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Linearizability problem of persistent centers

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Abstract. The concepts of persistent and weakly persistent centers were introduced in 2009 and the same concept was applied in the study of some families of differential equations in 2013. Such concept was generalized for complex planar differential systems in 2014. In this paper we extend the notion of persistent center to a linearizable persistent center and a linearizable weakly persistent center. Using the methods and algorithms of computational algebra we characterize the planar cubic differential system having linearizable persistent and linearizable weakly persistent centers at the origin.

Keywords: linearizability problem, center problem, focus quantities, persistent center.

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1 Introduction

In the qualitative theory of differential systems there are many important open problems, for instance, the existence and number of limit cycles bifurcating from periodic orbits or singular points, problem of distinguishing between center and focus, linearizability problem, problem of critical periods etc. One of demanding problems is also characterizing centers and linearizable centers for a given planar polynomial differential system. A center of an analytic system is called a linearizable center if and only if there is an analytic change of coordinates which brings the original system to a linear system. To obtain conditions for linearizable centers one can compute the so called linearizability quantities whose vanishing ensures necessary and sufficient conditions for a center being linearizable (see Section 2 for more details). In spite

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of the capability of nowadays computers only the first few linearizability quantities can be computed. The problem also becomes more difficult with increasing of the number of parameters of the considered system. Thus, linearizability problem as well as also center problem only for some special families of polynomial systems have been investigated (see for instance [2,3,7,9–15,18,27,28,30,31]).

In [4] the authors generalize the concept of a center by introducing the concept of persistent and weakly persistent centers for real differential systems. Such kind of problem is equivalent to find the systems with centers which are structurally stable under perturbations inside the same class of systems (i.e. polynomial systems, analytic systems, smooth systems, etc). The same concept also appears in [5] in the study of some Abel equations. In [3] the authors generalized the notion of persistent center and weakly persistent center to complex planar differential systems and found all conditions to get a persistent center in planar cubic systems and all conditions to get a weakly persistent center for complex cubic Lotka–Volterra system.

In this paper we introduce the concepts of linearizable persistent and linearizable weakly persistent centers and find necessary and sufficient conditions for planar cubic systems to have a linearizable persistent center. We also present the necessary and sufficient conditions for the existence of a linearizable weakly persistent center for the family of systems which define a kind of a generalization of the Kolmogorov systems.

2 Preliminaries

For planar analytic differential systems of the form

$$\dot{u} = -v + P(u,v), \quad \dot{v} = u + Q(u,v), \tag{2.1}$$

where $u, v \in \mathbb{R}$ and P, Q are analytic functions whose series expansions start from degree at least two, it is well-known that the origin can be either a focus or a center. The problem to distinguish between a center or a focus is called the center-focus problem. The origin $O = (0,0)$ is called a center of system (2.1) if it is surrounded by a family of periodic orbits. By the Poincaré–Lyapunov theorem [20,25] O is a center of (2.1) if and only if it admits a first integral of the form

$$\Phi(u,v) = u^2 + v^2 + \sum_{j+k\geq 3} \phi_{j,k} u^j v^k.$$

One of the main difficulties in the study of the center problem comes from the complexity in computing the irreducible decomposition of the affine variety* of the ideal generated by the Lyapunov quantities, which are the coefficients of the Poincaré first return map (see e.g. [29]). Since it is easier to study complex varieties than real ones it is common to complexify the real system as follows. First setting $x = u + iv$ from system (2.1) we obtain the equation

$$\dot{x} = ix + F(x,\overline{x}), \tag{2.2}$$

where $i = \sqrt{-1}$, and $F(x,\overline{x}) = (P + iQ)((x+\overline{x})/2, (x-\overline{x})/2i)$. Adjoining to equation (2.2) its complex conjugate we get the system

$$\dot{x} = ix + F(x,\overline{x}), \quad \dot{\overline{x}} = -i\overline{x} + \overline{F(x,\overline{x})}.$$

*The affine variety of an ideal $I = \langle f_1, \dots, f_s \rangle \in k[x_1, \dots, x_n]$, where k is a field is the set $\mathbf{V}(I) = \{\mathbf{a} = (a_1, \dots, a_n) \in k^n : f_j(\mathbf{a}) = 0 \text{ for each } f_j \in I\}$.

Consider $y := \overline{x}$ as a new variable and $G = \overline{F(x, \overline{x})}$ as a new function we obtain a system of two complex differential equations which we write in the form

$$\dot{x} = ix + F(x, y), \quad \dot{y} = -iy + G(x, y), \tag{2.3}$$

where $x, y \in \mathbb{C}$ and F and G are analytic complex functions whose series expansions start from terms of degree at least two.

The next definition is natural in view of the Poincaré–Lyapunov theorem and the complexification procedure described above.

Definition 2.1 ([29])**.** System (2.3) has a complex center at the origin if it admits an analytic first integral of the form

$$\Psi(x, y) = xy + \sum_{j+k \geq 3} \varphi_{j,k} x^j y^k. \tag{2.4}$$

There are several generalizations of the classical center-focus problem. One of these notions is the following. The singular point O in a complex systems of the form

$$\dot{x} = px + P(x, y), \quad \dot{y} = -qy + Q(x, y),$$

where $x, y \in \mathbb{C}$, $p, q \in \mathbb{N}$, and $P, Q \in \mathbb{C}[x, y]$ is called a $p : -q$ resonant center if there exists a local analytic first integral of the form

$$\Phi(x, y) = x^q y^p + \sum_{j+k > p+q+1} \phi_{j-q,k-p} x^j y^k.$$

Some results for $p : -q$ resonant centers can be found in [2,9,10,12,13,21,22,28,33]. A different approach to similar systems is presented in [26]. Some other generalizations are discussed also in [11] and [23]. The classical center-focus problem for a class of cubic systems was recently considered in [31].

Another generalization of the concept of center to real system was presented by [4].

Definition 2.2 ([4])**.** The origin of (2.2) is a persistent center (respectively, weakly persistent center) if it is a center for

$$\dot{x} = ix + \lambda F(x, \overline{x}), \quad x \in \mathbb{C}$$

for all $\lambda \in \mathbb{C}$ (respectively, $\lambda \in \mathbb{R}$).

Such concept was extended to the complex case by [3].

Definition 2.3 ([3])**.** The origin of (2.3) is a persistent center (respectively, weakly persistent center) if it is a center for

$$\dot{x} = ix + \lambda F(x, y), \quad \dot{y} = -iy + \mu G(x, y), \quad x, y \in \mathbb{C} \tag{2.5}$$

for all $\lambda, \mu \in \mathbb{C}$ (respectively, $\lambda = \mu \in \mathbb{C}$).

In [4] authors found some general systems of type (2.2) where the origin is a persistent center: a) $\dot{x} = ix + Ax^2 + C\overline{x}^2$ (quadratic), b) $\dot{x} = ix + F(x)$, with $F(0) = F'(0) = 0$ (holomorphic), c) $\dot{x} = ix + F(\overline{x})$, with $F(0) = F'(0) = 0$ (Hamiltonian), d) $\dot{x} = ix + x\overline{x}F(\overline{x})$ (separated), and e) $\dot{x} = ix + Bx^k\overline{x}^l\Psi(x\overline{x})$, with $k \neq l + 1$ (reversible), where $A, B, C \in \mathbb{C}$ and Ψ a real analytic function such that the series expansion for $x^k\overline{x}^l\Psi(x\overline{x})$ starts with second order terms. Furthermore, for some special cases of (2.2), where

$$F(x, \overline{x}) = Ax^2 + Bx\overline{x} + C\overline{x}^2 + Dx^3 + Ex^2\overline{x} + Fx\overline{x}^2 + G\overline{x}^3,$$

the origin is proven to be a persistent center (see [4, Theorem 1.2]). This result was generalized in [3] to systems (2.3), see [3, Theorem 2.2].

If the singular point at the origin of system (2.1) is known to be a center we say that this center is *isochronous* if all periodic solutions of (2.1) in a neighbourhood of the origin have the same period. Moreover, system (2.1) is said to be *linearizable* if there is an analytic change of coordinates

$$u = u_1 + Z(u_1, v_1), \quad v = v_1 + W(u_1, v_1),$$

that reduces (2.1) to the canonical linear center

$$\dot{u_1} = -v_1, \quad \dot{v_1} = u_1.$$

Of course that there are similar definition for complex systems (2.3). The following theorem (see e.g. [29] for the details) shows us that there is an intimate relation between linearizability and isochronicity.

Theorem 2.4. *Assume that the origin of* (2.1) *is a center. Then the origin is an isochronous center if and only if system* (2.1) *is linearizable.*

It follows from Theorem 2.4 that solving the isochronicity problem is equivalent to solving the linearizability problem which is, from a computational point of view, much simpler.

Generalizing these concepts we introduce some new definitions concerning linearizability of persistent and weakly persistent centers of complex system (2.3).

Definition 2.5. The origin O is called an *linearizable persistent center* of system (2.3) if it is a persistent center and there exists an analytic change of coordinates of the form

$$x = x_1 + \sum_{j+k\geq 2} c_{j,k} x_1^j y_1^k, \quad y = y_1 + \sum_{j+k\geq 2} d_{j,k} x_1^j y_1^k,$$

that reduces (2.5) to the system

$$\dot{x}_1 = i x_1, \quad \dot{y}_1 = -i y_1 \tag{2.6}$$

for all $\lambda, \mu \in \mathbb{C}$.

Definition 2.6. The origin O is called a *linearizable weakly persistent center* of system (2.3) if it is a weakly persistent center and system (2.5) is linearizable for all $\lambda, \mu \in \mathbb{C}$.

The main goal of this paper is to study the linearizability of some planar cubic differential systems. In Section 3 some computational techniques for the linearizability of persistent centers and wealky persistent centers are given. Applying these techniques and some other ideas, in Section 4 we present two main results. First, in Subsection 4.1 we give the necessary and sufficient conditions for the existence of an linearizable persistent center for the family of systems of the form

$$\begin{aligned} \dot{x} &= ix + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}y^2x + a_{03}y^3, \\ \dot{y} &= -iy + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}y^2x + b_{03}y^3, \end{aligned} \tag{2.7}$$

and then, in Subsection 4.2 we present the necessary and sufficient conditions for the existence of an linearizable weakly persistent center for the family of systems which define a kind of a generalization of the Kolmogorov systems (which we call "semi-Kolmogorov" systems), i.e. systems of the form

$$\begin{aligned} \dot{x} &= ix + x\left(a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2\right), \\ \dot{y} &= -iy + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3. \end{aligned} \tag{2.8}$$

3 Linearizability of persistent centers

In this section a general approach used to study the classical linearizability problem and the linearizability problem for persistent centers and weakly persistent centers for complex systems (2.3) is described. There are different algorithms to find the necessary conditions for linearizability (see e.g. [7,27,29]). Here we present a brief description of one of these algorithms. The first step is the calculation of the linearizability quantities, denoted by i_k and j_k, which are polynomials in coefficients of $F(x,y)$ and $G(x,y)$. For example, in case of (2.7) this means $i_k, j_k \in \mathbb{C}[a,b]$, where $a = (a_{20}, a_{11}, a_{02}, a_{30}, a_{21}, a_{12}, a_{03})$ and $b = (b_{20}, b_{11}, b_{02}, b_{30}, b_{21}, b_{12}, b_{03})$. For any polynomial system (2.3) we use the parameters notation (a,b) in a similar way. We want to decide whether system (2.3) can be transformed into the linear system (2.6) by means of the inverse linearizing transformation

$$x_1 = x + \sum_{m+j=2}^{\infty} c_{m-1,j}(a,b)\, x^m y^j, \qquad y_1 = y + \sum_{m+j=2}^{\infty} d_{m-1,j}(a,b)\, x^m y^j. \tag{3.1}$$

