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Existence of positive solutions for nonlinear Dirichlet problems with gradient dependence and arbitrary growth

Nikolaos S. Papageorgiou[1], **Calogero Vetro**[✉ 2] and **Francesca Vetro**[3]

[1]Department of Mathematics, National Technical University, Zografou campus, 15780, Athens, Greece
[2]Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy
[3]Department of Energy, Information Engineering and Mathematical Models (DEIM), University of Palermo, Viale delle Scienze ed. 8, 90128, Palermo, Italy

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Abstract. We consider a nonlinear elliptic problem driven by the Dirichlet p-Laplacian and a reaction term which depends also on the gradient (convection). No growth condition is imposed on the reaction term $f(z,\cdot,y)$. Using topological tools and the asymptotic analysis of a family of perturbed problems, we prove the existence of a positive smooth solution.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following nonlinear Dirichlet problem with gradient dependence

$$-\Delta_p u(z) = f(z, u(z), \nabla u(z)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0, \ u > 0. \tag{1.1}$$

In this problem Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W_0^{1,p}(\Omega),\ 1 < p < \infty.$$

The dependence of the reaction term $f(z,x,y)$ on the gradient ∇u of the unknown function u, precludes the use of variational methods in the study of problem (1.1). Instead our approach is topological and uses the theory of nonlinear operators of monotone type and the asymptotic analysis of a perturbation of the original problem. We prove the existence of a

✉Corresponding author. Email: calogero.vetro@unipa.it

positive smooth solution, without imposing any growth condition on $f(z,\cdot,y)$. Instead, we employ a Hartman-type condition on $f(z,\cdot,y)$ which leads to an a priori bound for the positive solutions. This absence of any growth condition on $f(z,\cdot,y)$ distinguishes our work from previous ones on elliptic equations with convection. We refer to the papers of de Figueiredo–Girardi–Matzeu [3], Girardi–Matzeu [6] (semilinear problems driven by the Laplacian) and Faraci–Motreanu–Puglisi [2], Ruiz [9] (nonlinear problems driven by the p-Laplacian). We mention also the recent work of Gasiński–Papageorgiou [5] on Neumann problems driven by a differential operator of the form $\operatorname{div}(a(u)\nabla u)$. In all the aforementioned works $f(z,\cdot,y)$ exhibits the usual subcritical polynomial growth.

2 Mathematical background – hypotheses

Let X be a reflexive Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). A map $V : X \to X^*$ is said to be "pseudomonotone", if it has the following property:

$$\text{“}u_n \xrightarrow{w} u \text{ in X}\,,\ V(u_n) \xrightarrow{w} u^* \text{ in } X^* \text{ and } \limsup_{n\to+\infty}\langle V(u_n), u_n - u\rangle \leq 0$$
$$\text{imply that}\quad u^* = V(u) \quad \text{and} \quad \langle V(u_n), u_n\rangle \to \langle V(u), u\rangle\text{”}.$$

A maximal monotone, everywhere defined operator is pseudomonotone. Moreover, if $V = A + K$ with A maximal monotone and everywhere defined and K is completely continuous (that is, $u_n \xrightarrow{w} u$ in $X \Rightarrow K(u_n) \to K(u)$ in X^*), then V is pseudomonotone (see Gasiński–Papageorgiou [4], p. 336).

Pseudomonotone operators exhibit remarkable surjectivity properties. More precisely we have (see Gasiński–Papageorgiou [4], p. 336).

Proposition 2.1. *If $V : X \to X^*$ is pseudomonotone and strongly coercive, that is,*

$$\lim_{\|u\|\to+\infty}\frac{\langle V(u), u\rangle}{\|u\|} = +\infty,$$

then V is surjective (that is, $R(V) = X^*$).

The following two spaces will be used in the analysis of problem (1.1):
- the Sobolev space $W_0^{1,p}(\Omega)$;
- the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$.

