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Necessary and sufficient conditions for the existence of non-constant solutions generated by impulses of second order BVPs with convex potential

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Abstract. This paper concerns solutions generated by impulses for a class of second order BVPs with convex potential. Necessary and sufficient conditions for the existence of non-constant solutions are derived via variational methods and critical point theory.

Keywords: necessary and sufficient condition, convex potential, solutions generated by impulses, variational method.

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1 Introduction

Consider the following second-order boundary value problem

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], & \text{(1.1a)} \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, & \text{(1.1b)} \end{cases}$$

where $u(t) = (u^1(t), u^2(t), \dots, u^N(t))^T$, $\nabla F(t, x)$ is the gradient of $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ with respect to x and F satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \le a(|x|)b(t), \qquad |\nabla F(t, x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

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In particular, when $N = 1$, (1.1) is reduced to the following scalar problem

$$\begin{cases} \ddot{u}(t) = F_x(t, u(t)) \quad \text{a.e. } t \in [0, T], & \text{(1.2a)} \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. & \text{(1.2b)} \end{cases}$$

Different convexity hypotheses on the potential are employed to study the existence of solutions of the problems (1.1) and (1.2), such as, convexity [11], subconvexity [14,22,26], $\mu(t)$-convexity and $k(t)$-concavity [20,25,26]. In particular, Mawhin and Willem [11] obtained, by using the variational methods, the sufficient and necessary conditions on the solvability of the above two problems as follows.

Theorem A ([11, Theorem 1.8]). *Assume that F satisfies condition (A) and $F(t, \cdot)$ is strictly convex for a.e. $t \in [0, T]$. Then the following conditions are equivalent:*

(α) Problem (1.1) is solvable.

(β) There exists $\bar{x} \in \mathbb{R}^N$ such that

$$\int_0^T \nabla F(t, \bar{x}) dt = 0.$$

(γ) $\int_0^T F(t, x) dt \to +\infty$ as $|x| \to \infty$.

Theorem B ([11, Theorem 1.9]). *If $F_x(t, \cdot)$ is nondecreasing for a.e. $t \in [0, T]$, then the problem (1.2) has at least one solution if and only if there exists some $\bar{a} \in \mathbb{R}$ satisfying*

$$\int_0^T F_x(t, \bar{a}) dt = 0.$$

By Theorem A, the problem (1.1) does not possess any solutions provided the equations $\int_0^T \nabla F(t, x) dt = 0$ are not solvable in $\mathbb{R}^N$. For instance, the following boundary value problem possesses no solution:

$$\begin{cases} \ddot{u}(t) = \nabla(2t(\exp(u^1(t)) + \exp(u^2(t)))) \quad \text{a.e. } t \in [0, 1], \\ u(0) - u(1) = \dot{u}(0) - \dot{u}(1) = 0, \end{cases} \tag{1.3}$$

where $u(t) = (u^1(t), u^2(t))^T$. When the second order BVPs (1.1) and (1.2) have no solution, the present paper concerns about generating solutions by impulses.

More precisely, in this paper we will consider the necessary and sufficient conditions for the existence of solutions generated by impulses of the above two BVPs, that is, solutions of the problem (1.1) generated by

$$\Delta(\dot{u}^i(t_1)) = I_i(u^i(t_1)), \qquad i = 1, 2, \ldots, N \tag{1.4}$$

and solutions of the problem (1.2) generated by

$$\Delta(\dot{u}(t_1)) = I(u(t_1)), \tag{1.5}$$

where $0 < t_1 < T$ is the instant where the impulse occurs, $\Delta(\dot{u}^i(t_1)) = \dot{u}^i(t_1^+) - \dot{u}^i(t_1^-)$ with $\dot{u}^i(t_1^{\pm}) = \lim_{t \to t_1^{\pm}} \dot{u}^i(t)$, the impulsive functions $I, I_i \in C^1(\mathbb{R}, \mathbb{R})$ for each $i = 1, 2, \ldots, N$. Here, a solution of the problem (1.1) (resp. (1.2)) with the impulsive condition (1.4) (resp. (1.5)) is said to be generated by impulses if the problem (1.1) (resp. (1.2)) does not possess any solution.