If such a transformation exists, we say that the system is *linearizable*. Differentiation of each part of (3.1) with respect to t, applying (2.3) and (2.6), and then equating the coefficients of the terms $x^{q_1+1}y^{q_2}$ and $x^{q_1}y^{q_2+1}$ yields the following recurrence formulae [29],

$$(q_1 - q_2)\, c_{q_1,q_1} = \sum_{s_1+s_2=0}^{q_1+q_2-1} \left[(s_1+1)\, c_{s_1,s_2} a_{q_1-s_1,q_2-s_2} - s_2 c_{s_1,s_2} b_{q_1-s_1,q_2-s_2} \right], \tag{3.2}$$

$$(q_1 - q_2)\, d_{q_1,q_1} = \sum_{s_1+s_2=0}^{q_1+q_2-1} \left[s_1 d_{s_1,s_2} a_{q_1-s_1,q_2-s_2} - (s_2+1)\, d_{s_1,s_2} b_{q_1-s_1,q_2-s_2} \right], \tag{3.3}$$

where $s_1, s_2 \geq -1$, $q_1, q_2 \geq -1$, $q_1 + q_2 \geq 0$, $c_{1,-1} = c_{-1,1} = d_{1,-1} = d_{-1,1} = 0$, $c_{0,0} = d_{0,0} = 1$, and we set $a_{p,q} = b_{p,q} = 0$ if $p + q < 1$. Using the recurrence formulae (3.2), (3.3) one can compute the coefficients c_{q_1,q_2}, d_{q_1,q_2}, where $q_1 \neq q_2$. For $q_1 = q_2 = k$ the coefficients $c_{k,k}$, $d_{k,k}$ may be chosen arbitrary [29]. We set $c_{k,k} = d_{k,k} = 0$. The system is linearizable if and only if all the polynomials for $q_1 = q_2 = k \in \mathbb{N}$ on the right hand side of (3.2), called i_k, and on the right hand side of (3.3), called j_k, are equal to zero. The quantities i_k and j_k are called *linearizability quantities* for polynomial system (2.3). This means that system (2.3) is linearizable if and only if

$$i_k(a,b) = j_k(a,b) = 0, \qquad \forall k \in \mathbb{N}.$$

In the space of the parameters of a given polynomial family of systems (2.3) the set of all linearizable systems is the affine variety of the ideal $\langle i_1, j_1, i_2, j_2, \ldots \rangle$, i.e. $\mathbf{V}(\langle i_1, j_1, i_2, j_2, \ldots \rangle)$.

Now, we restrict our attention to the cubic polynomial systems (2.7) and its respective family

$$\begin{aligned} \dot{x} &= ix + \lambda\left(a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}y^2x + a_{03}y^3\right), \\ \dot{y} &= -iy + \mu\left(b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}y^2x + b_{03}y^3\right). \end{aligned} \tag{3.4}$$

Obviously for any fixed λ and μ one can compute $i_k = i_k(\lambda, \mu, a, b)$ and $j_k = j_k(\lambda, \mu, a, b)$ for system (3.4) and obtain

$$i_k = \sum_{m,n} i_k^{(m,n)}(a,b)\lambda^m \mu^n, \qquad j_k = \sum_{m,n} j_k^{(m,n)}(a,b)\lambda^m \mu^n. \tag{3.5}$$

We look at these linearizability quantities as polynomials in λ and μ and denote the coefficient of $\lambda^m\mu^n$ as $i_k^{(m,n)}$ and $j_k^{(m,n)}$ for $i_k(\lambda,\mu,a,b)$ and $j_k(\lambda,\mu,a,b)$, respectively, and call it the $k_{(m,n)}-$th *persistent linearizability quantity* (according to i_k and j_k). If the origin is a linearizable center of (3.4) for all $\lambda,\mu\in\mathbb{C}$, then it is by Definition 2.5 a linearizable persistent center of (2.7).

Note, that by Theorem 2.1 of [3] one can always assume that $\lambda\mu\neq 0$ in (3.4). This gives rise to define the following sets of polynomials

$$L_k=\left\{i_k^{(m,n)},\, j_k^{(m,n)};\quad m,n\in\mathbb{N}_0,\; m+n\neq 0\right\},\quad k=1,2,3,\ldots$$

and the ideals

$$\begin{aligned}\mathcal{L}^p &:= \langle L_1,L_2,\ldots,L_k,\ldots\rangle\,,\\ \mathcal{L}_k^p &:= \langle L_1,L_2,\ldots,L_k\rangle\,.\end{aligned}$$

According to Definition 2.3 setting $\lambda=\mu$ to (2.5) one can consider the linearizability of a possibly weakly persistent center. Now, one can consider $i_k(\lambda,a,b)$ and $j_k(\lambda,a,b)$ and define the coefficients, $i_k^{(m)}$, $j_k^{(m)}$, corresponding to λ^m in the series expansion of the linearizabilizy quantities

$$i_k=\sum_m i_k^{(m)}(a,b)\lambda^m,\quad j_k=\sum_m j_k^{(m)}(a,b)\lambda^m, \tag{3.6}$$

and call them $k_{(m)}-$th *weakly persistent linearizability quantities*. Next, defining the following sets of polynomials

$$L_k^w=\left\{i_k^{(m)},\, j_k^{(m)};\quad m\in\mathbb{N}\right\},\quad k=1,2,3,\ldots$$

one can define the ideals

$$\begin{aligned}\mathcal{L}^{wp} &:= \langle L_1^w,L_2^w,\ldots,L_k^w,\ldots\rangle\,,\\ \mathcal{L}_k^{wp} &:= \langle L_1^w,L_2^w,\ldots,L_k^w\rangle\,.\end{aligned}$$

Note that the ideals $\mathcal{L}^{wp}$, $\mathcal{L}_k^{wp}$, $\mathcal{L}^p$ and $\mathcal{L}_k^p$ are ideals in the polynomial ring $\mathbb{C}[a,b]$. By the Hilbert Basis Theorem (see e.g. [6]) any of them is finitely generated and every ascending chain of ideals $\mathcal{L}_1^p\subset\mathcal{L}_2^p\subset\mathcal{L}_3^p\subset\cdots$ and $\mathcal{L}_1^{wp}\subset\mathcal{L}_2^{wp}\subset\mathcal{L}_3^{wp}\subset\cdots$ stabilizes, which means that there exists $N\geq 1$ such that for every $k>N$, $\mathcal{L}_k^p=\mathcal{L}_N^p$. Now one can define the corresponding persistent and weakly persistent linearizability variety for systems (2.5) in a natural way

$$V_{\mathcal{L}^p}=\mathbf{V}\left(\mathcal{L}^p\right),\quad V_{\mathcal{L}^{wp}}=\mathbf{V}\left(\mathcal{L}^{wp}\right).$$

In the rest of the work we will search for $V_{\mathcal{L}^p}$ for systems (3.4) and $V_{\mathcal{L}^{wp}}$ for systems (3.4) with $a_{02}=a_{03}=0$. Note that the persistent center variety and the persistent linearizability variety is much easier to obtain than the (regular) center variety and (regular) linearizability variety for a cubic system (2.7) since the corresponding persistent linearizability quantities, $i_k^{(m,n)}$, $j_k^{(m,n)}$, are split compared to (regular) linearizability quantities $i_k(\lambda,\mu,a,b)$ and $j_k(\lambda,\mu,a,b)$.

In Figure 3.1 the relation between the varieties of (regular) centers V_c, weakly persistent centers V_{wpc}, persistent centers V_{pc}, (regular) linearizable centers V_l, linearizable weakly persistent centers, and linearizable persistent centers for system (2.7) is presented.

In the next two propositions systems (2.3) and (2.5) are quasi-homogeneous systems of degree n, i.e. there exists $n\geq 2$ such that $F(tx,ty)=t^nF(x,y)$ and $G(tx,ty)=t^nG(x,y)$, $\forall x,y,t$.

Proposition 3.1. *If system* (2.3) *is quasi-homogeneous of degree n, $n \geq 2$, then the origin is a weakly persistent center if and only if it is a center.*

See [3, pp. 114–115], for the proof.

Proposition 3.2. *If system* (2.3) *is quasi-homogeneous of degree n, $n \geq 2$, then the origin is a linearizable weakly persistent center if and only if it is a linearizable center, i.e, $V_{\mathcal{L}^{wp}} = V_{\mathcal{L}}$, where $V_{\mathcal{L}^{wp}}$ and $V_{\mathcal{L}}$ are the weak persistent linearizability variety and the linearizability variety, respectively, of system* (2.3).

Proof. Under the assumption of the proposition we write systems (2.3) and (2.5) for $\lambda = \mu$ in the form

$$\dot{x} = ix + F_n(x,y), \quad \dot{y} = -iy + G_n(x,y), \tag{3.7}$$

and

$$\dot{x} = ix + \lambda F_n(x,y), \quad \dot{y} = -iy + \lambda G_n(x,y), \tag{3.8}$$

respectively, where $n \geq 2$, $F_n(x,y)$ and $G_n(x,y)$ are homogeneous polynomials of degree n in x and y.

We shall prove that system (3.7) is equivalent to system (3.8), up to a linear change of variables, so both systems must have the same linearizability varieties. In fact, for any $\gamma \neq 0$ consider the linear change of variables

$$X = \gamma x, \quad Y = \gamma y.$$

Applying this change to system (3.7) and using the homogeneity of F_n and G_n we obtain the system

$$\dot{X} = iX + \left(\frac{1}{\gamma}\right)^n F_n(X,Y), \quad \dot{Y} = -iY + \left(\frac{1}{\gamma}\right)^n G_n(X,Y).$$

Setting $\lambda = \left(\frac{1}{\gamma}\right)^n$ we arrive to (3.8), which completes the proof. □

Now let us consider the relation between systems

$$\dot{x} = Ax + \mathbf{X}(x), \tag{3.9}$$

and

$$\dot{x} = Ax + \lambda \mathbf{X}(x), \tag{3.10}$$

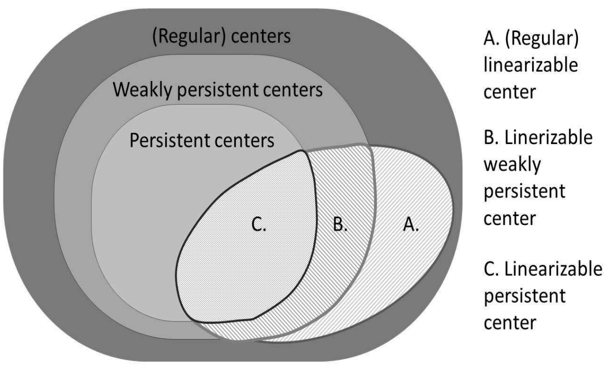

Figure 3.1: The relation between the varieties V_c, V_{pc}, V_{wpc} and V_l, $V_{\mathcal{L}^p}$, $V_{\mathcal{L}^{wp}}$ $V_{\mathcal{L}^p} \subset V_{\mathcal{L}^{wp}} \subset \mathbb{C}^{14}$ of the cubic system (2.7).

in terms of the linearizing transformation, where $x \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, A is a complex matrix and $\mathbf{X} : \mathbb{C} \to \mathbb{C}$ is an analytic map starting with quadratic terms. Suppose that (3.9) can be linearized to

$$\dot{y} = Ay, \tag{3.11}$$

by the linearizing transformation

$$x = y + \mathbf{h}(y).$$

Suppose that $\sigma_A = \{\kappa_1, \kappa_2, \ldots, \kappa_n\}$ and $A = \operatorname{diag}(\kappa_1, \kappa_2, \ldots, \kappa_n)$ (i.e. A is in the Jordan form). We follow the text in ([29, Sec. 2.2]) and denote

$$\begin{aligned}\mathbf{h}(y) &= \mathbf{h}^{(2)}(y) + \mathbf{h}^{(3)}(y) + \cdots + \mathbf{h}^{(k)}(y) + \cdots \\ \mathbf{X}(x) &= \mathbf{X}^{(2)}(x) + \mathbf{X}^{(3)}(x) + \cdots + \mathbf{X}^{(k)}(x) + \cdots\end{aligned}$$

and write $\mathcal{H}_k$ for the linear space of all possible (vectorial) monomials. For example for $x = (x_1, x_2) \in \mathbb{C}^2$ we have

$$\mathcal{H}_2 = \left\{ \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2^2 \end{pmatrix} \right\}.$$

The basis elements of $\mathcal{H}_k$ are usually written as $\left(x_1^{\alpha_1}x_2^{\alpha_2}, 0\right)^T$ and $\left(0, x_1^{\alpha_1}x_2^{\alpha_2}\right)^T$. Denote

$$\begin{aligned}\kappa &= (\kappa_1, \kappa_2, \ldots, \kappa_n), \\ \alpha &= (\alpha_1, \alpha_2, \ldots, \alpha_n).\end{aligned}$$

Now we define the homological operator $\mathcal{L} : \mathcal{H}_k \to \mathcal{H}_k$ as

$$\mathcal{L} : \mathbf{p}(y) \longmapsto d\mathbf{p}(y)\,Ay - A\mathbf{p}(y),$$

where $d\mathbf{p}$ is the Jacobian matrix of $\mathbf{p}$. In linear space $\mathcal{H}_k$ the eigenvalues, λ_k, of $\mathcal{L}$ are defined by

$$\lambda_m = \alpha \cdot \kappa - \kappa_m.$$

Any monomial $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$ in the m-th component on the right hand side (RHS) of the systems (3.9) and (3.10) for which $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfies the equation $\alpha \cdot \kappa - \kappa_m = 0$ is said to be *resonant*.