By $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we can take

$$\|u\| = \|\nabla u\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\right\},$$

where $\frac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Given $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$, we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u^{\pm} \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Consider the following nonlinear eigenvalue problem:

$$-\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{2.1}$$

We know that (2.1) has a smallest eigenvalue $\widehat{\lambda}_1$ which has the following properties:

- $\widehat{\lambda}_1 > 0$ and $\widehat{\lambda}_1$ is isolated (that is, if $\widehat{\sigma}(p)$ denotes the spectrum of (2.1), then we can find $\varepsilon > 0$ such that $(\widehat{\lambda}_1, \widehat{\lambda}_1 + \varepsilon) \cap \widehat{\sigma}(p) = \emptyset$);
- $\widehat{\lambda}_1$ is simple (that is, if $\widehat{u}$, $\widetilde{u}$ are eigenfunctions corresponding to $\widehat{\lambda}_1$, then $\widehat{u} = \xi\widetilde{u}$ for some $\xi \in \mathbb{R} \setminus \{0\}$);
- $$\widehat{\lambda}_1 = \inf\left[\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\right]. \tag{2.2}$$

The infimum in (2.2) is realized on the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1$. The above properties imply that the elements of this eigenspace, have fixed sign. By $\widehat{u}_1$ we denote the L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1$. The nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasiński–Papageorgiou [4], pp. 737–738) imply that $\widehat{u}_1 \in \operatorname{int} C_+$.

A function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is said to be "Carathéodory", if

- for all $(x, y) \in \mathbb{R} \times \mathbb{R}^N$, $z \to f(z, x, y)$ is measurable;
- for almost all $z \in \Omega$, $(x, y) \to f(z, x, y)$ is continuous.

Such a function is necessarily jointly measurable (see Hu–Papageorgiou [8], p. 142).

The hypotheses on the reaction term $f(z, x, y)$ are the following:

$H(f)$ $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0, y) = 0$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ and

(i) there exist $M > 0$ and $\delta > 0$ such that

$$f(z, x, y) \leq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } |x - M| \leq \delta, \text{ all } |y| \leq \delta;$$

(ii) there exists $a_M \in L^\infty(\Omega)$ such that

$$|f(z, x, y)| \leq a_M(z)[1 + |y|^{p-1}] \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq M, \text{ all } y \in \mathbb{R}^N;$$

(iii) for every $c > 0$, there exists $\eta_c \in L^\infty(\Omega)$ such that

$$\eta_c(z) \geq \widehat{\lambda}_1 \quad \text{for a.a. } z \in \Omega, \ \eta_c \not\equiv \widehat{\lambda}_1,$$
$$\liminf_{x \to 0^+} \frac{f(z, x, y)}{x^{p-1}} \geq \eta_c(z) \quad \text{uniformly for a.a. } z \in \Omega, \text{ all } |y| \leq c.$$

Remark 2.2. Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we assume that

$$f(z,x,y) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \le 0, \text{ all } y \in \mathbb{R}^N.$$

Hypothesis $H(f)\,(i)$ is essentially a condition due to Hartman [7] (p. 433). It was used by Hartman [7] for ordinary Dirichlet differential systems.

Example 2.3. The following function satisfies hypotheses $H(f)$. For the sake of simplicity we drop the z-dependence.

$$f(x,y) = \begin{cases} \eta x^{p-1} - 2\eta x^{r-1} + x^{\tau-1}|y|^{p-1} & \text{if } 0 \le x \le 1, \\ g_0(x,y) & \text{if } 1 < x, \end{cases}$$

with $g_0(\cdot,\cdot)$ any continuous function, $\eta > \widehat{\lambda}_1 > 0$, $p < r, \tau$. This particular $f(\cdot,\cdot)$ satisfies the Hartman condition $H(f)\,(i)$ with $M = 1$ and $\delta > 0$ small such that $\delta \le \eta^{\frac{1}{p-1}}$.

