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Oscillations of nonlinear differential equations with several deviating arguments

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Abstract. This article concerns the oscillatory behavior of first-order non-linear differential equations with several variable deviating arguments and non-negative coefficients. We study both delayed and advanced equations, and obtain sufficient conditions that guarantee the oscillation of all solutions. Our assumptions let us transform differential equalities into inequalities for which we use known techniques, and improve results in the literature. We also provide an example that illustrates our results.

Keywords: differential equation, non-monotone argument, oscillatory solution, non-oscillatory solution, Grönwall inequality.

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1 Introduction

In this article we consider the non-linear differential equation with several variable deviating arguments of either delay

$$x'(t) + \sum_{i=1}^{m} f_i(t, x(\tau_i(t))) = 0, \quad t \geq t_0 \tag{1.1}$$

or advanced type

$$x'(t) - \sum_{i=1}^{m} g_i(t, x(\sigma_i(t))) = 0, \quad t \geq t_0. \tag{1.2}$$

Here $f_i, g_i : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $\tau_i, \sigma_i : [t_0, \infty) \to \mathbb{R}$ are continuous functions for $i = 1, \ldots, m$. In addition to (1.1) we consider the initial condition

$$x(t) = \varphi(t), \quad t \leq t_0, \tag{1.3}$$

where $\varphi : (-\infty, t_0] \to \mathbb{R}$ is a bounded Borel measurable function.

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By a *solution to* (1.1) and (1.3) we mean an absolutely continuous function on $[t_0, \infty)$ satisfying (1.1) for almost all $t \geq t_0$ and (1.3) for all $t \leq t_0$. By *a solution of* (1.2) we mean an absolutely continuous function on $[t_0, \infty)$ satisfying (1.2) for almost all $t \geq t_0$.

A solution $x(t)$ of (1.1) (or (1.2)) is *oscillatory* if it has arbitrary large zeros. If there exists an eventually positive or an eventually negative solution, the equation is *non-oscillatory*. An equation is *oscillatory* if all its solutions are oscillatory.

In the previous decades, oscillatory behavior and stability of first-order differential equations with deviating arguments have been extensively studied, see for example [1–3], [7–17], [19–23] and references therein. Most of these papers concern the special case where $f_i(t,x) = p_i(t)x(\tau_i(t))$ and $g_i(t,x) = q_i(t)x(\sigma_i(t))$. For the general oscillation theory of differential equations the reader is referred to the monographs [1,6,18].

1.1 Linear differential equations

In this case we consider $f_i(t,x) = p_i(t)x(\tau_i(t))$ and $g_i(t,x) = q_i(t)x(\sigma_i(t))$.

In 1978 Ladde [17] and in 1982 Ladas and Stavroulakis [16] proved that if

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{e}, \tag{1.4}$$

where $\tau(t) = \max_{1\leq i\leq m}\{\tau_i(t)\}$, then all solutions of (1.1) are oscillatory; while if

$$\liminf_{t\to\infty} \int_{t}^{\sigma(t)} \sum_{i=1}^{m} q_i(s)ds > \frac{1}{e}, \tag{1.5}$$

where $\sigma(t) = \min_{1\leq i\leq m}\{\sigma_i(t)\}$, then all solutions of (1.2) are oscillatory. See also [18, Theorems 2.7.1 and 2.7.5].

In 1984, Hunt et al. [9] proved that if $t - \tau_i(t) \leq \tau_0$, $1 \leq i \leq m$, and

$$\liminf_{t\to\infty} \sum_{i=1}^{m} p_i(t)\left(t - \tau_i(t)\right) > \frac{1}{e}, \tag{1.6}$$

then all solutions of (1.1) are oscillatory.

In 1990, Zhou [23] proved that if $\sigma_i(t) - t \leq \sigma_0$ for $1 \leq i \leq m$, and

$$\liminf_{t\to\infty} \sum_{i=1}^{m} q_i(t)\left(\sigma_i(t) - t\right) > \frac{1}{e}, \tag{1.7}$$

then all solutions of (1.2) are oscillatory. See also [6, Corollary 2.6.12].

Assume that $\tau_i(t)$, $1 \leq i \leq m$ are not necessarily monotone, and set

$$h_i(t) = \sup_{t_0\leq s\leq t} \tau_i(s), \quad t \geq t_0 \quad \text{and} \quad h(t) = \max_{1\leq i\leq m} h_i(t), \quad t \geq t_0\,. \tag{1.8}$$

Clearly, $h_i(t)$, $h(t)$ are nondecreasing and $\tau_i(t) \leq h_i(t) \leq h(t) < t$ for all $t \geq t_0$.