Impulsive effects arise from the real world and are used to describe sudden, discontinuous jumps. Due to their significance, a number of papers [2, 10, 15] have provided the qualitative properties of such equations. Some efforts have been made in studying the existence of solutions of impulsive problems via variational methods, see, for instance, [1, 3, 4, 6–8, 12, 13, 16, 17, 21]. In these results, the nonlinear term plays a more important role than the impulsive terms do in guaranteeing the existence of solutions. While, by strengthening the role of impulses, some sufficient conditions for the existence of solutions generated by impulses are established. In 2011, Zhang and Li [24] established sufficient conditions for the following system to possess at least one non-zero periodic solution and at least one non-zero homoclinic solution and these solutions are generated by impulses when $f \equiv 0$.

$$\begin{cases} \ddot{q} + V_q(t,q) = f(t), & \text{for } t \in (s_{k-1}, s_k), \\ \Delta\dot{q}(s_k) = g_k(q(s_k)). \end{cases}$$

After that, Han and Zhang [5] considered the following asymptotically linear or sublinear Hamiltonian systems with impulsive conditions.

$$\begin{cases} \ddot{q}(t) = f(t, q(t)), & \text{for } t \in (s_{k-1}, s_k), \\ \Delta\dot{q}(s_k) = g_k(q(s_k)). \end{cases} \tag{1.6}$$

And sufficient conditions for the existence of periodic and homoclinic solutions generated by impulses are derived. In 2013, Sun, Chu and Chen [19] established sufficient conditions for the existence of a positive periodic solution generated by impulses for the following second-order singular differential equations with impulsive conditions.

$$\begin{cases} u''(t) - \dfrac{1}{u^\alpha(t)} = e(t), \\ \Delta u'(t_j) = I_j(u(t_j)). \end{cases}$$

In 2014, Zhang, Wu and Dai [23] obtained sufficient conditions to guarantee the system (1.6) has infinitely many non-zero periodic solutions generated by impulses. In 2015, by using Ricceri's Variational Principle, Heidarkhani, Ferrara and Salari [8] investigated sufficient conditions for the existence of infinitely many periodic solutions generated by impulses for the following perturbed second-order impulsive differential equations.

$$\begin{cases} \ddot{u}(t) + V_u(t, u(t)) = 0, & t \in (t_{j-1}, t_j), \\ \Delta\dot{u}(t_j) = \lambda f_j(u(t_j)) + \mu g_j(u(t_j)), \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

On the other hand, some attempts have been made on the necessary and sufficient conditions for the existence of solutions (not generated by impulses) for impulsive boundary value problems. By the method of upper and lower solutions, Hou and Yan [9] established some necessary and sufficient conditions for the existence of solutions for singular impulsive boundary value problems on the half-line; Using the variational method, Sun and Chu [18] recently established a necessary and sufficient condition for the existence of periodic solutions for a impulsive singular differential equation.

However, to the best of our knowledge, relatively little attention is paid to the necessary and sufficient conditions for the existence of solutions generated by impulses. As a result, the goal of this paper is to fill the gap in this area. Result of this paper for the problem (1.1) is presented as follows.

Theorem 1.1. *Assume that F satisfies the assumption (A), $F(t,\cdot)$ is strictly convex for a.e. $t \in [0,T]$ and the equations $\int_0^T \nabla F(t,x)dt = 0$ have no solution in $\mathbb{R}^N$. If $I_i' > 0$ for each $i = 1,2,\ldots,N$, then the following properties are equivalent:*

(α_1) *The problem (1.1) has at least one non-constant solution generated by impulses (1.4) in H_T^1.*

(β_1) *There exists $\overline{x} \in \mathbb{R}^N$ such that*

$$\int_0^T \nabla F(t,\overline{x})dt + \Big(I_1(\overline{x}^1), I_2(\overline{x}^2),\ldots,I_N(\overline{x}^N)\Big)^T = 0.$$

(γ_1) $\int_0^T F(t,x)dt + \sum_{i=1}^N \int_0^{x^i} I_i(s)ds \to +\infty$ *as* $|x| \to \infty$.

