Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **21**, 1–10; https://doi.org/10.14232/ejqtde.2018.1.21 www.math.u-szeged.hu/ejqtde/

Oscillation of a perturbed nonlinear third order functional differential equation

Said R. Grace[1] and **John R. Graef**[✉2]

[1]Cairo University, Department of Engineering Mathematics, Faculty of Engineering, Orman, Giza 12221, Egypt
[2]Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA

Received 7 September 2017, appeared 1 May 2018
Communicated by Ferenc Hartung

Abstract. In this paper, the authors present some new results on the oscillatory and asymptotic behavior of solutions of the perturbed nonlinear third order functional differential equation

$$\Big(b(t)\left(a(t)(x'(t))^{\alpha}\right)'\Big)' + p(t)f(x(\tau(t))) = h(t, x(t), x(\tau(t)), x'(t)).$$

In addition to other conditions, the authors assume that $uf(u) > 0$ for $u \neq 0$ and f is increasing. Examples to illustrate the results are included.

Keywords: third order differential equations, perturbed equations, delay argument, oscillation, asymptotic behavior.

2010 Mathematics Subject Classification: 34C10, 34C11, 34K11.

1 Introduction

We consider the third order nonlinear functional differential equation with a perturbation term

$$\Big(b(t)\left(a(t)(x'(t))^{\alpha}\right)'\Big)' + p(t)f(x(\tau(t))) = h(t, x(t), x(\tau(t)), x'(t)), \tag{1.1}$$

where $\alpha \geq 1$ is the ratio of odd positive integers, and we assume:

(H1) a, b, p, $\tau \in C\left([t_0, \infty)\right)$ are positive;

(H2) $f : \mathbb{R} \to \mathbb{R}$ and $h : [t_0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, $uf(u) > 0$ for $u \neq 0$, and f is nondecreasing;

(H3) $f(uv) \geq f(u)f(v)$ for $uv > 0$;

(H4) $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

✉Corresponding author. Email: John-Graef@utc.edu

By a *solution* of (1.1) we mean a function $x(t)$ whose quasi-derivatives $a(t)(x'(t))^\alpha$ and $(a(t)(x'(t))^\alpha)'$ are continuous on $[T_x, \infty)$, $T_x \geq t_0$, and which satisfies Eq. (1.1) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (1.1) that satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros, and *nonoscillatory* otherwise. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In two very nice papers Baculíková and Džurina studied the oscillatory and asymptotic behavior of solutions of some third order nonlinear delay differential equations. In [1], they considered the equation

$$(b(t)(x''(t))^\alpha)' + p(t)f(x(\tau(t))) = 0$$

under the same covering assumptions as those above and assumed that

$$\int_{t_0}^{\infty} \frac{1}{b(s)^{\frac{1}{\alpha}}} ds = \infty.$$

In [2], they considered the equation

$$\Big(b(t)\,\big(a(t)(x'(t))\big)'\Big)' + p(t)f(x(\tau(t))) - q(t)h(x(\sigma(t))) = 0$$

under conditions (H1)–(H4),

$$\int_{t_0}^{\infty} \frac{1}{b(s)} ds = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{a(s)} ds = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \int_{t}^{\infty} \frac{1}{b(s)} \int_{s}^{\infty} q(u)duds < \infty. \tag{1.3}$$

They employed a new technique to obtain some interesting results on the oscillatory and asymptotic behavior of solutions (see [2, Theorem 2.1]). They obtained another oscillation result (see [2, Theorem 2.6]) by replacing condition (1.3) with

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \int_{t_0}^{t} \frac{1}{b(s)} \int_{t_0}^{s} q(u)dudsdt < \infty. \tag{1.4}$$

Notice that condition (1.3) implies that q is small in that we must have

$$\int_{t_0}^{\infty} q(u)du < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{b(s)} \int_{s}^{\infty} q(u)duds < \infty.$$

Condition (1.4) requires q to be small is some sense relative to b and a.