In 2016, Braverman, Chatzarakis and Stavroulakis [3] proved that if for some $r \in \mathbb{N}$,

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)a_r(h(t), \tau_i(\zeta))d\zeta > 1, \tag{1.9}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta)a_r(h(t), \tau_i(\zeta))d\zeta > 1 - \frac{1-a-\sqrt{1-2a-a^2}}{2}, \tag{1.10}$$

or

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r(h(t), \tau_i(\zeta)) d\zeta > \frac{1}{e}, \tag{1.11}$$

where $a = \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s) ds$, then all solutions of (1.1) are oscillatory. Here $\tau_i(t) < t$, $\lim_{t\to\infty} \tau_i(t) = \infty$, for $i = 1, 2, \dots, m$ and $t \geq t_0$, and

$$\begin{aligned} a_1(t,s) &= \exp\left\{ \int_s^t \sum_{i=1}^{m} p_i(\zeta) d\zeta \right\} \\ a_{r+1}(t,s) &:= \exp\left\{ \int_s^t \sum_{i=1}^{m} p_i(\zeta) a_r(\zeta, \tau_i(\zeta)) d\zeta \right\}. \end{aligned} \tag{1.12}$$

Assume that $\sigma_i(t)$, $1 \leq i \leq m$ are not necessarily monotone, and set

$$\rho_i(t) = \inf_{s \geq t} \sigma_i(s), \quad t \geq t_0 \quad \text{and} \quad \rho(t) = \min_{1 \leq i \leq m} \rho_i(t), \quad t \geq t_0. \tag{1.13}$$

In the same paper [3], the authors proved that if for some $r \in \mathbb{N}$,

$$\limsup_{t\to\infty} \int_{t}^{\rho(t)} \sum_{i=1}^{m} q_i(\zeta) b_r(\rho(t), \sigma_i(\zeta))\, d\zeta > 1, \tag{1.14}$$

or

$$\limsup_{t\to\infty} \int_{t}^{\rho(t)} \sum_{i=1}^{m} q_i(\zeta) b_r(\rho(t), \sigma_i(\zeta))\, d\zeta > 1 - \frac{1 - b - \sqrt{1 - 2b - b^2}}{2}, \tag{1.15}$$

or

$$\liminf_{t\to\infty} \int_{t}^{\rho(t)} \sum_{i=1}^{m} q_i(\zeta) b_r(\rho(t), \sigma_i(\zeta))\, d\zeta > \frac{1}{e}, \tag{1.16}$$

where $b = \liminf_{t\to\infty} \int_{t}^{\sigma(t)} \sum_{i=1}^{m} q_i(s) ds$, then all solutions of (1.2) are oscillatory. Here $t < \sigma_i(t)$ for $i = 1, 2, \dots, m$ and $t \geq t_0$, and

$$\begin{aligned} b_1(t,s) &= \exp\left\{ \int_t^s \sum_{i=1}^{m} q_i(\zeta) d\zeta \right\} \\ b_{r+1}(t,s) &:= \exp\left\{ \int_t^s \sum_{i=1}^{m} q_i(\zeta) b_r(t, \sigma_i(\zeta)) d\zeta \right\}. \end{aligned} \tag{1.17}$$

Akca, Chatzarakis and Stavroulakis [2, Theorem 2] proved oscillation for (1.1) if

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r(h(\zeta), \tau_i(\zeta)) d\zeta > \frac{1 + \ln \lambda_0}{\lambda_0}, \tag{1.18}$$

where λ_0 is the smaller root of the transcendental equation

$$e^{a\lambda} = \lambda, \quad \text{and} \quad a = \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s) ds.$$

Using hypotheses (H1)–(H4) below, we transform differential equalities into inequalities, and then follow some ideas from [4,5] to study (1.1), (1.2). We derive new sufficient conditions for the oscillation of all solutions. These conditions involve lim sup and lim inf, and essentially improve all the previous results. Also, we give examples that illustrate the significance of our results.

2 Basic lemmas

2.1 Delay differential equations

We study (1.1) under the hypotheses:

(H1) $\tau_i(t) < t$ and $\lim_{t\to\infty} \tau_i(t) = \infty$, for $i = 1, 2, \ldots, m$, and $t \geq t_0$;

(H2) $x f_i(t, x) \geq 0$ and there exists a continuous non-negative function p_i such that

$$|f_i(t, x)| \geq p_i(t)|x| \quad \forall x \in \mathbb{R},\ t \geq t_0 .$$

The proofs of our main results are essentially based on the following lemmas.

Lemma 2.1. *Assume that* (H1) *and* (H2) *hold and $h(t)$ is defined by* (1.8).