When $N = 1$, Theorem 1.1 is also valid for the problem (1.2). However, for the scalar problem, the convexity of $F(t,\cdot)$ implies that $F_x(t,\cdot)$ is nondecreasing in $\mathbb{R}$, so the strictness of convexity of $F(t,\cdot)$ may be dropped, and a better result is obtained.

Theorem 1.2. *Assume that F satisfies the assumption (A) where $N = 1$, $F(t,\cdot)$ is convex for a.e. $t \in [0,T]$ and the equation $\int_0^T F_x(t,x)dt = 0$ has no solution in $\mathbb{R}$. If $I' \geq 0$, then problem (1.2) has at least one non-constant solution generated by impulse (1.5) in H_T^1 if and only if there exists $\overline{x} \in \mathbb{R}$ such that*

$$\int_0^T F_x(t,\overline{x})dt + I(\overline{x}) = 0. \tag{1.7}$$

In the following, an example is given to illustrate Theorem 1.1.

Example 1.3. It has been shown above that the boundary value problem (1.3) is not solvable. However, after adding the following impulses

$$\Delta(\dot{u}^1(0.5)) = u^1(0.5) \quad \text{and} \quad \Delta(\dot{u}^2(0.5)) = 2u^2(0.5), \tag{1.8}$$

the problem (1.3) has at least one non-constant solution generated by impulses (1.8) in H_T^1.

Indeed, $I_1(x^1) = x^1$, $I_2(x^2) = 2x^2$ and it is clear that the following equations are solvable.

$$0 = \int_0^1 \nabla F(t,x)dt + \Big(I_1(x^1), I_2(x^2)\Big)\Big)^T = \begin{cases} \exp(x^1) + x^1, \\ \exp(x^2) + 2x^2. \end{cases}$$

This proves the assertion by Theorem 1.1.

2 Preliminaries

Let C_T^∞ be the space of indefinitely differentiable T-periodic functions from $\mathbb{R}$ to $\mathbb{R}^N$.

$$H_T^1 \equiv \left\{ u : [0,T] \to \mathbb{R}^N \;\middle|\; \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0) = u(T) \text{ and } \dot{u} \in L^2(0,T;\mathbb{R}^N) \end{array} \right\}$$

is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_0^T (\dot{u}(t),\dot{v}(t))dt + \int_0^T (u(t),v(t))dt, \qquad \forall u,v \in H_T^1,$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$, and the corresponding norm is

$$\|u\| = \left(\|\dot{u}\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

Let $\widetilde{u}(t) \equiv u(t) - \overline{u}$, where $\overline{u} = (1/T)\int_0^T u(t)dt$.

Consider the functional $\Phi : H_T^1 \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t,u(t))dt + \sum_{i=1}^{N}\int_0^{u^i(t_1)} I_i(s)ds.$$

The assumption (A) and all $I_i \in C^1(\mathbb{R},\mathbb{R})$ imply that $\Phi \in C^1(H_T^1,\mathbb{R})$ and

$$(\Phi'(u),v) = \int_0^T (\dot{u}(t),\dot{v}(t))dt + \int_0^T (\nabla F(t,u(t)),v(t))\,dt + \sum_{i=1}^{N} I_i(u^i(t_1))v^i(t_1).$$

Thus, if $u \in H_T^1$ is a critical point of Φ, then u is a solution of the problem (1.1)–(1.4). In fact, for any $v \in H_T^1$, we have

$$\int_0^T (\dot{u}(t),\dot{v}(t))dt + \int_0^T (\nabla F(t,u(t)),v(t))\,dt + \sum_{i=1}^{N} I_i(u^i(t_1))v^i(t_1) = 0. \tag{2.1}$$