Our goal here is to establish oscillation results for equation (1.1) without imposing a "smallness" condition on the perturbation term. We also present some results on the boundedness and oscillatory behavior of a special case of (1.1), namely,

$$\Big(b(t)\,\big(a(t)(x'(t))^\alpha\big)'\Big)' + p(t)x^\beta(t) = e(t) + q(t)x^\gamma(t), \tag{1.5}$$

where β and γ are the ratios of odd positive integers with $\beta > \gamma$ and $e : [t_0, \infty) \to \mathbb{R}$ is a continuous function. As was done in [1,2], we will use a comparison approach.

2 Oscillation of equation (1.1)

We assume that there exists a positive continuous function $q : [t_0, \infty) \to \mathbb{R}^+$ such that

(H5) $|h(t,u,v,w)| \le q(t)f(v)$ for all $(t,u,v,w) \in [t_0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

For any $t_1 \ge t_0$, we set

$$I(t;t_1) = \int_{t_1}^{\tau(t)} \left(\frac{1}{a(u)} \int_{t_1}^{u} \frac{1}{b(s)} ds \right)^{1/\alpha} du. \tag{2.1}$$

We also assume that there are functions $\xi, \eta \in C^1[t_0,\infty)$ satisfying

$$\tau(t) \le \xi(t) \le \eta(t) \le t \quad \text{for all large } t, \tag{2.2}$$

and set

$$I^*(t) = \left(\int_{\tau(t)}^{\xi(t)} \frac{1}{a^{1/\alpha}(s)} ds \right) \left(\int_{\xi(t)}^{\eta(t)} \frac{1}{b(s)} ds \right)^{1/\alpha}, \tag{2.3}$$

and

$$Q(t) = p(t) - q(t). \tag{2.4}$$

In some of our results we will also ask that

$$\int_{t_0}^{\infty} \frac{1}{b(s)} ds = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(s)} ds = \infty. \tag{2.5}$$

Our first oscillation result is contained in the following theorem.

Theorem 2.1. *Let $Q(t) > 0$ for large t, conditions (H1)–(H5), (2.2), and (2.5) hold, and assume that all solutions of the first order delay differential equations*

$$y'(t) + Q(t) f(I(t)) f(y^{1/\alpha}(\tau(t))) = 0 \tag{2.6}$$

and

$$z'(t) + Q(t) f(I^*(t)) f(z^{1/\alpha}(\eta(t))) = 0 \tag{2.7}$$

are oscillatory. Then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $x(t)$ and $x(\tau(t))$ are positive and condition (2.2) holds for $t \ge t_1$ for some $t_1 \ge t_0$. If $x(t)$ is eventually negative, a similar proof holds. From our assumptions and equation (1.1), we see that

$$\left(b(t) \left(a(t)(x'(t))^\alpha \right)' \right)' \le -Q(t) f(x(\tau(t))) < 0, \tag{2.8}$$

for all $t \ge t_1$.

It is easy to see that we need to consider the following two cases:

(I) $a(t)(x'(t))^\alpha > 0$ and $b(t) \left(a(t)(x'(t))^\alpha \right)' > 0$, or

(II) $a(t)(x'(t))^\alpha < 0$ and $b(t) \left(a(t)(x'(t))^\alpha \right)' > 0$

for $t \geq t_2$ for some $t_2 \geq t_1$. We will first examine Case (I). For $t \geq t_2$, we see that

$$a(t)(x'(t))^\alpha \geq \int_{t_2}^{t} \frac{1}{b(s)} b(s) \left(a(s)(x'(s))^\alpha\right)' ds \geq \left(\int_{t_2}^{t} \frac{1}{b(s)} ds\right) b(t) \left(a(t)(x'(t))^\alpha\right)',$$

or

$$x'(t) \geq \left(\frac{1}{a(t)} \int_{t_2}^{t} \frac{1}{b(s)} ds\right)^{1/\alpha} \left(b(t) \left(a(t)(x'(t))^\alpha\right)'\right)^{1/\alpha}.$$