Let $M > 0$ be as in hypothesis $H(f)\,(i)$ and consider the nonexpansive function (that is, Lipschitz continuous with Lipschitz constant 1) $p_M : \mathbb{R} \to \mathbb{R}$ defined by

$$p_M(x) = \begin{cases} x^+ & \text{if } x \le M, \\ M & \text{if } M < x. \end{cases}$$

Clearly $|p_M(x)| \le |x|$ for all $x \in \mathbb{R}$. We introduce the following function

$$\widehat{f}(z,x,y) = f(z, p_M(x), y) + p_M(x)^{p-1}. \tag{2.3}$$

This is a Carathéodory function. Let $e \in \operatorname{int} C_+$ and $\varepsilon > 0$. We consider the following auxiliary Dirichlet problem:

$$-\Delta_p u(z) + |u(z)|^{p-2}u(z) = \widehat{f}(z, u(z), \nabla u(z)) + \varepsilon e(z) \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = 0. \tag{2.4}$$

Proposition 2.4. *If hypotheses $H(f)$ hold and $\varepsilon > 0$ is small, then problem (2.4) admits a solution $u_\varepsilon \in \operatorname{int} C_+$ and $0 \le u_\varepsilon(z) \le M$ for all $z \in \overline{\Omega}$.*

Proof. Let $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ ($\frac{1}{p} + \frac{1}{p'} = 1$) be the nonlinear map defined by

$$\langle A(u), h \rangle = \int_\Omega |\nabla u|^{p-2} (\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

It is well-known (see, for example Gasiński–Papageorgiou [4]), that the map $A(\cdot)$ is bounded (that is, maps bounded sets to bounded ones), continuous, strictly monotone, hence maximal monotone too. Also let $\psi_p : W_0^{1,p}(\Omega) \to L^{p'}(\Omega)$ be

$$\psi_p(u) = |u|^{p-2}u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

This map too is bounded, continuous and maximal monotone (recall that $L^{p'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$).

Let $N_{\widehat{f}}$ denote the Nemitsky map corresponding to the Carathéodory function $\widehat{f}$, that is,

$$N_{\widehat{f}}(u)(\cdot) = \widehat{f}(\cdot, u(\cdot), \nabla u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Hypothesis $H(f)\,(ii)$ and the Krasnoselskii theorem (see Gasiński–Papageorgiou [4], p. 407) imply that

$$N_{\widehat{f}}(\cdot) \text{ is bounded, continuous.}$$

The compact embeddings of

$$W^{1,p}(\Omega) \text{ into } L^p(\Omega) \quad \text{and of} \quad L^{p'}(\Omega) \text{ into } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$$

(use the Sobolev embedding theorem and Lemma 2.2.27, p. 141 of Gasiński–Papageorgiou [4]), imply that ψ_p and $N_{\widehat{f}}$ are both completely continuous maps.

Let $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be defined by

$$V(u) = A(u) + \psi_p(u) - N_{\widehat{f}}(u) - \varepsilon e \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently $V(\cdot)$ is bounded, continuous and pseudomonotone. Also, for all $u \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}
\langle V(u), u\rangle &= \|\nabla u\|_p^p + \|u\|_p^p - \int_\Omega \widehat{f}(z, u, \nabla u) u \, dz - \varepsilon \int_\Omega e u \, dz \\
&\geq \|u\|^p - c_1 \|u\|^{p-1} - c_1 \quad \text{for some } c_1 > 0 \quad (\text{see hypothesis } H(f)\,(ii)), \\
\Rightarrow \quad & V(\cdot) \text{ is strongly coercive} \quad (\text{recall } p > 1).
\end{aligned}$$

Then we can use Proposition 2.1 and find $u_\varepsilon \in W_0^{1,p}(\Omega)$, $u_\varepsilon \neq 0$ such that

$$\begin{aligned}
& V(u_\varepsilon) = 0 \\
\Rightarrow \quad & \langle A(u_\varepsilon), h\rangle + \int_\Omega |u_\varepsilon|^{p-2} u_\varepsilon h \, dz = \int_\Omega \widehat{f}(z, u_\varepsilon, \nabla u_\varepsilon) h \, dz + \varepsilon \int_\Omega e h \, dz \\
& \qquad\qquad\qquad\qquad \text{for all } h \in W_0^{1,p}(\Omega).
\end{aligned} \tag{2.5}$$

In (2.5) we choose $h = -u_\varepsilon^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
& \|\nabla u_\varepsilon^-\|_p^p \leq 0 \quad (\text{see (2.3) and recall that } e \in \operatorname{int} C_+), \\
\Rightarrow \quad & u_\varepsilon \geq 0, \quad u_\varepsilon \neq 0.
\end{aligned}$$