Then for any $v \in H_T^1$ satisfying $v(t_1) = 0$, we get

$$\int_0^T (\dot{u}(t),\dot{v}(t))dt = -\int_0^T (\nabla F(t,u(t)),v(t))\,dt.$$

Since the behavior of a function on a set of measure zero does not affect its integral and $C_T^\infty \subset H_T^1$, we have

$$\int_0^T (\dot{u}(t),\dot{v}(t))dt = -\int_0^T (\nabla F(t,u(t)),v(t))\,dt, \quad \text{for any } v \in C_T^\infty.$$

So $\ddot{u}$ exists and (1.1a) holds. Moreover, the existence of weak derivative of u and $\dot{u}$ implies that (1.1b) holds. It follows from (1.1b) that

$$\begin{aligned}\int_0^T (\ddot{u}(t),v(t))\,dt &= \int_0^{t_1} (\ddot{u}(t),v(t))\,dt + \int_{t_1}^T (\ddot{u}(t),v(t))\,dt \\ &= (\dot{u}(t),v(t))\Big|_0^{t_1^-} + (\dot{u}(t),v(t))\Big|_{t_1^+}^T - \int_0^T (\dot{u}(t),\dot{v}(t))dt \\ &= -\sum_{i=1}^{N} \Delta(\dot{u}^i(t_1))v^i(t_1) - \int_0^T (\dot{u}(t),\dot{v}(t))dt,\end{aligned}$$

which combining with (1.1a) and (2.1) yields

$$\sum_{i=1}^{N}\Big(\Delta(\dot{u}^i(t_1)) - I_i(u^i(t_1))\Big)v^i(t_1) = 0, \quad \text{for any } v \in H_T^1,$$

which implies (1.4) holds.

For the reader's convenience, we now recall some facts.

Lemma 2.1 ([11, Proposition 1.4]). *Let $G \in C^1(\mathbb{R}^n, \mathbb{R})$ be a convex function. Then, for all $x, y \in \mathbb{R}^n$ we have*

$$G(x) \geq G(y) + (\nabla G(y), x - y).$$

Lemma 2.2 ([11, Proposition 1.5]). *Let $G \in C^1(\mathbb{R}^n, \mathbb{R})$ be a strictly convex function. The following properties are equivalent*

(α) *There exists $\overline{x} \in \mathbb{R}^N$ such that $\nabla G(\overline{x}) = 0$.*

(β) *$G(x) \to +\infty$ when $|x| \to \infty$.*

Lemma 2.3 ([11, Theorem 1.1]). *If φ is weakly lower semi-continuous on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

3 Main result

In this section, the main results of this paper are proved.

Lemma 3.1. *Assume that F satisfies the assumption (A) and $F(t, \cdot)$ is convex for a.e. $t \in [0, T]$. If $I_i' \geq 0$ for each $i = 1, 2, \ldots, N$ and*

$$H(x) \equiv \int_0^T F(t, x)dt + \sum_{i=1}^{N} \int_0^{x^i} I_i(s)ds \to +\infty \quad as\ |x| \to \infty, \tag{3.1}$$

then the problem (1.1)–(1.4) has at least one solution which minimizes Φ on H_T^1.

Proof. Let $\{u_n\}$ be a weakly convergence sequence to u_0 in H_T^1, then $\{u_n\}$ converges uniformly to u_0 on $[0, T]$. Then there exists a constant $C_1 > 0$ such that $\|u_n\|_\infty \leq C_1$ for $n = 0, 1, 2, \ldots$, so the continuity of I_i implies that

$$\begin{aligned}
&\left| \sum_{i=1}^{N} \int_0^{u_n^i(t_1)} I_i(s)ds - \sum_{i=1}^{N} \int_0^{u_0^i(t_1)} I_i(s)ds \right| \\
&\quad \leq \sum_{i=1}^{N} \left| \int_{u_0^i(t_1)}^{u_n^i(t_1)} I_i(s)ds \right| \leq NC_2 \|u_n - u_0\|_\infty \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