Integrating this inequality from t_2 to $\tau(t) \geq t_2$, we have

$$\begin{aligned} x(\tau(t)) &\geq \left(\int_{t_2}^{\tau(t)} \left(\frac{1}{a(u)} \int_{t_2}^{u} \frac{1}{b(s)} ds\right)^{1/\alpha} du\right) \left(b(t) \left(a(t)(x'(t))^\alpha\right)'\right)^{1/\alpha} \\ &= I(t; t_2) y^{1/\alpha}(t), \end{aligned} \tag{2.9}$$

where $y(t) = b(t) \left(a(t)(x'(t))^\alpha\right)'$. Using (2.9) in (2.8) and applying (H3), we obtain

$$y'(t) + Q(t) f(I(t)) f(y^{1/\alpha}(\tau(t))) \leq 0 \quad \text{for } t \geq t_2.$$

It follows from [11, Corollary 1] that the corresponding differential equation (2.6) also has a positive solution. This contradiction completes the proof for Case (I).

For Case (II), it is easy to see that

$$\begin{aligned} -a(\xi(t))(x'(\xi(t)))^\alpha &\geq \left(\int_{\xi(t)}^{\eta(t)} \frac{1}{b(s)} ds\right) b(\eta(t)) \left(a(\eta(t))(x'(\eta(t)))^\alpha\right)' \\ &= \left(\int_{\xi(t)}^{\eta(t)} \frac{1}{b(s)} ds\right) z(\eta(t)), \end{aligned} \tag{2.10}$$

for $t \geq t_2$, where $z(t) = b(t) \left(a(t)(x'(t))^\alpha\right)'$.

Now $x' = \left[\frac{a(x')^\alpha}{a}\right]^{1/\alpha} = \frac{1}{a^{1/\alpha}} \left[a(x')^\alpha\right]^{1/\alpha}$ so integrating for $v \geq u \geq t_2$, we have

$$x(v) - x(u) = \int_u^v \frac{1}{a^{1/\alpha}(s)} \left[a(s)(x'(s))^\alpha\right]^{1/\alpha} ds,$$

or

$$x(u) - x(v) = \int_u^v \frac{1}{a^{1/\alpha}(s)} \left[-a(s)(x'(s))^\alpha\right]^{1/\alpha} ds \geq \int_u^v \frac{1}{a^{1/\alpha}(s)} ds \left[-a(v)(x'(v))^\alpha\right]^{1/\alpha} ds.$$

Hence,

$$x(u) \geq \left(\int_u^v \frac{1}{a^{1/\alpha}(s)} ds\right) \left(-\left(a(v)(x'(v))^\alpha\right)\right)^{1/\alpha}.$$

Setting $u = \tau(t)$ and $v = \xi(t)$ in the above inequality, we obtain

$$x(\tau(t)) \geq \left(\int_{\tau(t)}^{\xi(t)} \frac{1}{a^{1/\alpha}(s)} ds\right) \left(-\left(a(\xi(t))(x'(\xi(t)))^\alpha\right)\right)^{1/\alpha}. \tag{2.11}$$

From (2.10) and (2.11) we see that

$$x(\tau(t)) \geq \left(\int_{\tau(t)}^{\xi(t)} \frac{1}{a^{1/\alpha}(s)} ds\right) \left(\int_{\xi(t)}^{\eta(t)} \frac{1}{b(s)} ds\right)^{1/\alpha} z^{1/\alpha}(\eta(t)) =: I^*(t) z^{1/\alpha}(\eta(t)). \tag{2.12}$$

Using (2.12) in equation (2.8), we have

$$z'(t) + Q(t) f(I^*(t)) f(z^{1/\alpha}(\eta(t))) \leq 0.$$

It folows from [11, Corollary 1] that the corresponding differential equation (2.7) also has a positive solution, which is a contradiction. This completes the proof of the theorem. □

The next two corollaries follow immediately from known oscillation criteria for first order delay differential equations; fo example, see [9, Theorem 2].

Corollary 2.2. *Let $f(x) = x^\alpha$, $Q(t) > 0$ for all large t, and conditions (H1), (H4), (2.2), and (2.5) hold. If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} Q(s) I^\alpha(s) ds > \frac{1}{e} \quad \textit{and} \quad \liminf_{t\to\infty} \int_{\eta(t)}^{t} Q(s) \left(I^*(s)\right)^\alpha ds > \frac{1}{e}, \tag{2.13}$$

then equation (1.1) is oscillatory.

