$ with $r < \frac{1}{p^+} m D^n \left[\min\left\{\kappa_D^{p^-}, \kappa_D^{p^+}\right\}(1 - s_D^n) + 1\right]\delta^{p^+}$ such that

$$\begin{aligned}\overline{\omega} :=& \frac{1}{r}\left(a_1 k_1 (p^+)^{1/p^-}[r]^{1/p} + \frac{a_2}{\alpha^-}[k_\alpha]^\alpha (p^+)^{\alpha^+/p^-}[[r]^{1/p}]^\alpha\right) \\ &< \frac{p^- s_D^n \inf_{x\in\Omega} G(x,\delta)}{(2[\kappa_D]^p(1 - s_D^n) + 1)\,\delta^{p^-}}.\end{aligned}$$

So, we establish the following result.

Theorem 4.2. *If hypotheses* (g_1), (g_3), (g_4), (g_5) *hold, then problem* (P_λ) *admits at least three weak solutions, for each* $\lambda \in \Lambda_{r,\delta} := \left]\frac{(2[\kappa_D]^p(1-s_D^n)+1)\delta^{p^-}}{p^- s_D^n \inf_{x\in\Omega} G(x,\delta)}, \frac{1}{\overline{\omega}}\right[$.

Proof. We adapt the proof of Theorem 3.2 to the new situation. So, we consider the same working space $W_0^{1,p(x)}(\Omega)$ with the norm $\|\cdot\|$ and the functionals $\Phi, \Psi : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$. This means that the regularity assumptions of Theorem 4.1 hold true.

Again, let s_D and κ_D as in Section 3. Let r and δ as in (g_5) and consider the function $w : \Omega \to \mathbb{R}$ given as

$$w(x) = \begin{cases} 0, & x \in \Omega \setminus B(x_0, D), \\ \delta, & x \in B(x_0, s_D D), \\ \delta\kappa_D(D - |x - x_0|), & x \in B(x_0, D) \setminus B(x_0, s_D D). \end{cases}$$

Following the same arguments in the proof of Theorem 3.2 (by taking in mind (g_4)), we obtain

$$\frac{\Psi(w)}{\Phi(w)} \geq \frac{p^- s_D^n \inf_{x\in\Omega} G(x,\delta)}{(2[\kappa_D]^p(1 - s_D^n) + 1)\delta^{p^-}}.$$

On the other hand, it turns out that

$$\Phi(w) \geq \frac{1}{p^+}\int_\Omega \left(|\nabla w(x)|^{p(x)} + |w(x)|^{p(x)}\right) dx \geq \frac{1}{p^+} m D^n \left[\min\left\{\kappa_D^{p^-}, \kappa_D^{p^+}\right\}(1 - s_D^n) + 1\right]\delta^{p^+}.$$

From $r < \frac{1}{p^+} m D^n \left[\min\left\{\kappa_D^{p^-}, \kappa_D^{p^+}\right\}(1 - s_D^n) + 1\right]\delta^{p^+}$, we deduce $r < \Phi(w)$. Thus, Proposition 2.1 and Theorem 2.2 imply that

$$\int_\Omega |u(x)|^{\alpha(x)} dx = \rho_\alpha(u) \leq \left[\|u\|_{L^{\alpha(x)}(\Omega)}\right]^\alpha \leq [k_\alpha\|u\|]^\alpha \tag{4.1}$$

for all $u \in W_0^{1,p(x)}(\Omega)$, where k_α is the best constant for the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$. For each $u \in \Phi^{-1}(]-\infty, r])$, by Theorem 2.2 we have

$$\begin{aligned} & \|u\| \leq \left[p^+\Phi(u)\right]^{1/p} \leq [p^+ r]^{1/p} = (p^+)^{1/p^-}[r]^{1/p}, \\ \Rightarrow \quad & \|u\| \leq (p^+)^{1/p^-}[r]^{1/p}. \end{aligned} \tag{4.2}$$

Moreover, the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^1(\Omega)$ (with best constant k_1), (g_1), (4.1) and (4.2) imply that, for each $u \in \Phi^{-1}(]-\infty, r])$, we have

$$\begin{aligned} \Psi(u) &\leq a_1 \int_\Omega |u(x)| dx + \frac{a_2}{\alpha^-} \int_\Omega |u(x)|^{\alpha(x)} dx \leq a_1 k_1 \|u\| + \frac{a_2}{\alpha^-}[k_\alpha]^\alpha [\|u\|]^\alpha \\ &\leq a_1 k_1 (p^+)^{1/p^-}[r]^{1/p} + \frac{a_2}{\alpha^-}[k_\alpha]^\alpha (p^+)^{\alpha^+/p^-}[[r]^{1/p}]^\alpha \\ \Rightarrow \quad & \frac{1}{r} \sup_{\Phi(u) \leq r} \Psi(u) \leq \frac{1}{r}\left(a_1 k_1 (p^+)^{1/p^-}[r]^{1/p} + \frac{a_2}{\alpha^-}[k_\alpha]^\alpha (p^+)^{\alpha^+/p^-}[[r]^{1/p}]^\alpha\right) \\ \Rightarrow \quad & \frac{1}{r} \sup_{\Phi(u) \leq r} \Psi(u) < \frac{\Psi(w)}{\Phi(w)}. \end{aligned}$$