(i) If $x(t)$ is an eventually positive solution of (1.1)*, then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau_i(t)) > 0$ and $x(t)$ is non-increasing for $t \geq t_1$. Furthermore*

$$- x'(t) \geq x(h(t)) \sum_{i=1}^{m} p_i(t). \tag{2.1}$$

(ii) If $y(t)$ is an eventually negative solution of (1.1)*, then there exists $t_1 \geq t_0$ such that $y(t) < 0$, $y(\tau_i(t)) < 0$ and $y(t)$ is nondecreasing for $t \geq t_1$. Furthermore*

$$y'(t) \leq y(h(t)) \sum_{i=1}^{m} p_i(t). \tag{2.2}$$

Proof. Since $x(t)$ is an eventually positive solution of (1.1) and $\lim_{t\to\infty} \tau_i(t) = \infty$, clearly there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau_i(t)) > 0$. By (1.1) we have

$$-x'(t) = \sum_{i=1}^{m} f_i(t, x(\tau_i(t)))$$

which means that $x(t)$ is non-increasing for $t \geq t_1$. Furthermore, in view of (H2), we have

$$- x'(t) \geq \sum_{i=1}^{m} p_i(t) x(\tau_i(t)) \geq x(h(t)) \sum_{i=1}^{m} p_i(t). \tag{2.3}$$

The proof of part (i) is complete.

Since $y(t)$ is an eventually negative solution of (1.1) and $\lim_{t\to\infty} \tau_i(t) = \infty$, clearly there exists $t_1 \geq t_0$ such that $y(t) < 0$, $y(\tau_i(t)) < 0$. By (1.1) we have

$$y'(t) = - \sum_{i=1}^{m} f_i(t, y(\tau_i(t)))$$

which means that $y(t)$ is nondecreasing for $t \geq t_1$. Furthermore, in view of (H2), we have

$$y'(t) \leq \sum_{i=1}^{m} p_i(t) y(\tau_i(t)) \leq y(h(t)) \sum_{i=1}^{m} p_i(t). \tag{2.4}$$

The proof of part (ii) is complete, and so is the proof of the lemma. □

To state the next lemma we define recursively the sequence:

$$\mathcal{P}_j(t) = \mathcal{P}_0(t)\left[1 + \int_{\tau_i(t)}^{t} \sum_{k=1}^{m} p_k(s) \exp\left(\int_{\tau_k(s)}^{t} \mathcal{P}_{j-1}(u)du\right)ds\right], \quad j \geq 1, \tag{2.5}$$

where $\mathcal{P}_0(t) = \sum_{i=1}^{m} p_i(t)$.

Lemma 2.2. *Assume that* (H1) *and* (H2) *hold, x is a positive solution of* (1.1)*, and $\mathcal{P}_j$ is defined by* (2.5)*. Then for every $j \geq 0$ we have*

$$x'(t) + \mathcal{P}_j(t)x(t) \leq 0$$

and by Grönwall's inequality,

$$x(s) \geq x(t) \exp\left(\int_{s}^{t} \mathcal{P}_j(u)du\right), \quad 0 \leq s \leq t. \tag{2.6}$$

Proof. In view of part (i) of Lemma 2.1, Equation (1.1) gives

$$x'(t) + \mathcal{P}_0(t)x(t) \leq 0. \tag{2.7}$$

Applying Grönwall's inequality, we obtain

$$x(s) \geq x(t) \exp\left(\int_{s}^{t} \mathcal{P}_0(u)du\right), \quad 0 \leq s \leq t. \tag{2.8}$$

Taking into account the fact that (H1) and (H2) hold, by integrating (1.1) from $\tau_i(t)$ to t, we have

$$x(t) - x(\tau_i(t)) + \int_{\tau_i(t)}^{t} \sum_{k=1}^{m} p_k(s)x(\tau_k(s))\, ds \leq 0. \tag{2.9}$$

Since $\tau_k(s) < t$, (2.8) guarantees that

$$x(\tau_k(s)) \geq x(t) \exp\left(\int_{\tau_k(s)}^{t} \mathcal{P}_0(u)du\right). \tag{2.10}$$

Combining (2.9) and (2.10) we have

$$x(t) - x(\tau_i(t)) + x(t) \int_{\tau_i(t)}^{t} \sum_{k=1}^{m} p_k(s) \exp\left(\int_{\tau_k(s)}^{t} \mathcal{P}_0(u)du\right) ds \leq 0.$$

Multiplying the above inequality by $p_i(t)$ and adding, we obtain

$$-\sum_{i=1}^{m} p_i(t)x(\tau_i(t)) + \sum_{i=1}^{m} p_i(t)\left[1 + \int_{\tau_i(t)}^{t} \sum_{k=1}^{m} p_k(s) \exp\left(\int_{\tau_k(s)}^{t} \mathcal{P}_0(u)du\right) ds\right] x(t) \leq 0.$$