where $C_2 = \max_{i \in \{1,2,\ldots,N\}, |s| \leq C_1} |I_i(s)|$. Therefore $\sum_{i=1}^N \int_0^{u^i(t_1)} I_i(s)ds$ is weakly continuous on H_T^1. What is more, $\int_0^T |\dot{u}(t)|^2 dt$ is a convex continuous function and $\int_0^T F(t, u(t))dt$ is weakly continuous on H_T^1. Thus Φ is weakly lower semi-continuous on H_T^1. Lemma 2.3 shows that it remains to prove that Φ is coercive. In view of (3.1), $H(x)$ has a minimum at some point $\overline{x} \in \mathbb{R}^N$ for which

$$\nabla H(\overline{x}) = \int_0^T \nabla F(t, \overline{x})dt + \left(I_1(\overline{x}^1), I_2(\overline{x}^2), \ldots, I_N(\overline{x}^N) \right)^T = 0.$$

Thus

$$\begin{aligned}
&\int_0^T (\nabla F(t,\overline{x}), u(t)-\overline{x})\,dt + \sum_{i=1}^N I_i(\overline{x}^i)(u^i(t_1)-\overline{x}^i) \\
&\quad= \int_0^T (\nabla F(t,\overline{x}), u(t)-\overline{u})\,dt + \sum_{i=1}^N I_i(\overline{x}^i)(u^i(t_1)-\overline{u}^i) \\
&\qquad+ \int_0^T (\nabla F(t,\overline{x}), \overline{u}-\overline{x})\,dt + \sum_{i=1}^N I_i(\overline{x}^i)(\overline{u}^i-\overline{x}^i) \\
&\quad= \int_0^T (\nabla F(t,\overline{x}), \widetilde{u}(t))\,dt + \sum_{i=1}^N I_i(\overline{x}^i)\widetilde{u}^i(t_1).
\end{aligned} \tag{3.2}$$

Since $F(t,\cdot)$ and $\int_0^z I_i(s)ds$ are convex, Lemma 2.1 and (3.2) imply that

$$\begin{aligned}
&\int_0^T F(t,u(t))dt + \sum_{i=1}^N \int_0^{u^i(t_1)} I_i(s)ds \\
&\quad\geq \int_0^T F(t,\overline{x})dt + \sum_{i=1}^N \int_0^{\overline{x}^i} I_i(s)ds + \int_0^T (\nabla F(t,\overline{x}), u(t)-\overline{x})\,dt + \sum_{i=1}^N I_i(\overline{x}^i)(u^i(t_1)-\overline{x}^i) \\
&\quad\geq \int_0^T F(t,\overline{x})dt + \sum_{i=1}^N \int_0^{\overline{x}^i} I_i(s)ds + \int_0^T (\nabla F(t,\overline{x}), \widetilde{u}(t))\,dt + \sum_{i=1}^N I_i(\overline{x}^i)\widetilde{u}^i(t_1) \\
&\quad\geq -\int_0^T |\nabla F(t,\overline{x})|dt\|\widetilde{u}\|_\infty - \left(\sum_{i=1}^N I_i^2(\overline{x}^i)\right)^{\frac{1}{2}} \|\widetilde{u}\|_\infty + C_3 \\
&\quad\geq -C_4\|\dot{u}\|_{L^2} + C_3,
\end{aligned}$$

where $C_3 = \int_0^T F(t,\overline{x})dt + \sum_{i=1}^N \int_0^{\overline{x}^i} I_i(s)ds$ and

$$C_4 = \int_0^T |\nabla F(t,\overline{x})|dt\sqrt{\frac{T}{12}} + \left(\sum_{i=1}^N I_i^2(\overline{x}^i)\right)^{\frac{1}{2}} \sqrt{\frac{T}{12}}.$$

Thus

$$\Phi(u) \geq \frac{1}{2}\|\dot{u}\|_{L^2}^2 - C_4\|\dot{u}\|_{L^2} + C_3. \tag{3.3}$$