It follows that Theorem 4.1 (i) holds true. Finally, we prove that Theorem 4.1 (ii) holds true too (i.e., $I_\lambda := \Phi - \lambda\Psi$ is coercive for each $\lambda > 0$). In fact, Proposition 2.1 and Theorem 2.2 imply that

$$\int_\Omega |u(x)|^{\gamma(x)} dx = \rho_\gamma(u) \leq \left[\|u\|_{L^{\gamma(x)}(\Omega)}\right]^\gamma \leq [k_\gamma \|u\|]^\gamma \tag{4.3}$$

for all $u \in W_0^{1,p(x)}(\Omega)$, where k_γ is the best constant for the compact embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\gamma(x)}(\Omega)$. Consequently, for each $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| \geq \max\{1, k_\gamma^{-1}\}$, using (g_3) and (4.3), we get

$$\begin{aligned} \Psi(u) &= \int_\Omega G(x, u(x)) dx \leq \int_\Omega c\left(1 + |u(x)|^{\gamma(x)}\right) dx \\ &\leq c\left(|\Omega| + [k_\gamma \|u\|]^\gamma\right) = c\left(|\Omega| + [k_\gamma]^\gamma \|u\|^{\gamma^+}\right). \end{aligned}$$

It follows that

$$\begin{aligned} I_\lambda(u) &\geq \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} dx + \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} dx - \lambda c\left(|\Omega| + [k_\gamma]^\gamma \|u\|^{\gamma^+}\right) \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - \lambda c\left(|\Omega| + [k_\gamma]^\gamma \|u\|^{\gamma^+}\right) \\ \Rightarrow \quad & I_\lambda \text{ is coercive.} \end{aligned}$$

Since $\Lambda_{r,\delta} \subset \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u) \leq r} \Psi(u)}\right[$, by an application of Theorem 4.1 with $\overline{u} = w$, we have that, for each $\lambda \in \Lambda_{r,\delta}$, I_λ admits at least three critical points in $W_0^{1,p(x)}(\Omega)$. Obviously, these critical points are three weak solutions of (P_λ). □

We conclude this article by dealing with a reaction term satisfying the hypotheses (g_1), (g_3), (g_4). Based on the sub-critical growth condition (g_1), we take the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$g(x,z) = \begin{cases} 1 + |z|^{q(x)-1}, & \text{if } (x,z) \in \Omega \times]-\infty, r], \\ 1 + z^{\gamma(x)-1} r^{q(x)-\gamma(x)}, & \text{if } (x,z) \in \Omega \times]r, +\infty[, \end{cases}$$

where r is a real positive number greater than 1, and $q, \gamma \in C(\overline{\Omega})$ with $1 < \gamma^- \leq \gamma^+ < \min\{q^-, p^-\} < p^*(x)$ for all $x \in \Omega$. Trivially, g is a Carathéodory function satisfying (g_1). Next, consider the function $G : \Omega \times \mathbb{R} \to \mathbb{R}$ given as $G(x,t) = \int_0^t g(x,z)dz$ for all $t \in \mathbb{R}$ and $x \in \Omega$, so (g_4) holds true as $g(x,z) \geq 1$ for all $(x,z) \in \Omega \times \mathbb{R}$. From

$$G(x,t) = \begin{cases} t + \frac{t^{q(x)}}{q(x)}, & \text{if } (x,t) \in \Omega \times [0,r], \\ t + \frac{t^{\gamma(x)}}{\gamma(x)} r^{q(x)-\gamma(x)} + r^{q(x)} \left(\frac{1}{q(x)} - \frac{1}{\gamma(x)} \right), & \text{if } (x,t) \in \Omega \times]r, +\infty[, \end{cases}$$

by routine calculations, we get $G(x,t) \leq (r + r^{q^+}/\gamma^-)(1 + t^{\gamma(x)})$ for all $(x,t) \in \Omega \times [0, +\infty[$ and so (g_3) holds true (indeed, $G(x,t) \leq 0$ for all $(x,z) \in \Omega \times]-\infty, 0]$).

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