From (2.5) we have

$$\begin{cases}
-\Delta_p u_\varepsilon(z) + u_\varepsilon(z)^{p-1} = f(z, p_M(u_\varepsilon(z)), \nabla u_\varepsilon(z)) + p_M(u_\varepsilon(z))^{p-1} + \varepsilon e(z) \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } z \in \Omega, \\
u_\varepsilon\big|_{\partial\Omega} = 0.
\end{cases} \tag{2.6}$$

From (2.6) and the nonlinear regularity theory (see Gasiński–Papageorgiou [4], pp. 737–738), we have

$$u_\varepsilon \in C_+ \setminus \{0\}.$$

Let $c = \|u_\varepsilon\|_{C_0^1(\overline{\Omega})}$. Hypotheses $H(f)\,(ii), (iii)$ imply that there exists $\widehat{\xi}_c > 0$ such that

$$f(z, x, y) + \widehat{\xi}_c x^{p-1} \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq M, \text{ all } |y| \leq c.$$

From (2.6) we have

$$\begin{aligned}
& \Delta_p u_\varepsilon(z) \leq [1 + \widehat{\xi}_c] u_\varepsilon(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \\
\Rightarrow \quad & u_\varepsilon \in \operatorname{int} C_+,
\end{aligned}$$

by the nonlinear strong maximum principle (see Gasiński–Papageorgiou [4], p. 738).

Finally we show that

$$0 \le u_\varepsilon(z) \le M \quad \text{for all } z \in \overline{\Omega}. \tag{2.7}$$

To show (2.7) we argue by contradiction. So, suppose that

$$u_\varepsilon(z_0) = \max_{\overline{\Omega}} u_\varepsilon > M.$$

Evidently $z_0 \in \Omega$. Then for $\vartheta > 0$ small we have

$$\frac{\partial u_\varepsilon}{\partial n}\Big|_{\partial B_\vartheta(z_0)} \le 0 \quad \text{and} \quad u_\varepsilon(z) > M \text{ for all } z \in \overline{B}_\vartheta(z_0) \subseteq \Omega.$$

Here $\overline{B}_\vartheta(z_0) = \{z \in \Omega : |z - z_0| \le \vartheta\}$. We have

$$p_M(u_\varepsilon(z)) = M \quad \text{for all } z \in \overline{B}_\vartheta(z_0) \text{ and } \nabla u_\varepsilon(z_0) = 0. \tag{2.8}$$

We choose $\vartheta > 0$ small so that

$$\begin{aligned} & |u_\varepsilon(z) - u_\varepsilon(z_0)| \le \delta \quad \text{and} \quad |\nabla u_\varepsilon(z)| \le \delta \quad \text{for all } z \in \overline{B}_\vartheta(z_0) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(recall } u_\varepsilon \in \operatorname{int} C_+\text{)}, \\ \Rightarrow\ & |p_M(u_\varepsilon(z)) - p_M(u_\varepsilon(z_0))| \le \delta \quad \text{and} \quad |\nabla u_\varepsilon(z)| \le \delta \quad \text{for all } z \in \overline{B}_\vartheta(z_0) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(recall } p_M(\cdot) \text{ is nonexpansive).} \end{aligned} \tag{2.9}$$

From (2.9) and hypothesis $H(f)$ (i), we have

$$f(z, p_M(u_\varepsilon(z)), \nabla u_\varepsilon(z)) p_M(u_\varepsilon(z)) \le 0 \quad \text{for all } z \in \overline{B}_\vartheta(z_0). \tag{2.10}$$