By part (i) of Lemma 2.1, the above inequality takes the form

$$x'(t) + \mathcal{P}_0(t)\left[1 + \int_{\tau_i(t)}^{t} \sum_{k=1}^{m} p_k(s) \exp\left(\int_{\tau_k(s)}^{t} \mathcal{P}_0(u)du\right) ds\right] x(t) \leq 0,$$

i.e.,

$$x'(t) + \mathcal{P}_1(t)x(t) \leq 0,$$

where

$$\mathcal{P}_1(t) = \mathcal{P}_0(t)\left[1 + \int_{\tau_i(t)}^{t} \sum_{k=1}^{m} p_k(s) \exp\left(\int_{\tau_k(s)}^{t} \mathcal{P}_0(u)du\right) ds\right].$$

Repeating the above argument leads to a new estimate

$$x'(t) + \mathcal{P}_2(t)x(t) \le 0,$$

where

$$\mathcal{P}_2(t) = \mathcal{P}_0(t)\left[1 + \int_{\tau_i(t)}^{t} \sum_{k=1}^{m} p_k(s) \exp\left(\int_{\tau_k(s)}^{t} \mathcal{P}_1(u)du\right) ds\right].$$

By induction, we obtain

$$x'(t) + \mathcal{P}_j(t)x(t) \le 0,$$

where

$$\mathcal{P}_j(t) = \mathcal{P}_0(t)\left[1 + \int_{\tau_i(t)}^{t} \sum_{k=1}^{m} p_k(s) \exp\left(\int_{\tau_k(s)}^{t} \mathcal{P}_{j-1}(u)du\right) ds\right], \quad j \ge 1.$$

The proof is complete. □

Lemma 2.3. *For the real-valued function* $f : [0,\infty) \to \mathbb{R}$ *defined as*

$$f(\lambda) = e^{\alpha\lambda} - \lambda$$

the following statements hold:

(i) If $0 < \alpha < 1/e$ *then the equation* $f(\lambda) = 0$ *has exactly two positive roots.*

(ii) If $\alpha = 1/e$ *then the equation* $f(\lambda) = 0$ *has exactly one root,* $\lambda_0 = e$.

(iii) If $\alpha > 1/e$ *then the equation* $f(\lambda) = 0$ *has no roots.*

Proof. (i) Observe that the function $f(\lambda) = e^{\alpha\lambda} - \lambda$ attains its unique minimum at $\lambda = -\ln(\alpha)/\alpha$ which equals $f_{\min} = (1 + \ln\alpha)/\alpha < 0$, since $0 < \alpha < 1/e$. In addition, $f(0) > 0$, $f'(\lambda) < 0$ for all $\lambda \in (0, 1/e)$, and $f'\lambda) > 0$ for all $\lambda > 1/e$. Therefore, the equation $f(\lambda) = 0$ has exactly two positive roots.

(ii) Observe that the function $f(\lambda) = e^{\lambda/e} - \lambda$ attains its unique minimum at $\lambda = e$ which equals $f_{\min} = 0$. In addition, $f(0) > 0$, $f'(\lambda) < 0$ for all $\lambda \in (0, e)$, and $f'(\lambda) > 0$ for all $\lambda > e$. Therefore, the equation $f(\lambda) = 0$ has exactly one positive root.

(iii) For $\alpha > 1/e$, the unique minimum $f_{\min} = (1 + \ln\alpha)/\alpha > 0$, therefore the equation $f(\lambda) = 0$ has no real roots.

The proof of the lemma is complete. □

The next lemma provides a lower estimate for the ratio $x(h(t))/x(t)$ in terms of the smaller root of $\lambda = e^{\alpha\lambda}$; see [15] and [6, Lemma 2.1.2].

Lemma 2.4. *Assume that* (H1) *and* (H2) *hold,* $h(t)$ *is defined by* (1.8)*,* x *is a positive solution of* (1.1) *and*

$$0 < \alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)ds \le \frac{1}{e}. \tag{2.11}$$

Then

$$\liminf_{t\to\infty} \frac{x(h(t))}{x(t)} \ge \lambda_0, \tag{2.12}$$

where λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\alpha\lambda}$.

The next lemma provides a lower estimate for the ratio $x(t)/x(h(t))$ in terms of the smaller root of $d^2 - (1-\alpha)d + \alpha^2/2 = 0$; see [6, Lemma 2.1.3].

Lemma 2.5. *Assume that* (H1) *and* (H2) *hold, $h(t)$ is defined by* (1.8)*, x is a positive solution of* (1.1) *and α is defined by* (2.11)*. Then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \geq \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{2.13}$$

2.2 Advanced differential equations

We study (1.2) under the hypotheses:

(H3) $t \leq \sigma_i(t)$ for $i = 1, 2, \dots, m$, and $t \geq t_0$;

(H4) $xg_i(t,x) \geq 0$ and there exists a continuous non-negative function q_i such that

$$|g_i(t,x)| \geq q_i(t)|x| \quad \forall x \in \mathbb{R},\ t \geq t_0\,.$$

Similar oscillation lemmas for the (dual) advanced differential equation (1.2) can be derived easily. The proofs of these lemmas are omitted, since they are quite similar to the delay equation.