Denote by $\mathcal{K}_k$ the complement to Image($\mathcal{L}$) in $\mathcal{H}_k$. Note that resonant terms can not be removed from the normal form. They are elements of $\mathcal{K}_k$. Suppose that $\mathcal{K}_k = \varnothing$ for any $k \geq 2$ for system (3.9). According to Theorem 2.2.3 of [29, Sec. 2.2] if for a system (3.9) for all $k \geq 2$ one has $\mathcal{K}_k = \varnothing$, then this system can be linearized to (3.11).

In the following lemma we find the relation between the linearizing transformation of (3.10) and the linearizing transformation of (3.9). By a let us denote all the parameters in $\mathbf{X}(x)$ from the RHS of (3.9) and (3.10). Note that $\mathbf{h} = \mathbf{h}(a, y)$.

Lemma 3.3. *Suppose that* $\mathbf{h} = \mathbf{h}(a, y)$ *is a linearizing transformation of* (3.9). *Then* $\widetilde{\mathbf{h}} = \mathbf{h}(\lambda a, y)$ *is the linearizing transformation of* (3.10).

Proof. Suppose that for all $\mathcal{H}_k$, $k \geq 2$, the corresponding set $\mathcal{K}_k$ is empty for (3.9) and that

$$d\mathbf{h}^{(k)}(a, y)\,Ay - A\mathbf{h}^{(k)}(a, y) = \mathbf{X}^{(k)}(y), \tag{3.12}$$

for all y, then

$$\mathbf{h}(a, y) = \mathbf{h}^{(2)}(a, y) + \mathbf{h}^{(3)}(a, y) + \cdots + \mathbf{h}^{(k)}(a, y) + \cdots.$$

It is obvious that systems (3.9) and (3.10) have the same resonant terms. Therefore, there exist also the linearizing transformation for (3.10) which we denote by $\widetilde{\mathbf{h}}$. For $\widetilde{\mathbf{h}}$ in $\mathcal{H}_k$ the set $\mathcal{K}_k$ is empty and $d\widetilde{\mathbf{h}}^{(k)}(y)\,Ay - A\widetilde{\mathbf{h}}^{(k)}(y) = \lambda \mathbf{X}^{(k)}(y)$ for all y.

Next we prove that $\widetilde{\mathbf{h}}$ can be set to the form $\widetilde{\mathbf{h}} = \mathbf{h}(\lambda a, y)$. First note that for all $k \geq 2$ and for all y and $\lambda \neq 0$ we have

$$\mathbf{h}^{(k)}(\lambda a, y) = \frac{1}{\lambda^{k-1}}\mathbf{h}^{(k)}(a, \lambda y) \quad \text{and} \quad \mathbf{X}^{(k)}(y) = \frac{1}{\lambda^k}\mathbf{X}^{(k)}(\lambda y). \tag{3.13}$$

Substituting y by λy in equation (3.12) and dividing it by λ^{k-1} yields

$$\frac{1}{\lambda^{k-2}} d\mathbf{h}^{(k)}(a, \lambda y) \cdot \frac{1}{\lambda} A\lambda y - \frac{1}{\lambda^{k-1}} A\mathbf{h}^{(k)}(a, \lambda y) = \frac{\lambda}{\lambda^k}\mathbf{X}^{(k)}(\lambda y).$$

Using equations (3.13) we rewrite the above equation in the form

$$d\mathbf{h}^{(k)}(\lambda a, y)\,Ay - A\mathbf{h}^{(k)}(\lambda a, y) = \lambda \mathbf{X}^{(k)}(y),$$

which proves that $\mathbf{h}(\lambda a, y)$ is a linearizing transformation for system (3.10). □

Finally, we recall some notions about the Darboux linearizability and integrability theory which is the main tool for proving the sufficiency of the main Theorems 4.1 and 4.2 of this paper. For (2.3) the *algebraic partial integral* f and its *cofactor* k are polynomials defined by the equation

$$\frac{\partial f}{\partial x}(ix + F(x,y)) + \frac{\partial f}{\partial y}(-iy + G(x,y)) = k \cdot f,$$

where the degree of the cofactor k is at most one less than the degree of system (2.3). We are looking for algebraic partial integrals $f_0, f_1, \ldots, f_s$ and $g_0, g_1, \ldots, g_t$ and their cofactors $k_0, k_1, \ldots, k_s$ and $l_1, l_2, \ldots, l_t$ and, in particular, for constants $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t$ such that

$$k_0 + \alpha_1 k_1 + \cdots + \alpha_s k_s = i, \tag{3.14}$$

and

$$l_0 + \beta_1 l_1 + \cdots + \beta_t l_t = -i. \tag{3.15}$$

If such algebraic partial integrals, cofactors and constants are found, the linearizing transformation (3.1) is of the form

$$x_1 = f_0 f_1^{\alpha_1} \cdots f_t^{\alpha_t}, \quad y_1 = g_0 g_1^{\beta_1} \cdots g_t^{\beta_t}. \tag{3.16}$$

Sometimes we can not find enough algebraic partial integrals to construct both Darboux linearizing transformations in (3.16). Let say that we can find only transformation x_1 which linearizes first equation of (2.3). In such case we can use first integral of (2.3). If we can find or at least prove the existence of first integral Ψ of the form (2.4) then the second equation of system (2.3) can be linearized by transformation

$$y_1 = \frac{\Psi}{x_1}. \tag{3.17}$$

In some cases the first integral can be also found (or be proven to exist) using the Darboux theory. If we can find a solution to the equation

$$\sum_{j=1}^{s} \psi_j K_j = 0, \tag{3.18}$$

where $K_1, \ldots, K_s$ are cofactors of some partial first integrals $p_1, \ldots, p_s$ of system (2.3) then

$$\Psi(x,y) = p_1^{\psi_1} p_2^{\psi_2} \cdots p_s^{\psi_s},$$

is the first integral of (2.3), which is called Darboux first integral.

If first integral $\Psi(x,y)$ for (2.3) can not be found as described above, we can look for a differentiable function $\mu(x,y)$ defined on an open set $\Omega \subseteq \mathbb{C}^2$, called an integrating factor. In particular, for (2.3) the integrating factor is any solution (defined on Ω) to the following partial differential equation

$$\frac{\partial \mu(x,y)}{\partial x}(ix + F(x,y)) + \frac{\partial \mu(x,y)}{\partial y}(-iy + G(x,y)) \equiv -\mu(x,y)\left(\frac{\partial(ix + F(x,y))}{\partial x} + \frac{\partial(-iy + G(x,y))}{\partial y}\right).$$

If we have s algebraic partial first integrals $p_1, \ldots, p_s$ with the corresponding cofactors $K_1, \ldots, K_s$ and we are able to find constants $\gamma_1, \ldots, \gamma_s$ such that

$$\sum_{j=1}^{s} \gamma_j K_j + \frac{\partial(ix + F(x,y))}{\partial x} + \frac{\partial(-iy + G(x,y))}{\partial y} \equiv 0, \tag{3.19}$$

then

$$\mu = p_1^{\gamma_1} \cdots p_s^{\gamma_s},$$

is an integrating factor of (2.3), which is called the Darboux integrating factor. Using the obtained integrating factor we can then construct first integral of system (2.3). More details about Darboux integrability and linearizability an interesting reader can find in [29].

4 Main results

In this section we characterize the linearizable persistent center of type (2.7) and linearizable weakly persistent center of type (2.8)

4.1 Linearizable persistent centers of system (2.7)

Theorem 4.1. *System (2.7) has a linearizable persistent center at the origin if and only if one of the following conditions holds:*

(1) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{30} = a_{21} = a_{20} = 0$,

(2) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{30} = a_{21} = a_{11} = 0$,

(3) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{21} = a_{12} = a_{11} = a_{03} = a_{02} = 0$,

(4) $b_{12} = b_{11} = b_{03} = a_{21} = a_{12} = a_{11} = a_{03} = a_{02} = 0$,

(5) $b_{12} = b_{03} = b_{02} = a_{21} = a_{12} = a_{11} = a_{03} = a_{02} = 0$.

Proof. The linearizability quantities (3.5) of a complex cubic system (3.4) are polynomials in λ and μ with coefficients $i_k^{(m,n)}(a,b)$, $j_k^{(m,n)}(a,b)$ being polynomials in a and b. So (2.7) has a linearizable center at the origin for all $\lambda, \mu \in \mathbb{C}$ if and only if all polynomials $i_k^{(m,n)}(a,b)$'s and $j_k^{(m,n)}(a,b)$'s vanish (i.e. $(a,b) \in V_{\mathcal{L}^p}$). We compute the first seven pairs, i_1, $j_1, \dots, i_7$, j_7, of linearizability quantities (3.5) of system (3.4). As the quantities are very large we present below only the first pair

$$\begin{aligned} i_1 &= a_{21}\lambda + ia_{11}a_{20}\lambda^2 - ia_{11}b_{11}\lambda\mu - \frac{2}{3}ia_{02}b_{20}\lambda\mu, \\ j_1 &= b_{12}\mu + ia_{11}b_{11}\lambda\mu + \frac{2}{3}ia_{02}b_{20}\lambda\mu - ib_{02}b_{11}\mu^2. \end{aligned}$$

The next computational step is to compute the irreducible decomposition of the variety of ideal $\mathcal{L}^p = \langle L_1, L_2, \dots \rangle$. We use the routine *minAssGTZ*[**] [8] of the computer algebra system SINGULAR [17]. Performing computations the same decomposition is obtained for $\mathbf{V}(\mathcal{L}_5^p)$, $\mathbf{V}(\mathcal{L}_6^p)$ and $\mathbf{V}(\mathcal{L}_7^p)$, but for $\mathbf{V}(\mathcal{L}_4^p)$ a different decomposition is obtained. This lead us to expect that $\mathbf{V}(\mathcal{L}^p) = \mathbf{V}(\mathcal{L}_5^p)$.

The decomposition of $\mathbf{V}(\mathcal{L}_5^p)$ consists of five components listed in Theorem 4.1 and these are necessary conditions for a linearizable persistent center at the origin of system (2.7).

Now we need to prove that each of these five conditions is also sufficient.