Corollary 2.3. *Let $f(x) = x^\beta$, $\beta/\alpha \in (0,1)$, $Q(t) > 0$ for all large t, and conditions (H1), (H4), (2.2), and (2.5) hold. If*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} Q(s) I^\beta(s) ds > 0 \quad \textit{and} \quad \limsup_{t\to\infty} \int_{\eta(t)}^{t} Q(s) \left(I^*(s)\right)^\beta ds > 0, \tag{2.14}$$

then equation (1.1) is oscillatory.

The following example illustrates the above results.

Example 2.4. Consider the equation

$$\left(t\left(t^3(x'(t))^3\right)'\right)' + p(t)x^3(\lambda_1 t) = q(t)\frac{x^3(\lambda_1 t)x'(t)}{(1+x^2(\lambda_1 t))(1+|x'(t)|)}, \qquad t \geq 1. \tag{2.15}$$

Here $a(t) = t^3$, $b(t) = t$, $\tau(t) = \lambda_1 t$, $\xi(t) = \lambda_2 t$ and $\eta(t) = \lambda_3 t$, where $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq 1$, $h(t, x(t), x(\tau(t)), x'(t)) = q(t)\frac{x^3(\tau(t))x'(t)}{(1+x^2(\tau(t)))(1+|x'(t)|)}$, $f(x) = x^3$, and $\alpha = 3$. Let $p(t)$ and $q(t)$ be positive continuous functions with $Q(t) = p(t) - q(t)$ positive for all large t. Now,

$$I(t;1) = \int_1^{\lambda_1 t} \left(\frac{1}{u^3}\int_1^u \frac{1}{s} ds\right)^{1/3} du = \frac{3}{4}(\ln \lambda_1 t)^{4/3}$$

and

$$I^*(t) = \left(\int_{\lambda_1 t}^{\lambda_2 t} \frac{1}{s} ds\right)\left(\int_{\lambda_2 t}^{\lambda_3 t} \frac{1}{s} ds\right)^{1/3} = \left(\ln\frac{\lambda_2}{\lambda_1}\right)\left(\ln\frac{\lambda_3}{\lambda_2}\right)^{1/3}.$$

If both of the equations

$$y'(t) + Q(t)I^3(t)y(\lambda_1 t) = 0$$

and

$$z'(t) + Q(t)\left(I^*(t)\right)^3 z(\lambda_3 t) = 0$$

are oscillatory, then equation (2.15) is oscillatory.

Instead of condition (2.2), we assume that there exists a function $\rho(t) \in C^1([t_0, \infty))$ satisfying

$$\rho'(t) > 0, \quad \rho(t) > t, \quad \omega(t) = \rho(\rho(\tau(t))) < t, \tag{2.16}$$

and we set

$$I^{**}(t) = \int_{\tau(t)}^{\rho(\tau(t))} \left(\frac{1}{a(u)}\int_u^{\rho(u)} \frac{1}{b(s)} ds\right)^{1/\alpha} du. \tag{2.17}$$

We can then obtain the following theorem.

Theorem 2.5. *Let $Q(t) > 0$ for large t, conditions (H1)–(H5), (2.5), and (2.16) hold. If all solutions of equations (2.6) and*

$$z'(t) + Q(t)f(I^{**}(t))f(z^{1/\alpha}(\omega(t))) = 0 \tag{2.18}$$

are oscillatory, then equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1) such that $x(t)$ and $x(\tau(t))$ are positive and condition (2.16) holds for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Theorem 2.1 we again obtain (2.8). The proof for Case (I) holding is similar to that of Theorem 2.1 and hence is omitted.