On the other hand, it follows from the assumption (A) and the convexity of $F(t,\cdot)$ and $\int_0^z I_i(s)ds$ that

$$\begin{aligned}
\Phi(u) &\geq \frac{1}{2}\|\dot{u}\|_{L^2}^2 + 2\int_0^T F\left(t,\frac{\overline{u}}{2}\right)dt + 2\sum_{i=1}^N \int_0^{\frac{\overline{u}^i}{2}} I_i(s)ds \\
&\quad - \int_0^T F(t,-\widetilde{u}(t))dt - \sum_{i=1}^N \int_0^{-\widetilde{u}^i(t_1)} I_i(s)ds \\
&\geq \frac{1}{2}\|\dot{u}\|_{L^2}^2 + 2H\left(\frac{\overline{u}}{2}\right) - \int_0^T a(|\widetilde{u}(t)|)b(t)dt - \sum_{i=1}^N \int_0^{-\widetilde{u}^i(t_1)} I_i(s)ds.
\end{aligned} \tag{3.4}$$

By Sobolev's inequality, we have

$$|-\widetilde{u}^i(t)| \leq |\widetilde{u}(t)| \leq \|\widetilde{u}\|_\infty \leq \sqrt{\frac{T}{12}}\|\dot{u}\|_{L^2} \quad \text{for all } t\in[0,T].$$

As $\|u\| \to \infty$ if and only if $(|\overline{u}|^2 + \int_0^T |\dot{u}(t)|^2 dt)^{1/2} \to \infty$, the above inequality, (3.3), (3.4) and (3.1) imply that Φ is coercive. □

Based on the above lemma, the impulsive differential system (1.1)–(1.4) and the impulsive differential equation (1.2)–(1.5) will be considered respectively.

3.1 Impulsive differential systems

Theorem 3.2. *Assume that F satisfies the assumption (A). If $F(t,\cdot)$ is strictly convex for a.e. $t \in [0,T]$ and $I_i' > 0$ for each $i = 1, 2, \ldots, N$. Then the following properties are equivalent:*

(α_2) *The problem (1.1)–(1.4) is solvable.*

(β_1) *There exists $\overline{x} \in \mathbb{R}^N$ such that*

$$\int_0^T \nabla F(t,\overline{x})dt + \Big(I_1(\overline{x}^1), I_2(\overline{x}^2), \ldots, I_N(\overline{x}^N)\Big)^T = 0.$$

(γ_1) $\int_0^T F(t,x)dt + \sum_{i=1}^N \int_0^{x^i} I_i(s)ds \to +\infty$ *as* $|x| \to \infty$.

Proof. If u_0 is a solution of the problem (1.1)–(1.4), integrating both sides of (1.1a) over $[0,T]$ and using the boundary condition (1.1b) and the impulsive condition (1.4), we have

$$\begin{aligned}\int_0^T \nabla F(t,u_0(t))dt &= \int_0^{t_1} \ddot{u}_0(t)dt + \int_{t_1}^T \ddot{u}_0(t)dt \\ &= \dot{u}_0(t_1^-) - \dot{u}_0(0) + \dot{u}_0(T) - \dot{u}_0(t_1^+) \\ &= -\Big(I_1(u_0^1(t_1)), I_2(u_0^2(t_1)), \ldots, I_N(u_0^N(t_1))\Big)^T. \end{aligned} \tag{3.5}$$

Define the strictly convex function $\widetilde{H} : \mathbb{R}^N \to \mathbb{R}$ by

$$\widetilde{H}(x) \equiv \int_0^T F(t, x + \widetilde{u}_0(t))dt + \sum_{i=1}^N \int_0^{x^i + \widetilde{u}_0^i(t_1)} I_i(s)ds.$$