Case (1). Systems (2.7) and (3.4) are

$$\dot{x} = ix + a_{11}xy + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3, \quad \dot{y} = -iy + b_{02}y^2 + b_{03}y^3, \tag{4.1}$$

and

$$\dot{x} = ix + \lambda(a_{11}xy + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3), \quad \dot{y} = -iy + \mu(b_{02}y^2 + b_{03}y^3), \tag{4.2}$$

respectively. System (4.2) has partial first integrals

$$\begin{aligned} g_0 &= y, \\ g_1 &= 1 + \frac{1}{2}i\left(\mu b_{02} - \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y, \\ g_2 &= 1 + \frac{1}{2}i\left(\mu b_{02} + \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y, \end{aligned}$$

with respective cofactors

$$\begin{aligned} l_0 &= -i + \mu b_{02}y + \mu b_{03}y^2, \\ l_1 &= \frac{1}{2}\left(\mu b_{02} - \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y + \mu b_{03}y^2, \\ l_2 &= \frac{1}{2}\left(\mu b_{02} + \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y + \mu b_{03}y^2. \end{aligned}$$

It is easy to verify that equation (3.15) is satisfied for

$$\beta_1 = \frac{\sqrt{\mu}b_{02} - \sqrt{4ib_{03} + \mu b_{02}^2}}{2\sqrt{4ib_{03} + \mu b_{02}^2}} \quad \text{and} \quad \beta_2 = -\frac{\sqrt{\mu}b_{02} + \sqrt{4ib_{03} + \mu b_{02}^2}}{2\sqrt{4ib_{03} + \mu b_{02}^2}},$$

so according to (3.16) we obtain that the second equation of (4.2) is linearizable by the change of coordinates

$$y_1 = y g_1^{\beta_1} g_2^{\beta_2}.$$

[**] *minAssGTZ* computes decomposition of the affine variety of the corresponding ideal into the irreducible components using the algorithm of [16].

In order to find a change of coordinates for the first equation of (4.2) we use (3.17). By Theorem 3.2 of [3], (4.2) possess a first integral $\Psi(x, y)$. Thus, by (3.17), the first equation of (4.2) is linearizable by the change of coordinates $x_1 = \Psi/y_1$. Therefore, O is a linearizable persistent center of (4.1).

Case (2). Systems (2.7) and (3.4) are

$$\dot{x} = ix + a_{20}x^2 + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3, \quad \dot{y} = -iy + b_{02}y^2 + b_{03}y^3, \tag{4.3}$$

and

$$\dot{x} = ix + \lambda(a_{20}x^2 + a_{02}y^2 + a_{12}xy^2 + a_{03}y^3), \quad \dot{y} = -iy + \mu(b_{02}y^2 + b_{03}y^3), \tag{4.4}$$

respectively. System (4.4) has partial first integrals

$$\begin{aligned} g_0 &= y, \\ g_1 &= 1 + \frac{1}{2}i\left(\mu b_{02} - \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y, \\ g_2 &= 1 + \frac{1}{2}i\left(\mu b_{02} + \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y, \end{aligned}$$

with respective cofactors

$$\begin{aligned} l_0 &= -i + \mu b_{02}y + \mu b_{03}y^2, \\ l_1 &= \frac{1}{2}\left(\mu b_{02} - \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y + \mu b_{03}y^2, \\ l_2 &= \frac{1}{2}\left(\mu b_{02} + \sqrt{\mu}\sqrt{4ib_{03} + \mu b_{02}^2}\right)y + \mu b_{03}y^2. \end{aligned}$$

It is easy to verify that equation (3.15) is satisfied for

$$\beta_1 = \frac{\sqrt{\mu}b_{02} - \sqrt{4ib_{03} + \mu b_{02}^2}}{2\sqrt{4ib_{03} + \mu b_{02}^2}} \quad \text{and} \quad \beta_2 = -\frac{\sqrt{\mu}b_{02} + \sqrt{4ib_{03} + \mu b_{02}^2}}{2\sqrt{4ib_{03} + \mu b_{02}^2}},$$

so according to (3.16) we obtain that the second equation of (4.2) is linearizable by the change of coordinates

$$y_1 = y g_1^{\beta_1} g_2^{\beta_2}.$$

In order to find a change of coordinates for the first equation of (4.2) we use (3.17). By Theorem 3.2 (condition 7) of [3], (4.4) possess a first integral $\Psi(x, y)$. Thus, by (3.17), the first equation of (4.4) is linearizable by the change of coordinates $x_1 = \Psi/y_1$. Therefore, O is a linearizable persistent center of (4.3).

Case (3). Systems (2.7) and (3.4) are

$$\dot{x} = ix + a_{20}x^2 + a_{30}x^3, \quad \dot{y} = -iy + b_{02}y^2 + b_{03}y^3, \tag{4.5}$$

and

$$\dot{x} = ix + \lambda(a_{20}x^2 + a_{30}x^3), \quad \dot{y} = -iy + \mu(b_{02}y^2 + b_{03}y^3), \tag{4.6}$$

respectively. System (4.6) has partial first integrals

$$\begin{aligned}
l_1 &= x,\\
l_2 &= 1-\frac{1}{2}i\left(\lambda a_{20}-\sqrt{\lambda}\sqrt{-4ia_{30}+\lambda a_{20}^2}\right)x,\\
l_3 &= 1-\frac{1}{2}i\left(\lambda a_{20}+\sqrt{\lambda}\sqrt{-4ia_{30}+\lambda a_{20}^2}\right)x,\\
l_4 &= y,\\
l_5 &= 1+\frac{1}{2}i\left(\mu b_{02}-\sqrt{\mu}\sqrt{4ib_{03}+\mu b_{02}^2}\right)y,\\
l_6 &= 1+\frac{1}{2}i\left(\mu b_{02}+\sqrt{\mu}\sqrt{4ib_{03}+\mu b_{02}^2}\right)y,
\end{aligned}$$

with respective cofactors

$$\begin{aligned}
k_1 &= i+\lambda a_{20}x+\lambda a_{30}x^2,\\
k_2 &= \frac{1}{2}\left(\lambda a_{20}-\sqrt{\lambda}\sqrt{-4ia_{30}+\lambda a_{20}^2}\right)x+\lambda a_{30}x^2,\\
k_3 &= \frac{1}{2}\left(\lambda a_{20}+\sqrt{\lambda}\sqrt{-4ia_{30}+\lambda a_{20}^2}\right)x+\lambda a_{30}x^2,\\
k_4 &= -i+\mu b_{02}y+\mu b_{03}y^2,\\
k_5 &= \frac{1}{2}\left(\mu b_{02}-\sqrt{\mu}\sqrt{4ib_{03}+\mu b_{02}^2}\right)y+\mu b_{03}y^2,\\
k_6 &= \frac{1}{2}\left(\mu b_{02}+\sqrt{\mu}\sqrt{4ib_{03}+\mu b_{02}^2}\right)y+\mu b_{03}y^2.
\end{aligned}$$

It is easy to verify that choosing $f_0 = l_1$, $f_1 = l_2$ and $f_2 = l_3$ equation (3.14) is satisfied for

$$\alpha_1 = \frac{\sqrt{\lambda}a_{20}-\sqrt{-4ia_{30}+\lambda a_{20}^2}}{2\sqrt{-4ia_{30}+\lambda a_{20}^2}} \quad\text{and}\quad \alpha_2 = -\frac{\sqrt{\lambda}a_{20}+\sqrt{-4ia_{30}+\lambda a_{20}^2}}{2\sqrt{-4ia_{30}+\lambda a_{20}^2}},$$

and choosing $g_0 = l_4$, $g_1 = l_5$ and $g_2 = l_6$, equation (3.15) is satisfied for

$$\beta_1 = \frac{\sqrt{\mu}b_{02}-\sqrt{4ib_{03}+\mu b_{02}^2}}{2\sqrt{4ib_{03}+\mu b_{02}^2}} \quad\text{and}\quad \beta_2 = -\frac{\sqrt{\mu}b_{02}+\sqrt{4ib_{03}+\mu b_{02}^2}}{2\sqrt{4ib_{03}+\mu b_{02}^2}}.$$

So we obtain that (4.6) is linearizable by the change of coordinates

$$x_1 = xf_1^{\alpha_1}f_2^{\alpha_2}, \quad y_1 = yg_1^{\beta_1}g_2^{\beta_2}.$$

Therefore, O is a linearizable persistent center of (4.5).

Case (4). System (2.7) satisfying conditions (4) of this theorem can be transformed into system (4.3), where

$$(a_{20}, a_{30}, b_{02}, b_{20}, b_{21}, b_{30}) = -(b_{02}, b_{03}, a_{20}, a_{02}, a_{12}, a_{03}),$$

by the change of coordinates $(x, y, t) \to (y, x, -t)$. Therefore, O is a linearizable persistent center for this case.

Case (5). System (2.7) satisfying conditions (5) of this theorem can be transformed into system (4.1), where

$$(a_{20}, a_{30}, b_{11}, b_{20}, b_{21}, b_{30}) = -(b_{02}, b_{03}, a_{11}, a_{02}, a_{12}, a_{03}),$$

by the change of coordinates $(x, y, t) \to (y, x, -t)$. Therefore, O is a linearizable persistent center for this case. □

4.2 Linearizable weakly persistent centers of system (2.8)

In order to perform the complete analysis of linearizbility problem for a weakly persistent center of system (2.7) even using powerful computer we were not able to carry out computations of decomposition of variety $\mathbf{V}(I)$, where I is an ideal generated by first seven pairs of linearizability quantities. We also tried to obtain decomposition over the finite field of characteristics 32003 but computations were again too laborious. Therefore, we restrict our attention to systems (2.7) with $a_{03} = 0$ and $a_{02} = 0$, i.e. system (2.8), which is also called semi-Kolmogorov system. Kolmogorov systems are an important class of systems due to its wide use in mathematical biology to describe the interaction of two populations. In fact Kolmogorov systems are a general model for the dynamics of biological species because it is the simplest model to describe the interaction of two species occupying the same ecological niche, see [19]. In that sense, they are generalizations of the so-called Lotka–Volterra systems, see [24,32]. We see that system (2.8) is larger family than Kolmogorov systems and it is the largest family for which we were able to find decomposition of variety $\mathbf{V}(I)$.

Note that the Lotka–Volterra systems considered in [3], [7] and [27] are all subcases of (2.8) for which we state the following result.

Theorem 4.2. *System (2.8) has a linearizable weakly persistent center at the origin if and only if one of the following conditions holds*

(1) $b_{21} = b_{20} = b_{12} = b_{11} = b_{03} = a_{30} = a_{21} = a_{20} = a_{12} = b_{02} - 3a_{11} = 0,$

(2) $b_{30} = b_{21} = b_{12} = b_{11} = b_{02} = a_{30} = a_{21} = a_{20} = a_{11} = 2a_{12} - b_{03} = 0,$

(3) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{30} = a_{21} = a_{20} = 0,$

(4) $b_{21} = b_{20} = b_{12} = b_{11} = a_{30} = a_{21} = a_{20} = 3a_{12} - b_{03} = 3a_{11} + b_{02} = 0,$

(5) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{30} = a_{21} = a_{11} = 0,$

(6) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{21} = a_{20} = a_{a12} = 2a_{11} + b_{02} = 0,$

(7) $b_{30} = b_{21} = b_{20} = b_{12} = b_{11} = a_{21} = a_{12} = a_{11} = 0,$

(8) $b_{30} = b_{21} = b_{20} = b_{12} = b_{02} = a_{30} = a_{21} = a_{11} = a_{20} - 2b_{11} = a_{12} - 2b_{03} = 0,$

(9) $b_{30} = b_{21} = b_{20} = b_{12} = b_{02} = a_{21} = a_{12} = a_{11} = a_{20} + 2b_{11} = 0,$

(10) $b_{30} = b_{12} = b_{11} = b_{03} = a_{21} = a_{20} = a_{12} = 2a_{30} - b_{21} = 2a_{11} - b_{02} = 0,$

(11) $b_{12} = b_{11} = b_{03} = a_{21} = a_{12} = a_{11} = 0,$

(12) $b_{30} = b_{12} = b_{11} = a_{21} = a_{11} = a_{30} - b_{21} = a_{12} - b_{03} = 0,$

(13) $b_{12} = b_{03} = b_{02} = a_{21} = a_{12} = a_{11} = 0,$

(14) $b_{30} = b_{20} = b_{12} = a_{21} = a_{30} - b_{21} = a_{20} - b_{11} = a_{12} - b_{03} = a_{11} - b_{02} = b_{03}b_{11}^2 + b_{02}^2 b_{21} = 0.$

Proof. The linearizability quantities (3.6) of the system

$$\begin{aligned} \dot{x} &= ix + \lambda x\left(a_{20}x + a_{11}y + a_{30}x^2 + a_{21}xy + a_{12}y^2\right), \\ \dot{y} &= -iy + \lambda\left(b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3\right), \end{aligned} \tag{4.7}$$

are polynomials in λ with coefficients $i_k^{(m)}(a,b)$, $j_k^{(m)}(a,b)$ being polynomials in a and b. So (2.8) has a linearizable center at the origin for all $\lambda \in \mathbb{C}$ if and only if all polynomials $i_k^{(m)}(a,b)$'s and $j_k^{(m)}(a,b)$'s vanish (i.e. $(a,b) \in \mathbf{V}\left(\mathcal{L}^{wp}\right)$).