We multiply (2.6) with $u_\varepsilon(z)$ and then integrate over $\overline{B}_\vartheta(z_0)$. Using the nonlinear Green's identity (see Gasiński–Papageorgiou [4], p. 211), we have

$$\begin{aligned} & \int_{\overline{B}_\vartheta(z_0)} |\nabla u_\varepsilon|^p dz - \int_{\partial B_\vartheta(z_0)} |\nabla u_\varepsilon|^{p-2} \frac{\partial u_\varepsilon}{\partial n} u_\varepsilon d\sigma + \int_{\overline{B}_\vartheta(z_0)} u_\varepsilon^p dz \\ & \quad = \int_{\overline{B}_\vartheta(z_0)} [f(z, p_M(u_\varepsilon), \nabla u_\varepsilon) + p_M(u_\varepsilon)^{p-1}] u_\varepsilon dz + \varepsilon \int_{\overline{B}_\vartheta(z_0)} e u_\varepsilon dz \\ & \quad = \int_{\overline{B}_\vartheta(z_0)} [f(z, p_M(u_\varepsilon), \nabla u_\varepsilon) + p_M(u_\varepsilon)^{p-1}] p_M(u_\varepsilon) \frac{u_\varepsilon}{M} dz + \varepsilon \int_{\overline{B}_\vartheta(z_0)} e u_\varepsilon dz \quad \text{(see (2.8))} \\ & \quad \le \int_{\overline{B}_\vartheta(z_0)} M^{p-1} u_\varepsilon dz + \varepsilon \int_{\overline{B}_\vartheta(z_0)} e u_\varepsilon dz \quad \text{(see (2.10) and (2.8))}, \\ \Rightarrow \quad & - \int_{\partial B_\vartheta(z_0)} |\nabla u_\varepsilon|^{p-2} \frac{\partial u_\varepsilon}{\partial n} u_\varepsilon d\sigma + \int_{\overline{B}_\vartheta(z_0)} [u_\varepsilon^{p-1} - M^{p-1} - \varepsilon e] u_\varepsilon dz \le 0. \end{aligned}$$

But from (2.8) we see that for all $\varepsilon > 0$ small (say for $\varepsilon \in (0, \varepsilon_0)$) the left hand side of the last inequality is strictly bigger than zero, a contradiction. Therefore (2.7) is true. □

Proposition 2.5. *If hypotheses $H(f)$ hold, then there exist $\gamma \in (0,1)$ and $c_2 > 0$ such that*

$$u_\varepsilon \in C_0^{1,\gamma}(\overline{\Omega}) \text{ and } \|u_\varepsilon\|_{C_0^{1,\gamma}(\overline{\Omega})} \le c_2 \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

Proof. Let $u_\varepsilon \in \operatorname{int} C_+$ ($\varepsilon \in (0, \varepsilon_0)$) be the solution of (2.4) produced in Proposition 2.4. We know that

$$0 \le u_\varepsilon(z) \le M \quad \text{for all } z \in \overline{\Omega}. \tag{2.11}$$

Also from (2.3) and (2.5), we have

$$-\Delta_p u_\varepsilon(z) = f(z, u_\varepsilon(z), \nabla u_\varepsilon(z)) + \varepsilon u_\varepsilon(z) \quad \text{for a.a. } z \in \Omega,\ u_\varepsilon\big|_{\partial\Omega} = 0. \tag{2.12}$$

From (2.11) and (2.12) and the nonlinear regularity theory (see Gasiński–Papageorgiou [4], Theorem 6.2.7, p. 738), we infer that there exist $\gamma \in (0,1)$ and $c_2 > 0$ such that

$$u_\varepsilon \in C_0^{1,\gamma}(\overline{\Omega}) \text{ and } \|u_\varepsilon\|_{C_0^{1,\gamma}(\overline{\Omega})} \le c_2 \quad \text{for all } \varepsilon \in (0, \varepsilon_0). \qquad \square$$

Now we are ready for the existence result.

Theorem 2.6. *If hypotheses $H(f)$ hold, then problem (1.1) admits a positive solution $\widehat{u} \in \operatorname{int} C_+$.*

Proof. Let $\varepsilon_n \in (0, \varepsilon_0)$, $n \in \mathbb{N}$, and assume that $\varepsilon_n \to 0^+$. Let $u_n = u_{\varepsilon_n} \in \operatorname{int} C_+$ for all $n \in \mathbb{N}$ (see Proposition 2.4). On account of Proposition 2.5 and since $C_0^{1,\gamma}(\overline{\Omega})$ is embedded compactly into $C_0^1(\overline{\Omega})$, we may assume that

$$u_n \to \widehat{u} \quad \text{in } C_0^1(\overline{\Omega}).$$

We claim that $\widehat{u} \neq 0$.