Since $\nabla\widetilde{H}(\overline{u}_0) = 0$ by (3.5), Lemma 2.2 implies that $\widetilde{H}(x) \to +\infty$ as $|x| \to \infty$. It follows from the convexity of $F(t,\cdot)$ and $I_i' > 0$ that

$$\begin{aligned}\widetilde{H}(x) &\le \frac{1}{2}\left[\int_0^T F(t,2x)dt + \sum_{i=1}^N \int_0^{2x^i} I_i(s)ds\right] + \frac{1}{2}\left[\int_0^T F(t,2\widetilde{u}_0(t))dt + \sum_{i=1}^N \int_0^{2\widetilde{u}_0^i(t_1)} I_i(s)ds\right] \\ &= \frac{1}{2}H(2x) + C_5,\end{aligned}$$

where $C_5 = \big[\int_0^T F(t,2\widetilde{u}_0(t))dt + \sum_{i=1}^N \int_0^{2\widetilde{u}_0^i(t_1)} I_i(s)ds\big]/2$. So $H(x) \to +\infty$ as $|x| \to \infty$. Thus it follows from Lemma 2.2 that there exists $\overline{x} \in \mathbb{R}^N$ such that $\nabla H(\overline{x}) = 0$ and (α_2) implies (β_1).

By Lemma 2.2 applied to the function H, (β_1) implies (γ_1).

It follows from Lemma 3.1 that (γ_1) implies (α_2). □

Proof of Theorem 1.1. Since the equations $\int_0^T \nabla F(t,x)dt = 0$ have no solution in $\mathbb{R}^N$, it follows from Theorem A that the problem (1.1) has no solution. So (α_2) implies that the problem (1.1) has at least one solution generated by impulses (1.4). What is more, the solution is not a constant. In fact, suppose that the solution $u(t) = C$, a.e. $t \in [0,T]$, then by (1.1a), this implies $\nabla F(t,C) = 0$, a.e. $t \in [0,T]$, then $\int_0^T \nabla F(t,C)dt = 0$, which is a contradiction, thus (α_1) holds. This proves the assertion by Theorem 3.2. □

3.2 Impulsive differential equations

We begin with the following lemma on impulsive linear boundary value problem.

Lemma 3.3. *Let $f : [0, T] \to \mathbb{R}$ and $\overline{x} \in \mathbb{R}$. The scalar problem*

$$\begin{cases} \ddot{u}(t) = f(t) \quad \text{a.e. } t \in [0,T], & (3.6a) \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, & (3.6b) \\ \Delta(\dot{u}(t_1)) = I(\overline{x}), & (3.6c) \end{cases}$$

is solvable if and only if

$$\int_0^T f(t)dt + I(\overline{x}) = 0. \tag{3.7}$$

Proof. If $u_0(t)$ is a solution of (3.6), then integrating (3.6a) over $[0, T]$ and using the boundary conditions and the impulsive condition, we have

$$\begin{aligned} \int_0^T f(t)dt &= \int_0^{t_1} \ddot{u}_0(t)dt + \int_{t_1}^T \ddot{u}_0(t)dt \\ &= \dot{u}_0(t_1^-) - \dot{u}_0(0) + \dot{u}_0(T) - \dot{u}_0(t_1^+) \\ &= -\Delta(\dot{u}_0(t_1)), \end{aligned}$$

which implies (3.7) holds. For the sufficiency, if (3.7) holds, it could be verified that (3.6) has the following solution.

$$u(t) = \begin{cases} \int_0^t \int_0^s f(\xi)d\xi ds - C_1 t - I(\overline{x})t + C_2, & 0 \le t \le t_1, \\ \int_0^t \int_0^s f(\xi)d\xi ds - C_1 t + C_2, & t_1 < t \le T, \end{cases}$$

where $C_1 = \int_0^T \int_0^s f(\xi)d\xi ds/T$, $C_2 = \overline{x} - \int_0^{t_1} \int_0^s f(\xi)d\xi ds + C_1 t_1 + I(\overline{x})t_1$. □

Theorem 3.4. *Assume that F satisfies the assumption (A) where $N = 1$. If $F(t, \cdot)$ is convex for a.e. $t \in [0, T]$ and $I' \ge 0$. Then the problem (1.2)–(1.5) is solvable if and only if there exists $\overline{x} \in \mathbb{R}$ such that (1.7) holds.*