We compute the first six pairs of linearizability quantities, $i_1, j_1, \ldots, i_6, j_6$ of system (4.7). As the quantities are very large we present only the first pair:

$$\begin{aligned} i_1 &= a_{21}\lambda + i\left(a_{11}a_{20} - a_{11}b_{11}\right)\lambda^2, \\ j_1 &= b_{12}\lambda + i\left(a_{11}b_{11} - b_{02}b_{11}\right)\lambda^2. \end{aligned}$$

Performing computations using routine *minAssGTZ* of SINGULAR the same decomposition is obtained for $\mathbf{V}(\mathcal{L}_5^{wp})$ and $\mathbf{V}(\mathcal{L}_6^{wp})$ but for $\mathbf{V}(\mathcal{L}_4^{wp})$ a different decomposition is obtained. It lead us to expect that $\mathbf{V}(\mathcal{L}^{wp}) = \mathbf{V}(\mathcal{L}_5^{wp})$.

The decomposition of $\mathbf{V}(\mathcal{L}_5^{wp})$ consists of 14 components listed in the statement of the theorem. Thus, the necessary conditions for existence of a linearizable weakly persistent center at the origin for (2.8) are obtained.

Now we prove that each of these 14 conditions is also sufficient.

Case (1). Systems (2.8) and (4.7) are

$$\dot{x} = ix + a_{11}xy, \quad \dot{y} = -iy + 3a_{11}y^2 + b_{30}x^3, \tag{4.8}$$

and

$$\dot{x} = ix + \lambda(a_{11}xy), \quad \dot{y} = -iy + \lambda(3a_{11}y^2 + b_{30}x^3), \tag{4.9}$$

respectively. System (4.9) has partial first integrals

$$\begin{aligned} l_1 &= x, \\ l_2 &= 1 + 3i\lambda a_{11}y - \lambda^2 a_{11}b_{30}x^3, \\ l_3 &= y + \frac{1}{4}i\lambda b_{30}x^3, \end{aligned}$$

with respective cofactors

$$\begin{aligned} k_1 &= i + \lambda a_{11}y, \\ k_2 &= 3\lambda a_{11}y, \\ k_3 &= -i + 3\lambda a_{11}y. \end{aligned}$$

It is easy to verify that choosing $f_0 = l_1$ and $f_1 = l_2$, equation (3.14) is satisfied for $\alpha_1 = -\frac{1}{3}$ and choosing $g_0 = l_3$ and $g_1 = l_2$, equation (3.15) is satisfied for $\beta_1 = -1$. So we obtain that (4.9) is linearizable by the change of coordinates

$$\begin{aligned} x_1 &= x\left(1 + 3i\lambda a_{11}y - \lambda^2 a_{11}b_{30}x^3\right)^{-1/3}, \\ y_1 &= \left(y + \frac{1}{4}i\lambda b_{30}x^3\right)\left(1 + 3i\lambda a_{11}y - \lambda^2 a_{11}b_{30}x^3\right)^{-1}. \end{aligned}$$

Therefore, O is a linearizable weakly persistent center of (4.8).

Case (2). Systems (2.8) and (4.7) are

$$\dot{x} = ix + a_{12}xy^2, \quad \dot{y} = -iy + b_{20}x^2 + 2a_{12}y^3, \tag{4.10}$$

and

$$\dot{x} = ix + \lambda(a_{12}xy^2), \quad \dot{y} = -iy + \lambda(b_{20}x^2 + 2a_{12}y^3), \tag{4.11}$$

respectively. System (4.11) has partial first integrals

$$\begin{aligned} l_1 &= x, \\ l_2 &= 1 + 2i\lambda a_{12}y^2 - 4\lambda^2 a_{12}b_{20}x^2y - i\lambda^3 a_{12}b_{20}^2x^4, \\ l_3 &= y + \frac{1}{3}i\lambda b_{20}x^2, \end{aligned}$$

with respective cofactors

$$\begin{aligned} k_1 &= i + \lambda a_{12}y^2, \\ k_2 &= 4\lambda a_{12}y^2, \\ k_3 &= -i + 2\lambda a_{12}y^2. \end{aligned}$$

It is easy to verify that choosing $f_0 = l_1$ and $f_1 = l_2$, equation (3.14) is satisfied for $\alpha_1 = -\frac{1}{4}$ and choosing $g_0 = l_3$ and $g_1 = l_2$, equation (3.14) is satisfied for $\beta_1 = -\frac{1}{2}$. So we obtain that (4.11) is linearizable by the change of coordinates

$$\begin{aligned} x_1 &= x\left(1 + 2i\lambda a_{12}y^2 - 4\lambda^2 a_{12}b_{20}x^2y - i\lambda^3 a_{12}b_{20}^2x^4\right)^{-1/4}, \\ y_1 &= \left(y + \frac{1}{3}i\lambda b_{20}x^2\right)\left(1 + 2i\lambda a_{12}y^2 - 4\lambda^2 a_{12}b_{20}x^2y - i\lambda^3 a_{12}b_{20}^2x^4\right)^{-1/2}. \end{aligned}$$

Therefore, O is a linearizable weakly persistent center of (4.10).

Case (3). Systems (2.8) and (4.7) are

$$\dot{x} = ix + a_{11}xy + a_{12}xy^2, \quad \dot{y} = -iy + b_{02}y^2 + b_{03}y^3, \tag{4.12}$$

and

$$\dot{x} = ix + \lambda(a_{11}xy + a_{12}xy^2), \quad \dot{y} = -iy + \lambda(b_{02}y^2 + b_{03}y^3), \tag{4.13}$$

respectively. System (4.12) is considered in [7], Theorem 2 (case 4), where the corresponding linearizable change of coordinates is obtained. So, by Lemma 3.3 system (4.13) is also linearizable. Therefore, O is a linearizable weakly persistent center of (4.12).

Case (4). Systems (2.8) and (4.7) are

$$\begin{aligned} \dot{x} &= x(i + a_{11}y + a_{12}y^2), \\ \dot{y} &= -iy + b_{30}x^3 - 3a_{11}y^2 + 3a_{12}y^3, \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= x(i + a_{11}\lambda y + a_{12}\lambda y^2) = P(x,y), \\ \dot{y} &= -iy + b_{30}\lambda x^3 - 3a_{11}\lambda y^2 + 3a_{12}\lambda y^3 = Q(x,y). \end{aligned} \tag{4.14}$$

For system (4.14) we find only one partial integral $f_0 = x$, with the corresponding cofactor $K_0 = i + x\,(a_{20} + a_{30}x)\,\lambda$, which is not enough to construct neither a Darboux first integral nor an integrating factor. To prove that there exists a first integral of the form (2.4) we follow the approach in [14] and first make the substitution $z = x/y$. In the new coordinates system (4.14) takes the form

$$\begin{aligned} \dot{z} &= z(2i + 4a_{11}\lambda y - 2a_{12}\lambda y^2 - b_{30}\lambda y^2 z^3), \\ \dot{y} &= -iy - 3a_{11}\lambda y^2 + 3a_{12}\lambda y^3 + b_{30}\lambda y^3 z^3. \end{aligned} \tag{4.15}$$

We look for a first integral for (4.15) of the form

$$F(z,y) = zy^2 \sum_{k=2}^{\infty} f_k(z) y^k.$$

The functions f_k are determined recursively by the differential equation

$$\begin{aligned} f_k'(z) - \frac{k}{2z} f_k(z) = {} & \frac{-\lambda((3k-2)a_{12} - (k-1)b_{30}z^3) f_{k-2}(z)}{2iz} - \frac{4\lambda a_{11} f_{k-1}'(z)}{2i} \\ & + \frac{(3n-1)\lambda a_{11} f_{k-1}(z)}{2iz} + \frac{\lambda(2a_{12} + b_{30}z^3) f_{k-2}'(z)}{2i}. \end{aligned} \tag{4.16}$$

For $k = 2, 3, 4, 5, 6, 7$ we find

$$\begin{aligned} f_2(z) &= z, \quad f_3(z) = 4i\lambda a_{11} z, \quad f_4(z) = \frac{1}{2} i\lambda z(b_{30}z^3 + 28i\lambda a_{11}^2 - 8a_{12}), \\ f_5(z) &= \frac{1}{3}\lambda^2 a_{11} z(84a_{12} - 13b_{30}z^3 - 140i\lambda a_{11}^2), \\ f_6(z) &= \frac{1}{48}\lambda^2 z\Big(-3(224a_{12}^2 - 96a_{12}b_{30}z^3 + b_{30}^2 z^6) + 40ia_{11}^2(168a_{12} - 31b_{30}z^3)\lambda + 7280a_{11}^4\lambda^2\Big), \\ f_7(z) &= \frac{1}{42} z\Big(-5880ia_{11}a_{12}^2\lambda^3 + 2870ia_{11}a_{12}b_{30}z^3\lambda^3 - 55ia_{11}b_{30}^2 z^6\lambda^3 - 25480a_{11}^3 a_{12}\lambda^4 \\ &\qquad + 5460a_{11}^3 b_{30}z^3\lambda^4 + 20384ia_{11}^5\lambda^5\Big). \end{aligned}$$

We claim that $f_k(z) = zp_{k-a}$ if $k \equiv a \pmod 3$, where $a \in \{0,1,2\}$ and p_{k-a} is a polynomial of degree $k - a$. Thus, we have three cases:

1. $k \equiv 0 \pmod 3$;
2. $k \equiv 1 \pmod 3$;
3. $k \equiv 2 \pmod 3$.

Assume that $f_k(z) = zp_{k-a}(z)$ for $k = 1, \ldots, n-1$ where $k \equiv a \pmod 3$ and $a \in \{0,1,2\}$. We compute $f_k(z)$ for $k = n$. To this end we solve (4.16) for $k = n$. Let us first consider case (a). If $k = n \equiv 0 \pmod 3$ then $n - 1 \equiv 2 \pmod 3$ and $n - 2 \equiv 1 \pmod 3$ and $f_{n-1}(z) = zq_{n-3}(z)$, $f_{n-2}(z) = zr_{n-3}(z)$, where $q_{n-3}(z)$ and $r_{n-3}(z)$ are polynomials of degree $n - 3$. We want to show that $f_n(z) = zp_n(z)$, where $p_n(z)$ is polynomial of degree n. Using the induction assumption about f_{n-1} and f_{n-2} differential equation (4.16) becomes

$$f_n'(z) - \frac{n}{2z} f_n(z) = P_n(z) + zQ_{n-1}(z) = C_n(z),$$

where $P_n(z)$, $C_n(z)$ and $Q_{n-1}(z)$ are polynomials of degree n, n and $n-1$, respectively and, if $P_n(z) = A_0 + A_1 z + \cdots + A_n z^n$ and $Q_n(z) = B_0 + B_1 z + \cdots + B_n z^n$, then $C_n(z) = c_0 + c_1 z + \cdots + c_n z^n$, where $c_0 = A_0$, $c_1 = A_1 + B_0, \ldots, c_n = A_n + B_{n-1}$.