Arguing by contradiction, assume that $\widehat{u} = 0$. If $c = \sup_{n\in\mathbb{N}} \|u_n\|_{C_0^1(\overline{\Omega})}$, then hypothesis $H(f)\,(iii)$ implies that given $\varepsilon > 0$, we can find $\widehat{\delta} = \widehat{\delta}(\varepsilon) > 0$ such that

$$f(z,x,y) \ge [\eta_c(z) - \varepsilon]\, x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } 0 \le x \le \widehat{\delta}, \text{ all } |y| \le c. \tag{2.13}$$

Since $u_n \to 0$ in $C^1(\overline{\Omega})$ (recall that we have assumed that $\widehat{u} = 0$), we can find $n_0 \in \mathbb{N}$ such that

$$0 \le u_n(z) \le \widehat{\delta} \quad \text{for all } z \in \overline{\Omega}, \text{ all } n \ge n_0. \tag{2.14}$$

For $n \ge n_0$, we consider the function

$$R(\widehat{u}_1, u_n)(z) = |\nabla \widehat{u}_1(z)|^p - |\nabla u_n(z)|^{p-2} \left(\nabla u_n(z), \nabla \left(\frac{\widehat{u}_1^p}{u_n^{p-1}} \right)(z) \right)_{\mathbb{R}^N}.$$

From the nonlinear Picone's identity of Allegretto–Huang [1], for $n \ge n_0$ we have

$$\begin{aligned}
0 &\le \int_\Omega R(\widehat{u}_1, u_n) dz \\
&= \|\nabla \widehat{u}_1\|_p^p - \int_\Omega |\nabla u_n(z)|^{p-2} \left(\nabla u_n, \nabla \left(\frac{\widehat{u}_1^p}{u_n^{p-1}} \right) \right)_{\mathbb{R}^N} dz \\
&= \widehat{\lambda}_1 - \int_\Omega f(z, u_n, \nabla u_n) \frac{\widehat{u}_1^p}{u_n^{p-1}} dz - \varepsilon_n \int_\Omega e \frac{\widehat{u}_1^p}{u_n^{p-1}} dz \\
&\le \widehat{\lambda}_1 - \int_\Omega (\eta_c(z) - \varepsilon) \widehat{u}_1^p dz \quad \text{(see (2.13), (2.14) and recall that } e \in \operatorname{int} C_+) \\
&= \int_\Omega (\widehat{\lambda}_1 - \eta_c(z)) \widehat{u}_1^p dz + \varepsilon \quad \text{(recall that } \|\widehat{u}_1\|_p = 1).
\end{aligned} \tag{2.15}$$

Since $\widehat{u}_1 \in \operatorname{int} C_+$ and hypothesis $H(f)$ (iii) says that

$$\eta_c(z) \geq \widehat{\lambda}_1 \quad \text{for a.a. } z \in \Omega,\ \eta_c \neq \widehat{\lambda}_1,$$

we infer that

$$\xi^* = \int_\Omega (\eta_c(z) - \widehat{\lambda}_1)\widehat{u}_1^p dz > 0.$$

So, choosing $\varepsilon \in (0, \xi^*)$, from (2.15) we have

$$0 \leq \int_\Omega R(\widehat{u}_1, u_n)dz < 0 \qquad \text{for all } n \geq n_0,$$

a contradiction. Therefore $\widehat{u} \neq 0$. Moreover, passing to the limits as $n \to +\infty$ (see (2.12) with $\varepsilon = \varepsilon_n$, $n \in \mathbb{N}$), we obtain

$$-\Delta_p \widehat{u}(z) = f(z, \widehat{u}(z), \nabla \widehat{u}(z)) \quad \text{for a.a. } z \in \Omega, \quad \widehat{u}\big|_{\partial\Omega} = 0.$$

As before the nonlinear regularity theory and the nonlinear maximum principle, imply that $\widehat{u} \in \operatorname{int} C_+$. □

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