3 Main results

3.1 Delay differential equations

We derive new sufficient oscillation conditions, involving lim sup and lim inf, which essentially improve well-known results in the literature.

Theorem 3.1. *Assume that* (H1) *and* (H2) *hold. If for some $j \geq 0$,*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s) \exp\left(\int_{\tau_i(s)}^{h(t)} \mathcal{P}_j(u)du\right) ds > 1, \tag{3.1}$$

where $h(t)$ is defined by (1.8) *and $\mathcal{P}_j$ by* (2.5)*, then all solutions of* (1.1) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that (1.1) has a non-oscillatory solution x. First we consider eventually positive solutions. Note that the conditions of Lemmas 2.1 and 2.2 are satisfied; thus we have

$$x'(t) + \sum_{i=1}^{m} p_i(t) x\left(\tau_i(t)\right) \leq 0, \tag{3.2}$$

$$x\left(\tau_i(s)\right) \geq x(h(t)) \exp\left(\int_{\tau_i(s)}^{h(t)} \mathcal{P}_j(u)du\right). \tag{3.3}$$

The rest of the proof is similar to the proof of [4, Theorem 1.1]; so we omit it here. □

Theorem 3.2. *Assume that* (H1) *and* (H2) *hold and α is defined by* (2.11)*. If for some $j \geq 0$*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s) \exp\left(\int_{\tau_i(s)}^{t} \mathcal{P}_j(u)du\right) ds > \frac{2}{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}, \tag{3.4}$$

where $h(t)$ is defined by (1.8) *and $\mathcal{P}_j$ by* (2.5)*, then all solutions of* (1.1) *are oscillatory.*

Proof. As in the proof of Lemma 2.2, we have

$$x(s) \geq x(t) \exp\left(\int_s^t \mathcal{P}_j(u)du\right), \qquad 0 \leq s \leq t. \tag{3.5}$$

Integrating (3.2) from $h(t)$ to t, we have

$$x(t) - x(h(t)) + \int_{h(t)}^t \sum_{i=1}^m p_i(s)x(\tau_i(s))ds \leq 0,$$

which in view of (3.5) gives

$$x(t) - x(h(t)) + \int_{h(t)}^t \sum_{i=1}^m p_i(s)x(t)\exp\left(\int_{\tau_i(s)}^t \mathcal{P}_j(u)du\right) ds \leq 0;$$

equivalently

$$x(t) - x(h(t)) + x(h(t))\int_{h(t)}^t \sum_{i=1}^m p_i(s)\frac{x(t)}{x(h(t))}\exp\left(\int_{\tau_i(s)}^t \mathcal{P}_j(u)du\right) ds \leq 0.$$

The strict inequality is valid if we omit $x(t) > 0$ in the left-hand side:

$$-x(h(t)) + x(h(t))\int_{h(t)}^t \sum_{i=1}^m p_i(s)\frac{x(t)}{x(h(t))}\exp\left(\int_{\tau_i(s)}^t \mathcal{P}_j(u)du\right) ds \leq 0,$$

or

$$x(h(t))\left[\frac{x(t)}{x(h(t))}\int_{h(t)}^t \sum_{i=1}^m p_i(s)\exp\left(\int_{\tau_i(s)}^t \mathcal{P}_j(u)du\right) ds - 1\right] < 0.$$

Thus

$$\int_{h(t)}^t \sum_{i=1}^m p_i(s)\exp\left(\int_{\tau_i(s)}^t \mathcal{P}_j(u)du\right) ds < \frac{x(h(t))}{x(t)}$$

and therefore

$$\limsup_{t\to\infty}\int_{h(t)}^t \sum_{i=1}^m p_i(s)\exp\left(\int_{\tau_i(s)}^t \mathcal{P}_j(u)du\right) ds \leq \limsup_{t\to\infty}\frac{x(h(t))}{x(t)}.$$

From the above inequality and the fact that $x(h(t))/x(t)$ is bounded above by 1 and below by the positive bound in Lemma 2.5, we have

$$\limsup_{t\to\infty}\int_{h(t)}^t \sum_{i=1}^m p_i(s)\exp\left(\int_{\tau_i(s)}^t \mathcal{P}_j(u)du\right) ds \leq \frac{2}{1-a-\sqrt{1-2a-a^2}},$$

which contradicts (3.4). The proof is complete. ☐

Theorem 3.3. *Assume that* (H1) *and* (H2) *hold and* α *is defined by* (2.11). *If for some* $j \geq 0$

$$\limsup_{t\to\infty}\int_{h(t)}^t \sum_{i=1}^m p_i(s)\exp\left(\int_{\tau_i(s)}^{h(s)} \mathcal{P}_j(u)du\right) ds > \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{3.6}$$

where $h(t)$ *is defined by* (1.8), $\mathcal{P}_j$ *by* (2.5) *and* λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\alpha\lambda}$, *then all solutions of* (1.1) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that (1.1) has a non-oscillatory solution x. First we consider eventually positive solutions. Note that the conditions of Lemmas 2.1 and 2.2 are satisfied. Clearly (3.5) is satisfied.