As the general solution of linear differential equation of the form $f'(z) + p(z)f(z) = q(z)$ is

$$f(z) = Ce^{-\int p(z)dz} + e^{-\int p(z)dz} \int e^{\int p(z)dz} q(z)dz,$$

and in our case we have $p(z) = -\frac{n}{2z}$ and $q(z) = C_n(z)$, it follows that $e^{-\int p(z)dz} = z^{\frac{n}{2}}$ and the solution is

$$\begin{aligned} f_n(z) = &\ Cz^{\frac{n}{2}} + z^{\frac{n}{2}} \int \frac{C_n(z)}{z^{\frac{n}{2}}} dz \\ = &\ Cz^{\frac{n}{2}} + z^{\frac{n}{2}} \int (c_0 z^{-\frac{n}{2}} + c_1 z^{1-\frac{n}{2}} + \cdots + c_n z^{\frac{n}{2}}) dz \\ = &\ Cz^{\frac{n}{2}} + z^{\frac{n}{2}} (c_0 z^{1-\frac{n}{2}} + c_1 z^{2-\frac{n}{2}} + \cdots + c_n z^{\frac{n}{2}+1}) \\ = &\ Cz^{\frac{n}{2}} + c_0 z + c_1 z^2 + \cdots + c_n z^{n+1} \\ = &\ Cz^{\frac{n}{2}} + z(c_0 + c_1 z + \cdots + c_n z^n). \end{aligned}$$

Choosing $C = 0$ we obtain $f_n(z) = zp_n(z)$.

Now we shortly consider case (b). If $n - 1 \equiv 0 \pmod 3$ then $n - 2 \equiv 2 \pmod 3$ and $n \equiv 1 \pmod 3$. Assuming that $f_{n-1}(z) = zq_{n-1}(z)$ and $f_{n-2}(z) = zr_{n-4}(z)$, we want to show that $f_n = zp_{n-1}$, where q_{n-1} and p_{k-1} are polynomials of degree $n-1$ and r_{n-4} is polynomial of degree $n-4$. Using the induction assumption about f_{n-1} and f_{n-2} differential equation (4.16) becomes

$$f_n'(z) - \frac{n}{2z} f_n(z) = P_{n-1}(z) + zQ_{n-2}(z) = C_{n-1}(z),$$

where $P_{n-1}(z)$, $C_{n-1}(z)$ and $Q_{n-2}(z)$ are polynomials of degree $n-1$, $n-1$ and $n-2$, respectively. Then the solution to (4.16) is

$$\begin{aligned} f_n(z) = &\ Cz^{\frac{n}{2}} + z^{\frac{n}{2}} \int \frac{C_{n-1}(z)}{z^{\frac{n}{2}}} dz \\ = &\ Cz^{\frac{n}{2}} + z^{\frac{n}{2}} \int \frac{c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}}{z^{\frac{n}{2}}} dz \\ = &\ Cz^{\frac{n}{2}} + z^{\frac{n}{2}} (c_0 z^{1-\frac{n}{2}} + c_1 z^{2-\frac{n}{2}} + \cdots + c_{n-1} z^{\frac{n}{2}}) \\ = &\ Cz^{\frac{n}{2}} + z(c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}). \end{aligned}$$

Choosing again $C = 0$ we have shown that $f_n(z) = zp_{n-1}(z)$.

In the case (c) when $k - 2 = n - 2 \equiv 0 \pmod 3$ the proof is similar as in the previous two cases. We assume that $f_{n-2} = zr_{n-2}(z)$ and $f_{n-1} = zq_{n+1}(z)$ and we prove that $f_n = zp_{n+1}(z)$.

Thus, we have shown that $f_2(z) = z$, $f_3(z) = 4i\lambda a_{11} z$ and for $k > 3$ we have proven that $f_k = zp_{k-a}$, if $k \equiv a \pmod 3$. Therefore, (formal) first integral of system (4.15) is $F(z, y) = z^2 y^4 + \sum_{i+j=7}^{\infty} \phi_{ij} z^i y^j$ (and $\Psi(x, y) = \sqrt{F(x/y, y)}$ is a first integral of system (4.14) of the form (2.4)).

Now we look for a linearization $Y = Y(x, y)$ of the second equation of system (4.14). The function Y should satisfy the equation

$$\frac{\partial Y}{\partial x} P(x, y) + \frac{\partial Y}{\partial y} Q(x, y) + iY = 0.$$

After the substitution $z = x/y$, this equation is written as

$$\frac{1}{y}\frac{\partial Y}{\partial z}P(x,y) + \left(\frac{\partial Y}{\partial y} - \frac{z}{y}\frac{\partial Y}{\partial z}\right)Q(x,y) + iY = 0.$$

We look for a transformation of the form

$$Y(z,y) = \sum_{k=1}^{\infty} g_k(z)y^k,$$

where $g_k(z)$ are polynomials satisfying a first-order linear differential equation,

$$\begin{aligned}(k-2)\lambda(3a_{12} + b_{30}z^3)g_{k-2}(z) - 3(k-1)\lambda a_{11}g_{k-1}(z) - (k-1)ig_k(z)\\ + z(-\lambda(2a_{12} + b_{30}z^3)g'_{k-2}(z) + 4\lambda a_{11}g'_{k-1}(z) + 2ig'_k(z)) = 0.\end{aligned} \quad (4.17)$$

Solving the equation we obtain

$$g_1(z) = 1, \quad g_2(z) = 3i\lambda a_{11}, \quad g_3(z) = -\frac{3}{2}i\lambda a_{12} + \frac{1}{4}i\lambda b_{30}z^3 - 9\lambda^2 a_{11}^2,$$
$$g_4(z) = \frac{3}{4}\lambda^2 a_{11}(14a_{12} - 3b_{30}z^3 - 36i\lambda a_{11}^2).$$

We assume that $g_k(z) = p_{k-a}(z)$ where $k \equiv a\,(\mathrm{mod}\,3)$ and $a \in \{0,1,2\}$. We prove this again by induction. We assume that equation $g_k(z) = p_{k-a}(z)$ holds for $k = 1,2,\ldots,n-1$ and we want to show that this is the case also for $k = n$. To end this task one must consider three cases: $a = 0$, $a = 1$ and $a = 2$. The proof is very similar as the proof of integrability for system (4.14), thus we only briefly consider the case $a = 0$, i.e. $k = n \equiv 0\,(\mathrm{mod}\,3)$. By the induction assumption we have $g_{n-2}(z) = r_{n-3}(z)$ and $g_{n-1}(z) = q_{n-3}(z)$, and we want to show that $g_n(z) = r_n(z)$. Differential equation (4.17) becomes

$$g'_n(z) - \frac{n-1}{2z}g_n(z) = \frac{C_n(z)}{z},$$

where $C_n(z)$ is polynomial of degree n. The solution of the latter equation is

$$\begin{aligned}g_n(z) &= Cz^{\frac{n-1}{2}} + z^{\frac{n-1}{2}}\int \frac{C_n(z)}{z^{\frac{n+1}{2}}}dz\\ &= Cz^{\frac{n-1}{2}} + c_0 + c_1 z + \cdots + c_n z^n.\end{aligned}$$

Choosing $C = 0$ we obtain $g_n(z) = p_n(z)$, where $p_n(z)$ is polynomial of degree n. In a similar way one can prove that if $n-2 \equiv 0\,(\mathrm{mod}\,3)$ ($n-1 \equiv 0\,(\mathrm{mod}\,3)$) then $g_n = p_{n-2}$ ($g_n = p_{n-1}$).

Then $Y(\frac{x}{y}, y)$ is a series in x and y of the form $Y(x,y) = y + \sum_{i+j=2}^{\infty} Y_{ij}x^i y^j$ and the second equation of (4.14) can be linearized by the transformation $Y = Y(x,y)$. The first equation of (4.14) can be linearized by the transformation $X = X(x,y) = \frac{\Psi(x,y)}{Y(x,y)}$.

Case (5). Systems (2.8) and (4.7) are

$$\dot{x} = ix + x(a_{20}x + a_{12}y^2), \quad \dot{y} = -iy + y(b_{02}y + b_{03}y^2), \quad (4.18)$$

and

$$\dot{x} = ix + \lambda x(a_{20}x + a_{12}y^2), \quad \dot{y} = -iy + \lambda y(b_{02}y + b_{03}y^2), \quad (4.19)$$

respectively. System (4.18) is considered in [7, Theorem 2, case 3], where the corresponding linearizable change of coordinates is obtained. So, by Lemma 3.3 system (4.19) is linearizable. Therefore, O is a linearizable weakly persistent center of (4.18).

Case (6). Systems (2.8) and (4.7) are

$$\dot{x} = ix + x(a_{30}x^2 + a_{11}y), \quad \dot{y} = -iy + y(-2a_{11}y + b_{03}y^2), \tag{4.20}$$

and

$$\dot{x} = ix + \lambda x(a_{30}x^2 + a_{11}y), \quad \dot{y} = -iy + \lambda y(-2a_{11}y + b_{03}y^2), \tag{4.21}$$

respectively. System (4.20) is considered in [7, Theorem 2, case 5], where the corresponding linearizable change of coordinates is obtained. So, by Lemma 3.3 system (4.21) is linearizable. Therefore, O is a linearizable weakly persistent center of (4.20).

Case (7). Systems (2.8) and (4.7) are

$$\dot{x} = ix + x(a_{20}x + a_{30}x^2), \quad \dot{y} = -iy + y(b_{02}y + b_{03}y^2), \tag{4.22}$$

and

$$\dot{x} = ix + \lambda x(a_{20}x + a_{30}x^2), \quad \dot{y} = -iy + \lambda y(b_{02}y + b_{03}y^2), \tag{4.23}$$

respectively. System (4.22) is a subcase of the case considered in [27, Theorem 3, case VIII (3)], where the corresponding linearizable change of coordinates is obtained. So, by Lemma 3.3 system (4.23) is linearizable. Therefore, O is a linearizable weakly persistent center of (4.22).

Case (8). Systems (2.8) and (4.7) are

$$\dot{x} = ix + x(2b_{11}x + 2b_{03}y^2), \quad \dot{y} = -iy + y(b_{11}x + b_{03}y^2), \tag{4.24}$$

and

$$\dot{x} = ix + \lambda x(2b_{11}x + 2b_{03}y^2), \quad \dot{y} = -iy + \lambda y(b_{11}x + b_{03}y^2), \tag{4.25}$$

respectively. System (4.24) is considered in [7, Theorem 2, case 6], where the corresponding linearizable change of coordinates is obtained. So, by Lemma 3.3 system (4.25) is linearizable. Therefore, O is a linearizable weakly persistent center of (4.24).

Case (9). System (2.8) satisfying conditions (9) of this theorem can be transformed into system (2.8) satisfying conditions (6) of this theorem, where

$$(a_{30}, b_{03}, b_{11}) = -(b_{03}, a_{30}, a_{11}),$$

by the change of coordinates $(x, y, t) \to (y, x, -t)$. Therefore, O is a linearizable weakly persistent center for this case

Case (10). Systems (2.8) and (4.7) are

$$\dot{x} = ix + x(a_{11}y + a_{30}x^2), \quad \dot{y} = -iy + b_{20}x^2 + 2a_{11}y^2 + 2a_{30}x^2y, \tag{4.26}$$

and

$$\dot{x} = ix + \lambda x(a_{11}y + a_{30}x^2), \quad \dot{y} = -iy + \lambda(b_{20}x^2 + 2a_{11}y^2 + 2a_{30}x^2y), \tag{4.27}$$

respectively. System (4.27) has partial first integrals

$$\begin{aligned} l_1 &= x, \\ l_2 &= 1 + 2i\lambda a_{11}y - \left(i\lambda a_{30} + \lambda^2 a_{11}b_{20}\right)x^2, \\ l_3 &= y + \frac{1}{3}i\lambda b_{20}x^2, \end{aligned}$$

with respective cofactors

$$\begin{aligned} k_1 &= i + \lambda a_{11} y + \lambda a_{30} x^2, \\ k_2 &= 2\lambda a_{11} y + 2\lambda a_{30} x^2, \\ k_3 &= -i + 2\lambda a_{11} y + 2\lambda a_{30} x^2. \end{aligned}$$

It is easy to verify that choosing $f_0 = l_1$ and $f_1 = l_2$, equation (3.14) is satisfied for $\alpha_1 = -\frac{1}{2}$ and choosing $g_0 = l_3$ and $g_1 = l_2$, equation (3.15) is satisfied for $\beta_1 = -1$. So we obtain that (4.27) is linearizable by the change of coordinates

$$\begin{aligned} x_1 &= x \left(1 + 2i\lambda a_{11} y - \left(i\lambda a_{30} + \lambda^2 a_{11} b_{20}\right) x^2\right)^{-1/2}, \\ y_1 &= \left(y + \frac{1}{3} i\lambda b_{20} x^2\right) \left(1 + 2i\lambda a_{11} y - \left(i\lambda a_{30} + \lambda^2 a_{11} b_{20}\right) x^2\right)^{-1}. \end{aligned}$$

Therefore, O is a linearizable weakly persistent center of (4.26).