Proof. If $u_0(t)$ is a solution of the problem (1.2)–(1.5), similarly as (3.5) we have

$$\int_0^T F_x(t, u_0(t))dt + I(u_0(t_1)) = 0.$$

Since $F(t, \cdot)$ is convex, we see that $F_x(t, \cdot)$ is nondecreasing. What is more, $I' \ge 0$. Thus, if $m \le u_0(t) \le M$ for $t \in [0, T]$, we have

$$\int_0^T F_x(t, m)dt + I(m) \le 0 \le \int_0^T F_x(t, M)dt + I(M).$$

And (1.7) is derived follows from the intermediate value theorem.

For the sufficiency, consider first the following problem

$$\begin{cases} \ddot{w}(t) = F_x(t, \overline{x}) \quad \text{a.e. } t \in [0,T], \\ w(0) - w(T) = \dot{w}(0) - \dot{w}(T) = 0, \\ \Delta(\dot{w}(t_1)) = I(\overline{x}). \end{cases} \tag{3.8}$$

By (1.7), Lemma 3.3 implies that the problem (3.8) has a solution $w_*(t)$.

The subsequent discussions on the problem (1.2)–(1.5) will be divided into three cases.

Case I. $\int_0^T F_x(t,x)dt + I(x) = 0$ for all $x \geq \overline{x}$.

In this case, condition (1.7) implies

$$\int_0^T F_x(t,x) - F_x(t,\overline{x})dt + I(x) - I(\overline{x}) = 0 \quad \text{for all } x \geq \overline{x}.$$

What is more, $F_x(t,\cdot)$ and I are nondecreasing functions, so we have

$$\int_0^T F_x(t,x) - F_x(t,\overline{x})dt = 0 \quad \text{for all } x \geq \overline{x} \tag{3.9}$$

and

$$I(x) = I(\overline{x}) \quad \text{for all } x \geq \overline{x}. \tag{3.10}$$

Thus, we obtain, by (3.9) and the monotonicity of $F_x(t,\cdot)$,

$$F_x(t,x) = F_x(t,\overline{x}) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \geq \overline{x}. \tag{3.11}$$

Let $\eta \in \mathbb{R}$ sufficiently large so that

$$u_*(t) \equiv w_*(t) + \eta > \overline{x}, \quad t \in [0,T],$$

then $u_*(0) - u_*(T) = \dot{u}_*(0) - \dot{u}_*(T) = 0$ and $u_*(t)$ is a solution of the problem (1.2)–(1.5). In fact, in view of (3.11), we have

$$\ddot{u}_*(t) = \ddot{w}_*(t) = F_x(t,\overline{x}) = F_x(t,u_*(t)) \quad \text{for a.e. } t \in [0,T].$$

And it follows from (3.10) that

$$\Delta(\dot{u}_*(t_1)) = \Delta(\dot{w}_*(t_1)) = I(\overline{x}) = I(u_*(t_1)).$$

Case II. $\int_0^T F_x(t,x)dt + I(x) = 0$ for all $x \leq \overline{x}$.

The proof of Case II is similar to that of Case I and will be omitted.

Case III. There exist $x_1 < \overline{x} < x_2$ such that

$$\int_0^T F_x(t,x_1)dt + I(x_1) < 0 < \int_0^T F_x(t,x_2)dt + I(x_2).$$

Since $\widehat{H}(x) \equiv \int_0^T F(t,x)dt + \int_0^x I(s)ds$ is convex, it follows that, by Lemma 2.1,

$$\widehat{H}(x) - \widehat{H}(x_i) \geq \left[\int_0^T F_x(t,x_i)dt + I(x_i)\right](x - x_i),$$

for each $i = 1,2$. So

$$\int_0^T F(t,x)dt + \int_0^x I(s)ds \to +\infty \quad \text{as } |x| \to \infty.$$

Thus the solvability of the problem (1.2)–(1.5) follows from Lemma 3.1. □

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, it follows from Theorem 3.4 and Theorem B that Theorem 1.2 holds. □

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