If Case (II) holds, it is easy to see that

$$\begin{aligned} -a(t)(x'(t))^{\alpha} &\geq \int_t^{\rho(t)} \frac{1}{b(s)} \left(b(s) \left(a(s)(x'(s))^{\alpha} \right)' \right) ds \\ &\geq \left(\int_t^{\rho(t)} \frac{1}{b(s)} ds \right) \left(b(\rho(t)) \left(a(\rho(t))(x'(\rho(t)))^{\alpha} \right)' \right) \\ &= \left(\int_t^{\rho(t)} \frac{1}{b(s)} ds \right) z(\rho(t)), \end{aligned} \tag{2.19}$$

where $z(t) = b(t)\left(a(t)(x'(t))^{\alpha}\right)'$. Dividing by $a(t)$ and integrating from $\tau(t)$ to $\rho(\tau(t))$, we obtain

$$\begin{aligned} x(\tau(t)) &\geq \int_{\tau(t)}^{\rho(\tau(t))} \left(\frac{z(\rho(u))}{a(u)} \right)^{1/\alpha} \left(\int_u^{\rho(u)} \frac{1}{b(s)} ds \right)^{1/\alpha} du \\ &\geq I^{**}(t) z^{1/\alpha}(\omega(t))), \end{aligned} \tag{2.20}$$

for all large t. Using (2.20) in (2.8) and proceeding as in the proof of Case (II) in Theorem 2.1, we arrive at the desired contradiction. This completes the proof of the theorem. $\square$

To illustrate this result we have the following example.

Example 2.6. Consider Example 2.4 with $\rho(t) = \theta t$ and $\theta > 1$. Now $\omega(t) = \theta^2 \lambda_1 t$ and

$$I^{**}(t) = \int_{\lambda_1 t}^{\theta \lambda_1 t} \left(\frac{1}{u^3} \int_u^{\theta u} \frac{1}{s} ds \right)^{1/3} du = \int_{\lambda_1 t}^{\theta \lambda_1 t} \frac{1}{u} \left(\ln \frac{\theta u}{u} \right)^{1/3} du = (\ln \theta)^{4/3}.$$

If $\theta^2 \lambda_1 t \leq 1$ and the equations

$$y'(t) + Q(t) I^3(t) y(\lambda_1 t) = 0$$

and

$$z'(t) + Q(t) \left(\ln \theta \right)^4 z(\omega(t)) = 0$$

are oscillatory, then equation (2.15) is oscillatory by Theorem 2.5.

3 Boundedness and oscillation of equation (1.5)

In order to obtain our results in this section, we need the following lemma.

Lemma 3.1 (Young's inequality)**.** *Let X and Y be nonnegative, $n > 1$, and $1/n + 1/m = 1$. Then*

$$XY \le \frac{1}{n}X^n + \frac{1}{m}Y^m, \tag{3.1}$$

and equality holds if and only if $Y = X^{n-1}$.

Theorem 3.2. *In addition to condition (H1), assume that*

$$\int_{t_0}^{\infty} \frac{1}{b(s)}ds < \infty, \int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(s)}ds < \infty, \int_{t_0}^{\infty} \frac{1}{b(u)} \int_{t_0}^{u} q^{\frac{\beta}{\beta-\gamma}}(s)p^{\frac{\gamma}{\gamma-\beta}}(s)dsdu < \infty, \tag{3.2}$$

and

$$\int_{t_0}^{\infty} \frac{1}{b(u)} \int_{t_0}^{u} |e(s)|\, dsdu < \infty. \tag{3.3}$$

Then every nonoscillatory solution of equation (1.5) is bounded.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.5) such that $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$. Applying (3.1) to $[q(s)x^{\gamma}(s) - p(s)x^{\beta}(s)]$ with

$$n = \frac{\beta}{\gamma} > 1, \qquad X(s) = x^{\gamma}(s), \qquad Y = \frac{\gamma}{\beta}\left(\frac{q(s)}{p(s)}\right), \quad \text{and} \quad m = \frac{\beta}{\beta-\gamma},$$