Case (11). Systems (2.8) and (4.7) are

$$\begin{aligned} \dot{x} &= ix + x\left(a_{20} x + a_{30} x^2\right), \\ \dot{y} &= -iy + b_{20} x^2 + b_{30} x^3 + b_{21} x^2 y + b_{02} y^2, \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= ix + \lambda x\left(a_{20} x + a_{30} x^2\right) = \bar{P}(x,y), \\ \dot{y} &= -iy + \lambda(b_{20} x^2 + b_{30} x^3 + b_{21} x^2 y + b_{02} y^2) = \bar{Q}(x,y). \end{aligned} \tag{4.28}$$

We find partial first integrals

$$\begin{aligned} f_0 &= x, \\ f_1 &= 1 + \frac{1}{2} x \left(-i a_{20} \lambda - \sqrt{\lambda\left(4i a_{30} - a_{20}^2 \lambda\right)}\right), \\ f_2 &= 1 + \frac{1}{2} x \left(-i a_{20} \lambda + \sqrt{\lambda\left(4i a_{30} - a_{20}^2 \lambda\right)}\right), \end{aligned}$$

with the corresponding cofactors

$$\begin{aligned} k_0 &= i + x\left(a_{20} + a_{30} x\right)\lambda, \\ k_1 &= \frac{1}{2} x \left(a_{20}\lambda + 2a_{30} x \lambda - i\sqrt{\lambda\left(4i a_{30} - a_{20}^2 \lambda\right)}\right), \\ k_2 &= \frac{1}{2} x \left(a_{20}\lambda + 2a_{30} x \lambda + i\sqrt{\lambda\left(4i a_{30} - a_{20}^2 \lambda\right)}\right). \end{aligned}$$

We found the solution to (3.14) for k_1 and k_2 to be

$$\alpha_1 = -\frac{1}{2} - \frac{a_{20}\sqrt{\lambda\left(4i a_{30} - a_{20}^2 \lambda\right)}}{8a_{30} + 2i a_{20}^2 \lambda}, \quad \alpha_2 = -\frac{1}{2} + \frac{a_{20}\sqrt{\lambda\left(4i a_{30} - a_{20}^2 \lambda\right)}}{8a_{30} + 2i a_{20}^2 \lambda}.$$

Therefore the first equation of (4.28) is linearizable by the change of coordinates

$$x_1 = x f_1^{\alpha_1} f_2^{\alpha_2}.$$

Using partial first integrals $p_1 = f_1$ and $p_2 = f_2$ with corresponding cofactors $K_1 = k_1$ and $K_2 = k_2$ we can construct an integrating factor $\mu = p_1^{\gamma_1} p_2^{\gamma_2}$ by solving equation (3.19) for γ_1, γ_2 and obtain

$$\begin{aligned}\gamma_1 = &-\frac{12a_{30} + \left(4b_{21} + a_{20}\left(\sqrt{\lambda\left(4ia_{30} - a_{20}^2\lambda\right)} + 3ia_{20}\lambda\right)\right)}{2\left(4a_{30} + ia_{20}^2\lambda\right)}\\ &+\frac{a_{20}b_{21}\left(\sqrt{\lambda\left(4ia_{30} - a_{20}^2\lambda\right)} - ia_{20}\lambda\right)}{2a_{30}\left(4a_{30} + ia_{20}^2\lambda\right)},\\ \gamma_2 = &-\frac{12a_{30} + \left(4b_{21} + a_{20}\left(3ia_{20}\lambda - \sqrt{\lambda\left(4ia_{30} - a_{20}^2\lambda\right)}\right)\right)}{2\left(4a_{30} + ia_{20}^2\lambda\right)}\\ &+\frac{a_{20}b_{21}\left(\sqrt{\lambda\left(4ia_{30} - a_{20}^2\lambda\right)} + ia_{20}\lambda\right)}{2a_{30}\left(4a_{30} + ia_{20}^2\lambda\right)},\end{aligned}$$

which implies that system (4.28) has an analytic first integral of the form (2.4) according to [1]. By Theorem 4.4.3 of [29] the second equation of (4.28) is linearizable by the change of coordinates

$$y_1 = \frac{\Psi}{x_1}.$$

Case (12). Systems (2.8) and (4.7) are

$$\begin{aligned}\dot{x} &= ix + x\left(a_{20}x + a_{30}x^2 + a_{12}y^2\right),\\ \dot{y} &= -iy + b_{20}x^2 + a_{30}x^2y + b_{02}y^2 + a_{12}y^3,\end{aligned}$$

and

$$\begin{aligned}\dot{x} &= ix + \lambda x\left(a_{20}x + a_{30}x^2 + a_{12}y^2\right),\\ \dot{y} &= -iy + \lambda(b_{20}x^2 + a_{30}x^2y + b_{02}y^2 + a_{12}y^3).\end{aligned} \tag{4.29}$$

We find partial first integrals

$$\begin{aligned}f_0 &= x,\\ f_1 &= 1 + \frac{1}{2}i\Big(y(b_{02}\lambda - \sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}) - x(a_{20}\lambda + A_+)\Big),\\ f_2 &= 1 + \frac{1}{2}i\Big(y(b_{02}\lambda - \sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}) + x(-a_{20}\lambda + A_+)\Big),\\ f_3 &= 1 + \frac{1}{2}i\Big(y(b_{02}\lambda + \sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}) - x(a_{20}\lambda + A_-)\Big),\\ f_4 &= 1 + \frac{1}{2}i\Big(y(b_{02}\lambda + \sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}) + x(-a_{20}\lambda + A_-)\Big),\end{aligned}$$

with corresponding cofactors

$$\begin{aligned}k_0 &= i + (x(a_{20} + a_{30}x) + a_{12}y^2)\lambda,\\ k_1 &= \frac{1}{2}\Big(\lambda\big(y(b_{02} + 2a_{12}y) + x(a_{20} + 2a_{30}x)\big) - y\sqrt{\lambda}\sqrt{b_{02}^2\lambda + 4ia_{12}} + xA_+\Big),\\ k_2 &= \frac{1}{2}\Big(\lambda\big(y(b_{02} + 2a_{12}y) + x(a_{20} + 2a_{30}x)\big) - y\sqrt{\lambda}\sqrt{b_{02}^2\lambda + 4ia_{12}} - xA_+\Big),\\ k_3 &= \frac{1}{2}\Big(\lambda\big(y(b_{02} + 2a_{12}y) + x(a_{20} + 2a_{30}x)\big) + y\sqrt{\lambda}\sqrt{b_{02}^2\lambda + 4ia_{12}} + xA_-\Big),\\ k_4 &= \frac{1}{2}\Big(\lambda\big(y(b_{02} + 2a_{12}y) + x(a_{20} + 2a_{30}x)\big) + y\sqrt{\lambda}\sqrt{b_{02}^2\lambda + 4ia_{12}} - xA_-\Big),\end{aligned}$$

where

$$A_{\pm} = \sqrt{\lambda\Big(-4ia_{30} + a_{20}^2\lambda - 2b_{02}b_{20}\lambda \pm 2b_{20}\sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}\Big)}.$$

We find the solution to (3.14) for α_1, α_2, α_3, and α_4 to be

$$\begin{aligned}
\alpha_1 &= -\frac{b_{02}\sqrt{\lambda}}{2\sqrt{4ia_{12} + b_{02}^2\lambda}},\\
\alpha_2 &= -\frac{1}{2},\\
\alpha_3 &= \frac{-2a_{20}\lambda\sqrt{4ia_{12} + b_{02}^2\lambda} + \left(b_{02}\sqrt{\lambda} - \sqrt{4ia_{12} + b_{02}^2\lambda}\right)\left(A_- + A_+\right)}{4A_-\sqrt{4ia_{12} + b_{02}^2\lambda}},\\
\alpha_4 &= \frac{2a_{20}\lambda\sqrt{4ia_{12} + b_{02}^2\lambda} + \left(b_{02}\sqrt{\lambda} - \sqrt{4ia_{12} + b_{02}^2\lambda}\right)\left(A_- - A_+\right)}{4A_-\sqrt{4ia_{12} + b_{02}^2\lambda}}.
\end{aligned}$$

Therefore the first equation of (4.29) is linearizable by the change of coordinates

$$x_1 = x f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} f_4^{\alpha_4}.$$

Now, using partial integrals $p_1 = f_1, \ldots, p_4 = f_4$ and the corresponding cofactors we construct the Darboux first integral. First, we solve equation (3.18) for $\psi_1, \ldots, \psi_4$ using $k_1, \ldots, k_4$ and obtain $\psi_1 = A_-$, $\psi_2 = -A_-$, $\psi_3 = -A_+$, and $\psi_4 = A_+$. Therefore, the Darboux first integral for system (4.29) is

$$\Psi = p_1^{\psi_1} p_2^{\psi_2} p_3^{\psi_3} p_4^{\psi_4}.$$

Actually, $\Psi_1(x, y) = \frac{\Psi - 1}{A_- A_+ \sqrt{\lambda}\sqrt{4ia_{12} + b_{02}^2\lambda}}$, is the first integral whose series expansion is of the form (2.4) and, therefore, the second equation of system (4.29) can be linearized by the change of coordinates

$$y_1 = \frac{\Psi_1}{x_1}.$$

Case (13). Systems (2.8) and (4.7) are

$$\begin{aligned}
\dot{x} &= x(i + a_{20}x + a_{30}x^2),\\
\dot{y} &= -iy + b_{20}x^2 + b_{30}x^3 + b_{11}xy + b_{21}x^2y,
\end{aligned}$$

and

$$\begin{aligned}
\dot{x} &= x(i + a_{20}\lambda x + a_{30}\lambda x^2) = P_1(x, y),\\
\dot{y} &= -iy + b_{20}\lambda x^2 + b_{30}\lambda x^3 + b_{11}\lambda xy + b_{21}\lambda x^2 y = Q_1(x, y).
\end{aligned} \tag{4.30}$$

We find partial first integrals

$$\begin{aligned}
f_0 &= x,\\
f_1 &= \frac{1}{2}\left(2 + 2ia_{30}\lambda x^2 + x\left(\sqrt{\lambda\left(a_{20}^2\lambda - 4ia_{30}\right)} - \lambda a_{20}\right)\left(x\lambda a_{20} + 2i\right)\right),\\
f_2 &= \frac{1}{2}\left(2 + 2ia_{30}\lambda x^2 - x\left(\sqrt{\lambda\left(a_{20}^2\lambda - 4ia_{30}\right)} + \lambda a_{20}\right)\left(x\lambda a_{20} + 2i\right)\right),
\end{aligned}$$

with the corresponding cofactors

$$K_0 = i + x\lambda(a_{20} + a_{30}x),$$
$$K_1 = x\left(a_{20}\lambda + 2a_{30}\lambda x + \sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}\right),$$
$$K_2 = x\left(a_{20}\lambda + 2a_{30}\lambda x - \sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}\right).$$

We found the solution to (3.14) for K_1 and K_2 to be

$$\beta_1 = \frac{1}{4}\left(-1 - \frac{a_{20}\lambda}{\sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}}\right), \quad \beta_2 = \frac{1}{4}\left(-1 + \frac{a_{20}\lambda}{\sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}}\right).$$