By Lemma 2.4, (2.12) implies: for each $\epsilon > 0$ there exists a t_ϵ such that

$$\frac{x(h(t))}{x(t)} > \lambda_0 - \epsilon \quad \text{for all } t \geq t_\epsilon . \tag{3.7}$$

The rest of the proof is as in [4, Theorem 2]. □

Theorem 3.4. *Assume that* (H1) *and* (H2) *hold. If for some* $j \geq 0$,

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s) \exp\left(\int_{\tau_i(s)}^{h(s)} \mathcal{P}_j(u)du \right) ds > \frac{1}{e}, \tag{3.8}$$

where $\mathcal{P}_j$ *is defined by* (2.5) *and* $h(t)$ *by* (1.8), *then all solutions of* (1.1) *are oscillatory.*

The proof of the above lemma is similar to the proof of [5, Theorem 3.2] and is omitted here.

3.2 Advanced differential equations

Similar oscillation theorems for the (dual) advanced differential equation (1.2) can be derived easily. The proofs of these theorems are omitted, since they are quite similar to the ones for delay equations.

Set $Q_0(t) = \sum_{i=1}^{m} q_i(t)$ and

$$Q_j(t) = Q_0(t) \left[1 + \int_{t}^{\sigma_i(t)} \sum_{k=1}^{m} q_k(s) \exp\left(\int_{t}^{\sigma_k(s)} Q_{j-1}(u)du \right) ds \right], \qquad j \geq 1. \tag{3.9}$$

Theorem 3.5. *Assume that* (H3) *and* (H4) *hold. If for some* $j \geq 0$

$$\limsup_{t\to\infty} \int_{t}^{\rho(t)} \sum_{i=1}^{m} q_i(s) \exp\left(\int_{\rho(t)}^{\sigma i(s)} Q_j(u)du \right) ds > 1, \tag{3.10}$$

where $\rho(t)$ *is defined by* (1.13) *and* Q_j *by* (3.9), *then all solutions of* (1.2) *are oscillatory.*

Theorem 3.6. *Assume that* (H3) *and* (H4) *hold and*

$$0 < b := \liminf_{t\to\infty} \int_{t}^{\sigma(t)} \sum_{i=1}^{m} q_i(s)ds \leq \frac{1}{e}. \tag{3.11}$$

If for some $j \geq 0$

$$\limsup_{t\to\infty} \int_{t}^{\rho(t)} \sum_{i=1}^{m} q_i(s) \exp\left(\int_{t}^{\sigma_i(s)} Q_j(u)du \right) ds > \frac{2}{1 - b - \sqrt{1 - 2b - b^2}}, \tag{3.12}$$

where $\rho(t)$ *is defined by* (1.13) *and* Q_j *by* (3.9), *then all solutions of* (1.2) *are oscillatory.*

Theorem 3.7. *Assume that* (H3) *and* (H4) *hold and* b *is defined by* (3.11). *If for some* $j \geq 0$

$$\limsup_{t\to\infty} \int_{t}^{\rho(t)} \sum_{i=1}^{m} q_i(s) \exp\left(\int_{\rho(s)}^{\sigma_i(s)} Q_j(u)du \right) ds > \frac{1 + \ln \lambda_0}{\lambda_0} - \frac{1 - b - \sqrt{1 - 2b - b^2}}{2}, \tag{3.13}$$

where $\rho(t)$ *is defined by* (1.13), Q_j *by* (3.9) *and* λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\alpha\lambda}$, *then all solutions of* (1.2) *are oscillatory.*

Theorem 3.8. *Assume that* (H3) *and* (H4) *hold. If for some* $j \ge 0$,

$$\liminf_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^m q_i(s) \exp\left(\int_{\rho(s)}^{\sigma_i(s)} Q_j(u)du\right) ds > \frac{1}{e}, \tag{3.14}$$

where $\rho(t)$ *is defined by* (1.13) *and* Q_j *by* (3.9)*, then all solutions of* (1.2) *are oscillatory.*

4 Example

Consider the non-linear delay differential equation

$$x'(t) + \frac{1}{3t}\left(1.1x(\tau_1(t)) + 0.1\sin\left(x(\tau_1(t))\right)\right) + \frac{2}{3t}x(\tau_2(t)) = 0, \quad t \ge 1. \tag{4.1}$$