we obtain

$$\begin{aligned} q(s)x^{\gamma}(s) - p(s)x^{\beta}(s) &= \frac{\beta}{\gamma}p(s)\left[x^{\gamma}(s)\frac{\gamma}{\beta}\frac{q(s)}{p(s)} - \frac{\gamma}{\beta}\left(x^{\gamma}(s)\right)^{\beta/\gamma}\right] \\ &= \frac{\beta}{\gamma}p(s)\left[XY - \frac{1}{n}X^n\right] \\ &\le \frac{\beta}{\gamma}p(s)\left(\frac{1}{m}Y^m\right) \\ &= \left(\frac{\beta-\gamma}{\gamma}\right)\left[\frac{\gamma}{\beta}q(s)\right]^{\frac{\beta}{\beta-\gamma}} (p(s))^{\frac{\gamma}{\gamma-\beta}}. \end{aligned} \tag{3.4}$$

From equation (1.5) we then have

$$\left(b(t)\left(a(t)(x'(t))^{\alpha}\right)'\right)' \le |e(t)| + \left(\frac{\beta-\gamma}{\gamma}\right)\left[\frac{\gamma}{\beta}q(t)\right]^{\frac{\beta}{\beta-\gamma}} (p(t))^{\frac{\gamma}{\gamma-\beta}} = |e(t)| + cq^{\frac{\beta}{\beta-\gamma}}(t)p^{\frac{\gamma}{\gamma-\beta}}(t),$$

where $c = \left(\frac{\beta-\gamma}{\gamma}\right)\left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta-\gamma}}$. Integrating this inequality from t_1 to t gives

$$\left(a(t)(x'(t))^{\alpha}\right)' \le \frac{c_1}{b(t)} + \frac{1}{b(t)}\int_{t_1}^{t} |e(s)|\, ds + \frac{c}{b(t)}\int_{t_1}^{t} q^{\frac{\beta}{\beta-\gamma}}(s)p^{\frac{\gamma}{\gamma-\beta}}(s)ds,$$

where $c_1 = b(t_1)\left(a(t_1)(x'(t_1))^{\alpha}\right)'$. Another integration yields

$$\begin{aligned} a(t)(x'(t))^{\alpha} \le c_2 + c_1\int_{t_1}^{t} \frac{1}{b(s)}ds + c\int_{t_1}^{t} \frac{1}{b(u)}\int_{t_1}^{u} q^{\frac{\beta}{\beta-\gamma}}(s)p^{\frac{\gamma}{\gamma-\beta}}(s)dsdu \\ + \int_{t_1}^{t} \frac{1}{b(u)}\int_{t_1}^{u} |e(s)|\, dsdu, \end{aligned}$$

where $c_2 = a(t_1)(x'(t_1))^\alpha$. From condition (3.2) and (3.3), there exists a constant C such that

$$x'(t) \le \left(\frac{C}{a(t)}\right)^{1/\alpha}.$$

Integrating this inequality from t_1 to t and using condition (3.2), we arrive at the desired conclusion. □

The following result is concerned with the oscillation of equation (1.5).

Theorem 3.3. *If*

$$\int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\alpha}}(s)} ds < \infty, \qquad \int_{t_0}^{\infty} \left[\frac{1}{a(u)} \int_{t_0}^{u} \frac{1}{b(s)} ds\right]^{\frac{1}{\alpha}} du < \infty, \tag{3.5}$$

$$\int_{t_0}^{\infty} \left[\frac{1}{a(v)} \int_{t_0}^{v} \frac{1}{b(u)} \int_{t_0}^{u} q^{\frac{\beta}{\beta-\gamma}}(s) p^{\frac{\gamma}{\gamma-\beta}}(s) ds du\right]^{\frac{1}{\alpha}} dv < \infty, \tag{3.6}$$

$$\liminf_{t\to\infty} \int_{t_0}^{t} \left[\frac{1}{a(v)} \int_{t_0}^{v} \frac{1}{b(u)} \int_{t_0}^{u} e(s) ds du\right]^{\frac{1}{\alpha}} dv = -\infty, \tag{3.7}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{1}{a(v)} \int_{t_0}^{v} \frac{1}{b(u)} \int_{t_0}^{u} e(s) ds du\right]^{\frac{1}{\alpha}} dv = +\infty, \tag{3.8}$$