Therefore the first equation of (4.30) is linearizable by the change of coordinates

$$x_1 = x f_1^{\beta_1} f_2^{\beta_2}.$$

Using cofactors K_1 and K_2 we can construct an integrating factor. We solve equation

$$\alpha_1 K_1 + \alpha_2 K_2 + \frac{\partial P_1(x,y)}{\partial x} + \frac{\partial Q_1(x,y)}{\partial y} = 0,$$

and obtain

$$\alpha_1 = \frac{\lambda(a_{20}b_{21} - 2a_{30}b_{11} - a_{20}a_{30})}{4a_{30}\sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}} - \frac{3}{4} - \frac{b_{21}}{4a_{30}},$$
$$\alpha_2 = -\frac{\lambda(a_{20}b_{21} - 2a_{30}b_{11} - a_{20}a_{30})}{4a_{30}\sqrt{\lambda(-4ia_{30} + a_{20}^2\lambda)}} - \frac{3}{4} - \frac{b_{21}}{4a_{30}}.$$

Therefore, the integrating factor of system (4.30) is

$$\mu = f_1^{\alpha_1} f_2^{\alpha_2},$$

which implies that system (4.30) has an analytic first integral of the form (2.4) [1]. By Theorem 4.4.3 of [29] the second equation of (4.30) is linearizable by the change of coordinates

$$y_1 = \frac{\Psi}{x_1}.$$

Case (14). Systems (2.8) and (4.7) are

$$\dot{x} = ix + x(a_{11}y + a_{30}x^2), \quad \dot{y} = -iy + y(-2a_{11}x + b_{03}y^2), \tag{4.31}$$

and

$$\dot{x} = ix + \lambda x(a_{11}y + a_{30}x^2), \quad \dot{y} = -iy + \lambda y(-2a_{11}x + b_{03}y^2), \tag{4.32}$$

respectively. System (4.31) is considered in [7, Theorem 2, case 1], where the corresponding linearizable change of coordinates is obtained. So, by Lemma 3.3 system (4.32) is linearizable. Therefore, O is a linearizable weakly persistent center of (4.31). □

Remark 4.3. We see that system (2.8) is system (2.7) with $a_{02} = a_{03} = 0$. Notice that if we apply to the conditions of Theorem 4.2 the involution $a_{ij} \leftrightarrow b_{ji}$, then we obtain conditions for linearizable weakly persistent center at the origin of the dual system of (2.8), i.e. system (2.7) with $b_{20} = b_{30} = 0$:

$$\dot{x} = ix + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3,$$
$$\dot{y} = -iy + y(b_{11}x + b_{02}y + b_{21}x^2 + b_{12}xy + b_{03}y^2).$$

5 Conclusions

In this paper we introduced the notions of linearizable persistent and linearizable weakly persistent centers. We found necessary and sufficient conditions for the existence of a linearizable persistent center to the general cubic system (2.7) and to the existence of a linearizable weakly persistent center to the 'semi-Kolmogorov' system (2.8). For obtaining necessary conditions for linearizable (weakly) persistent centers of cubic polynomial system (2.7) we used methods and algorithms of computational algebra. The sufficiency of the obtained conditions in most cases was proven using the Darboux linearization method, which is one of the main tools to find a linearizing change of coordinates. However, to simplify the proof in some cases we used Lemma 3.3, which provide a way to obtain a linearizing transformation for system (3.10) from system (3.9). In one special case we performed the change of coordinates, named blow up, in the system to obtain a linearizing transformation.

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References

[1] V. V. Amel'kin, N. A. Lukashevich, A. P. Sadovskii, *Nonlinear oscillations in second order systems*, Belarusian State University, Minsk, 1982. Zbl 0526.70024

[2] X. Chen, J. Giné, V. G. Romanovski, D. S. Shafer, The $1 : -q$ resonant center problem for certain cubic Lotka–Volterra systems, *Appl. Math. Comput.* **218**(2012), No. 23, 11620–11633. https://doi.org/10.1016/j.amc.2012.05.045; MR2944006; Zbl 1365.34006

[3] X. Chen, V. Romanovski, W. Zhang, Persistent centers of complex systems, *Bull. Sci. Math.* **138**(2014), No. 1, 110–123. https://doi.org/10.1016/j.bulsci.2013.10.003; MR3245496; Zbl 1291.34061

[4] A. Cima, A. Gasull, J. C. Medrado, On persistent centers, *Bull. Sci. Math.* **133**(2009), 644–657. https://doi.org/10.1016/j.bulsci.2008.08.007; MR2561367; Zbl 1183.34041

[5] A. Cima, A. Gasull, F. Mañosas, A simple solution of some composition conjectures for Abel equations, *J. Math. Anal. Appl.* **398**(2013), No. 2, 477–486. https://doi.org/10.1016/j.jmaa.2012.09.006; MR2990073; Zbl 1269.34035

[6] D. Cox, J. Little, D. O'Shea, *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra*, Springer, 3rd edition, New York, 2007. https://doi.org/10.1007/978-0-387-35651-8; MR2290010; Zbl 1118.13001

[7] D. Dolićanin, G. V. Milovanović, V. G. Romanovski, Linearizability conditions for a cubic system, *Appl. Math. Comput.* **190**(2007), No. 1, 937–945. https://doi.org/10.1016/j.amc.2007.01.084; MR2338765; Zbl 1133.34331

[8] W. Decker, S. Laplagne, G. Pfister, H. A. Schonemann, SINGULAR *3-1 library for computing the prime decomposition and radical of ideals, primdec.lib*, 2010.

[9] D. Dolićanin, J. Giné, R. Oliveira, V. G. Romanovski, The center problem for a $2 : -3$ resonant cubic Lotka–Volterra system, *Appl. Math. Comput.* **220**(2013), 12–19. https://doi.org/10.1016/j.amc.2013.06.007; MR3091826; Zbl 1329.34064

[10] M. Dukarić, B. Ferčec, J. Giné, The solution of the $1 : -3$ resonant center problem in the quadratic case, *Appl. Math. Comput.* **237**(2014), 501–511. https://doi.org/10.1016/j.amc.2014.03.147; MR3201150; Zbl 1334.34067

[11] V. Edneral, A. Mahdi, V. Romanovski, D. Shafer, The center problem on a center manifold in $\mathbb{R}^3$, *Nonlinear Anal.* **75**(2012), No. 4, 2614–2622. https://doi.org/10.1016/j.na.2011.11.006; MR2870943; Zbl 1259.34021

[12] B. Ferčec, G. Jaume, M. Mencinger, R. Oliveira, The center problem for a $1 : -4$ resonant quadratic system, *J. Math. Anal. Appl.* **420**(2014), No. 2, 1568–1591. https://doi.org/10.1016/j.jmaa.2014.06.060; MR3240094; Zbl 1301.34043

[13] J. Giné, C. Christopher, M. Prešern, V. G. Romanovski, N. L. Shcheglova, The resonant center problem for a $2 : -3$ resonant cubic Lotka–Volterra system, in: *Lecture Notes in Computer Science (CASC 2012, Maribor, Slovenia, September 3–6, 2012)*, **7442**(2012), 129–142. https://doi.org/10.1007/978-3-642-32973-911; Zbl 1375.34057

[14] J. Giné, V. G. Romanovski, Linearizability conditions for Lotka–Volterra planar complex cubic systems, *J. Phys. A* **42**(2009), 225206-1-225206-15. https://doi.org/10.1088/1751-8113/42/22/225206; MR2515574; Zbl 1188.34115

[15] J. Giné, J. Llibre, Analytic reducibility of nondegenerate centers: Cherkas systems, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 49, 1–10. https://doi.org/10.14232/ejqtde.2016.1.49; MR3533259; Zbl 1363.34092

[16] P. Gianni, B. Trager, G. Zacharias, Gröbner bases and primary decompositions of polynomial ideals, *J. Symbolic Comput.* **6**(1988), No. 2–3, 146–167. https://doi.org/10.1016/S0747-7171(88)80040-3; MR988410; Zbl 0667.13008

[17] G. M. Greuel, G. Pfister, H. A. Schönemann, SINGULAR 3.0 *A Computer Algebra System for Polynomial Computations*, Centre for Computer Algebra, University of Kaiserlautern, 2005. http://www.singular.uni-kl.de.

[18] W. Huang, X. Fan, X. Chen, Linearizability conditions of quasi-cubic systems, *Electron. J. Qual. Theory Differ. Equ.* **2012**, No. 77, 1–10. https://doi.org/10.14232/ejqtde.2012.1.77; MR2981907; Zbl 1340.34137

[19] A. Kolmogorov, Sulla teoria di Volterra della lotta per l'esistenza (in Italian), *Giorn. Ist. Ital. Attuari* **7**(1936), 74–80. Zbl 0013.31404

[20] A. M. Liapunov, *Stability of motion*, Academic Press, New York, 1966. MR0208093; Zbl 0161.06303

[21] C. Liu, G. Chen, C. Li, Integrability and linearizability of the Lotka–Volterra systems, *J. Differential Equations* **198**(2004), No. 2, 301–320. https://doi.org/10.1016/S0022-0396(03)00196-7; MR2038583; Zbl 1046.34075

[22] C. Liu, G. Chen, G. Chen, Integrability of Lotka–Volterra type systems of degree 4, *J. Math. Anal. Appl.* **388**(2012), No. 2, 1107–1116. https://doi.org/10.1016/j.jmaa.2011.10.056; MR2869810; Zbl 1259.34007

[23] J. Llibre, C. Pantazi, S. Walcher, First integrals of local analytic differential systems, *Bull. Sci. Math.* **136**(2012), No. 3, 342–359. https://doi.org/10.1016/j.bulsci.2011.10.003; MR2914953; Zbl 1245.34004

[24] A. J. Lotka, Analytical note on certain rhythmic relations in organic systems, *Proc. Natl. Acad. Sci. USA* **6**(1920), No. 7, 410–415. https://doi.org/10.1073/pnas.6.7.410

[25] H. Poincaré, Mémoire sur les courbes définies par une équation differentielle (in French), *J. Math. Pures Appl. (9)* (Sér. 3) **7**(1881) 375–422; (Sér. 3) **8**(1882), 251–296; (Sér. 4) **1**(1885) 167–244; (Sér. 4) **2**(1886), 151–217. Zbl 14.0666.01

[26] G. Revel, D. M. Alonso, J. L. Moiola, A degenerate 2:3 resonant Hopf–Hopf bifurcation as organizing center of the dynamics: numerical semiglobal results, *SIAM J. Appl. Dyn. Syst.* **14**(2015), No. 2, 1130–1164. https://doi.org/10.1137/140968197; MR3359672; Zbl 1336.34059

[27] V. G. Romanovski, M. Robnik, The centre and isochronicity problems for some cubic systems, *J. Phys. A* **34**(2001), 10267–10292. https://doi.org/10.1088/0305-4470/34/47/326; MR1872415; Zbl 1014.34028

[28] V. Romanovski, D. Shafer, On the center problem for $p : -q$ resonant polynomial vector fields, *Bull. Belg. Math. Soc. Simon Stevin* **115**(2008), No. 5, 871–887. MR2484138; Zbl 1163.34020

[29] V. G. Romanovski, D. S. Shafer, *The center and cyclicity problems: a computational algebra approach*. Birkhäuser, Boston, 2009. https://doi.org/10.1007/978-0-8176-4727-8; MR2500203; Zbl 1192.34003

[30] B. Sang, Center problem for a class of degenerate quartic systems, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 74, 1–17. https://doi.org/10.14232/ejqtde.2014.1.74; MR3304200; Zbl 1324.34073

[31] B. Sang, C. Niu, Solution of center-focus problem for a class of cubic systems, *Chin. Ann. Math. Ser. B* **37**(2016), No. 1, 149–160. https://doi.org/10.1007/s11401-015-0950-4; MR3436604; Zbl 1343.34078

[32] V. Volterra, *Leçons sur la théorie mathématique de la lutte pour la vie* (in French), Gauthier Villars, Paris, 1931. Zbl 0002.04202

[33] H. Żołądek, The problem of center for resonant singular points of polynomial vector fields, *J. Differential Equations* **137**(1997), No. 1, 94–118. https://doi.org/10.1006/jdeq.1997.3260; MR1451537; Zbl 0885.34034