Here $t_0 = 1$, $q_1(t) = 1/(3t)$ and $q_2(t) = 2/(3t)$. Using the constant $a = e^{1/e}$, we define recursively a sequence $\{t_k\}$:

$$\begin{aligned} &t_1 = t_0 + 1, \quad && t_2 = \frac{a+1}{2}t_1, \quad && t_3 = at_1, \quad && t_4 = (2a-1)t_1,\\ &t_5 = t_4 + 1, \quad && t_6 = \frac{a+1}{2}t_5, \quad && t_7 = at_5, \quad && t_8 = (2a-1)t_5, \ \dots \end{aligned}$$

The delayed arguments are defined on $[t_0, t_4]$ as (see Figure 4.1 (a))

$$\tau_1(t) = \begin{cases} t/a, & \text{if } t \in [t_0, t_1],\\ t_1/a - \frac{1}{a^2}(t - t_1), & \text{if } t \in [t_1, t_2],\\ t/a^2, & \text{if } t \in [t_2, t_3],\\ (t_1/a) + 2(t - t_3)/a, & \text{if } t \in [t_3, t_4]; \end{cases}$$

$$\tau_2(t) = \tau_1(t) - 0.1\,.$$

Similar definition are used on the intervals $[t_4, t_8]$, $[t_8, t_{12}]$, …

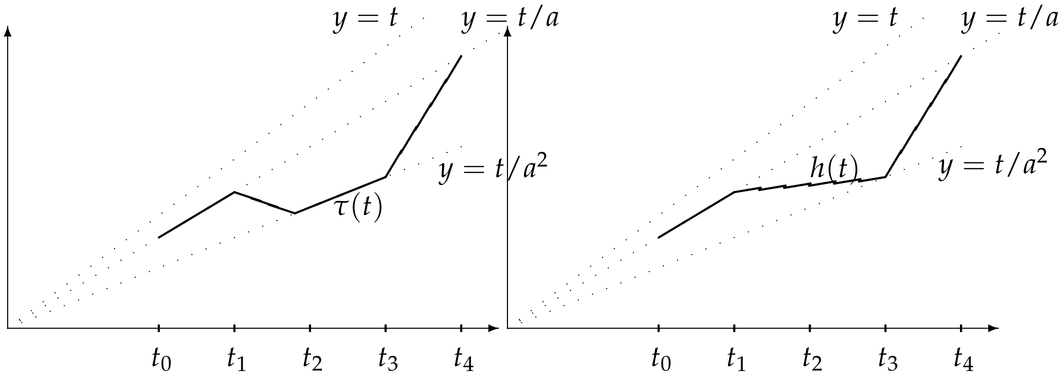

Figure 4.1: Graphs of $\tau_1(t)$ and $h_1(t)$

By (1.8), we see that

$$h_1(t) := \sup_{s\le t} \tau_1(s) = \begin{cases} ta, & \text{if } t \in [t_0, t_1],\\ t_1/a & \text{if } t \in [t_1, t_3],\\ (t_1/a) + 2(t - t_3)/a, & \text{if } t \in [t_3, t_4]; \end{cases}$$

$$h_2(t) = h_1(t) - 0.1$$

and consequently

$$h(t) = \max_{1\le i\le 2}\{h_i(t)\} = h_1(t), \qquad \tau(t) = \max_{1\le i\le 2}\{\tau_i(t)\} = \tau_1(t).$$

(Note that $t/a^2 \le \tau(t) \le h(t) \le t/a < t$). It is easy to see that

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)ds = \lim_{t\to\infty} \int_{t/a}^{t} \frac{1}{s} ds = \ln a = \frac{1}{e}$$

and $\lambda_0 = e$ (see Lemma 2.3). Thus

$$\begin{aligned}
&\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s) \exp\left(\int_{\tau_i(s)}^{h(s)} \mathcal{P}_j(u)du\right) ds \\
&\ge \limsup_{t\to\infty} \int_{h(t)}^{t} \mathcal{P}_0(s)ds = \limsup_{t\to\infty} \int_{t_1/a}^{at_1} \frac{1}{s} ds = \ln a^2 = \frac{2}{e} \\
&\simeq 0.7358 > \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-a-\sqrt{1-2a-a^2}}{2} \simeq 0.598;
\end{aligned}$$

that is, condition (3.6) of Theorem 3.3 is satisfied, and therefore all solutions of (4.1) are oscillatory. However, condition (1.4) is not satisfied.

We remark that similar examples can be constructed to illustrate the other theorems above.