then equation (1.5) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.5), say $x(t) > 0$ for $t \ge t_1 \ge t_0$. Proceeding as in the proof of Theorem 3.2, we obtain

$$\begin{aligned}\left(b(t)\left(a(t)(x'(t))^\alpha\right)'\right)' &\le e(t) + \left(\frac{\beta-\gamma}{\gamma}\right)\left[\frac{\gamma}{\beta} q(t)\right]^{\frac{\beta}{\beta-\gamma}} (p(t))^{\frac{\gamma}{\gamma-\beta}} \\ &= e(t) + c q^{\frac{\beta}{\beta-\gamma}}(t) p^{\frac{\gamma}{\gamma-\beta}}(t),\end{aligned}$$

where $c = \left(\frac{\beta-\gamma}{\gamma}\right)\left(\frac{\gamma}{\beta}\right)^{\frac{\beta}{\beta-\gamma}}$. Integrating this inequality from t_1 to t, we obtain

$$\left(a(t)(x'(t))^\alpha\right)' \le \frac{c_1}{b(t)} + \frac{1}{b(t)} \int_{t_1}^{t} e(s) ds + \frac{c}{b(t)} \int_{t_1}^{t} q^{\frac{\beta}{\beta-\gamma}}(s) p^{\frac{\gamma}{\gamma-\beta}}(s) ds,$$

where $c_1 = b(t_1)\left(a(t_1)(x'(t_1))^\alpha\right)'$. Integrating one more time, we have

$$a(t)(x'(t))^\alpha \le c_2 + c_1 \int_{t_1}^{t} \frac{1}{b(s)} ds + c \int_{t_1}^{t} \frac{1}{b(u)} \int_{t_1}^{u} q^{\frac{\beta}{\beta-\gamma}}(s) p^{\frac{\gamma}{\gamma-\beta}}(s) ds du + \int_{t_1}^{t} \frac{1}{b(u)} \int_{t_1}^{u} e(s) ds du,$$

where $c_2 = a(t_1)(x'(t_1))^\alpha$.

Dividing by $a(t)$ gives

$$\begin{aligned}x'(t) \le \Bigg\{ &\frac{c_2}{a(t)} + \frac{c_1}{a(t)} \int_{t_1}^{t} \frac{1}{b(s)} ds + \frac{c}{a(t)} \int_{t_1}^{t} \frac{1}{b(u)} \int_{t_1}^{u} q^{\frac{\beta}{\beta-\gamma}}(s) p^{\frac{\gamma}{\gamma-\beta}}(s) ds du \\ &+ \frac{1}{a(t)} \int_{t_1}^{t} \frac{1}{b(u)} \int_{t_1}^{u} e(s) ds du \Bigg\}^{\frac{1}{\alpha}}.\end{aligned}$$

Using the fact that $(u+v)^\gamma \le u^\gamma + v^\gamma$ for $u, v \ge 0$ and $0 < \gamma < 1$, and integrating again, we have

$$x(t) \le x(t_1) + \int_{t_1}^{t} \left[\frac{c_2}{a(s)}\right]^{\frac{1}{\alpha}} ds + \int_{t_1}^{t} \left[\frac{c_1}{a(u)} \int_{t_1}^{u} \frac{1}{b(s)} ds\right]^{\frac{1}{\alpha}} du$$
$$+ \int_{t_1}^{t} \left[\frac{c}{a(v)} \int_{t_1}^{v} \frac{1}{b(u)} \int_{t_1}^{u} q^{\frac{\beta}{\beta-\gamma}}(s) p^{\frac{\gamma}{\gamma-\beta}}(s) ds du\right]^{\frac{1}{\alpha}} dv + \int_{t_1}^{t} \left[\frac{1}{a(v)} \int_{t_1}^{v} \frac{1}{b(u)} \int_{t_1}^{u} e(s) ds du\right]^{\frac{1}{\alpha}} dv,$$

Taking $\liminf$ of both sides of the above inequality as $t \to \infty$ and applying conditions (3.5)–(3.7), we obtain a contradiction to $x(t)$ being a positive solution. The proof in case $x(t)$ is eventually negative is similar. □

The following examples illustrate the above results.