References

[1] R. P. Agarwal, L. Berezansky, E. Braverman, A. Domoshnitsky, *Nonoscillation theory of functional differential equations with applications*, Springer, New York, 2012. https://doi.org/10.1007/978-1-4614-3455-9; MR2908263

[2] H. Akca, G. E. Chatzarakis, I. P. Stavroulakis, An oscillation criterion for delay differential equations with several non-monotone arguments, *Appl. Math. Lett.* **59**(2016), 101–108. https://doi.org/10.1016/j.aml.2016.03.013; MR3494311

[3] E. Braverman, G. E. Chatzarakis, I. P. Stavroulakis, Iterative oscillation tests for differential equations with several non-monotone arguments, *Adv. Difference Equ.* **2016**, Paper No. 87, 18 pp. https://doi.org/10.1186/s13662-016-0817-3; MR3479781

[4] G. E. Chatzarakis, Differential equations with non-monotone arguments: Iterative oscillation results, *J. Math. Comput. Sci.* **6**(2016), No. 5, 953–964.

[5] J. P. Dix, J. G. Dix, Oscillation of solutions to nonlinear first-order delay differential equations, *Involve* **9**(2016), 465–482. https://doi.org/10.2140/involve.2016.9.465; MR3509339

[6] L. H. Erbe, Q. Kong, B. G. Zhang, *Oscillation theory for functional differential equations*, Marcel Dekker, New York, 1995. MR1309905

[7] L. H. Erbe, B. G. Zhang, Oscillation for first order linear differential equations with deviating arguments, *Differential Integral Equations* **1**(1988), 305–314. MR929918

[8] N. Fukagai, T. Kusano, Oscillation theory of first order functional-differential equations with deviating arguments, *Ann. Mat. Pura Appl. (4)* **136**(1984), 95–117. https://doi.org/10.1007/BF01773379; MR0765918

[9] B. R. Hunt, J. A. Yorke, When all solutions of $x'(t) = -\sum q_i(t)x(t - T_i(t))$ oscillate, *J. Differential Equations* **53**(1984), 139–145. https://doi.org/10.1016/0022-0396(84)90036-6; MR0748236

[10] J. Jaroš, I. P. Stavroulakis, Oscillation tests for delay equations, *Rocky Mountain J. Math.* **29**(1999), No. 1, 197–207. https://doi.org/10.1216/rmjm/1181071686 MR1687662

[11] M. Kon, Y. G. Sficas, I. P. Stavroulakis, Oscillation criteria for delay equations, *Proc. Amer. Math. Soc.* **128**(2000), No. 10, 2989–2997. https://doi.org/10.1090/S0002-9939-00-05530-1; MR1694869

[12] R. G. Koplatadze, T. A. Chanturija, Oscillating and monotone solutions of first-order differential equations with deviating argument (in Russian), *Differentsial'nye Uravneniya* **18**(1982), 1463–1465, 1472. MR671174

[13] R. G. Koplatadze, G. Kvinikadze, On the oscillation of solutions of first order delay differential inequalities and equations, *Georgian Math. J.* **3**(1994), 675–685. https://doi.org/10.1007/BF02254685; MR1296574

[14] T. Kusano, On even-order functional-differential equations with advanced and retarded arguments, *J. Differential Equations* **45**(1982), 75–84. https://doi.org/10.1016/0022-0396(82)90055-9 MR0662487

[15] M. K. Kwong, Oscillation of first-order delay equations, *J. Math. Anal. Appl.* **156**(1991), 274–286. https://doi.org/10.1016/0022-247X(91)90396-H; MR1102611

[16] G. S. Ladas, I. P. Stavroulakis, Oscillations caused by several retarded and advanced arguments, *J. Differential Equations* **44**(1982), 134–152. https://doi.org/10.1016/0022-0396(82)90029-8; MR0651691

[17] G. S. Ladde, Oscillations caused by retarded perturbations of first order linear ordinary differential equations, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **63**(1978), 351–359. MR0548601

[18] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 110, Marcel Dekker, Inc., New York, 1987. MR1017244

[19] X. Li, D. Zhu, Oscillation and nonoscillation of advanced differential equations with variable coefficients, *J. Math. Anal. Appl.* **269**(2002), 462–488. https://doi.org/10.1016/S0022-247X(02)00029-X; MR1907126

[20] A. D. Myshkis, Linear homogeneous differential equations of first order with deviating arguments (in Russian), *Uspekhi Mat. Nauk* **5**(1950), 160–162. MR0036423

[21] J. S. Yu, Z. C. Wang, B. G. Zhang, X. Z. Qian, Oscillations of differential equations with deviating arguments, *Panamer. Math. J.* **2**(1992), No. 2, 59–78. MR1160129

[22] B. G. Zhang, Oscillation of solutions of the first-order advanced type differential equations, *Sci. Exploration* **2**(1982), 79–82. MR0713776

[23] D. Zhou, On some problems on oscillation of functional differential equations of first order, *J. Shandong University* **25**(1990), 434–442. Zbl 0726.34060