Example 3.4. Consider the equation

$$\left(t^6 \left(t^6 (x'(t))^3\right)'\right)' + \frac{1}{t^4} x^5(t) = \frac{1}{t^6} x^3(t), \qquad t \ge 1. \tag{3.9}$$

All conditions of Theorem 3.2 are satisfied with $e(t) = 0$ and so every nonoscillatory solution of (3.9) is bounded. One such solution is $x(t) = 1/t$.

Example 3.5. Consider the equation

$$\left(t^3 \left(t^4 (x'(t))^3\right)'\right)' + \frac{1}{t^4} x^5(t) = \frac{1}{t^6} x^3(t) + t^6 \sin t. \tag{3.10}$$

It is easy to check that all the hypotheses of Theorem 3.3 are satisfied and hence all solutions of (3.10) are oscillatory.

Concluding Remarks. With suitable care, the nonlinearity $(x')^\alpha$ in equation (1.1) can be replaced with $|x'|^\alpha \operatorname{sgn}(x')$. There do not appear to be any criteria to ensure the nonoscillation of all solutions of equation (1.1). This would be an interesting topic to explore.

References

[1] B. Baculíková, J. Džurina, Oscillation of third-order nonlinear differential equations, *Appl. Math. Lett.* **24**(2011), 466–470. `https://doi.org/10.1016/j.aml.2010.10.043`; MR2749728

[2] J. Džurina, B. Baculíková, Oscillation of trinomial differential equations with positive and negative terms, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 43, 1–8. `https://doi.org/10.14232/ejqtde.2014.1.43`; MR3250034

[3] T. Candan, R. S. Dahiya, Oscillation of third-order functional differential equations with delay, in: *Proceedings of the Fifth Mississippi State Conference on Differential Equations and Computational Simulations (Mississippi State, MS, 2001), Electron. J. Differ. Equ. Conf.* Vol. 10, Southwest Texas State Univ., San Marcos, TX, 2003, pp. 79–88. MR198309

[4] S. R. Grace, R. P. Agarwal, R. Pavani, E. Thandapani, On the oscillation of certain third order nonlinear functional differential equations, *Appl. Math. Comput.* **202**(2008), 102–112. `https://doi.org/10.1016/j.amc.2008.01.025`; MR2437140

[5] J. R. Graef, S. M. Rankin, P. W. Spikes, Oscillation theorems for perturbed nonlinear differential equations, *J. Math. Anal. Appl.* **65**(1978), 375–390. https://doi.org/10.1016/0022-247X(78)90189-0; MR506315

[6] J. R. Graef, S. M. Rankin, P. W. Spikes, Oscillation results for nonlinear functional differential equations, *Funkcial. Ekvac.* **27**(1984), 255–260. MR775209

[7] I. T. Kiguradze, On the oscillation of solutions of the equation $d^m u/dt^m + a(t)|u(t)|^m \mathrm{sgn} u(t) = 0$ (in Russian), *Mat. Sb. (N.S.)* **65**(1964), 172–187. MR0173060

[8] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations*, Kluwer Acad. Publ., Dordrecht 1993. https://doi.org/10.1007/978-94-011-1808-8; MR1220223

[9] R. G. Koplatadze, T. A. Chanturiya, Oscillating and monotone solutions of first-order differential equations with deviating argument, *Differ. Uravn.* **18**(1982), 1463–1465. MR0671174

[10] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Marcel Dekker, New York, 1987. MR1017244

[11] Ch. G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Arch. Math. (Basel)* **36**(1981), 168–178. MR619435

[12] E. Thandapani, T. Li, On the oscillation of third-order quasi-linear neutral functional differential equations, *Arch. Math. (Brno)* **47**(2011), 181–199. https://doi.org/10.1007/BF01223686; MR2852380

[13] A. Tiryaki, M. F. Aktas, Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping, *J. Math. Anal. Appl.* **325**(2007), 54–68. https://doi.org/10.1016/j.jmaa.2006.01.001; MR2273028