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Singular solutions of a nonlinear elliptic equation in a punctured domain

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Abstract. We consider the following semilinear problem

$$\begin{cases} -\Delta u(x) = a(x)u^{\sigma}(x),\ x \in \Omega\backslash\{0\} \text{ (in the distributional sense)}, \\ u > 0, \text{ in } \Omega\backslash\{0\}, \\ \lim\limits_{|x|\to 0} |x|^{n-2}u(x) = 0, \\ u(x) = 0,\ x \in \partial\Omega, \end{cases}$$

where $\sigma < 1$, Ω is a bounded regular domain in $\mathbb{R}^n$ $(n \geq 3)$ containing 0 and a is a positive continuous function in $\Omega\backslash\{0\}$, which may be singular at $x = 0$ and/or at the boundary $\partial\Omega$. When the weight function $a(x)$ satisfies suitable assumption related to Karamata class, we prove the existence of a positive continuous solution on $\overline{\Omega}\backslash\{0\}$, which could blow-up at the origin. The global asymptotic behavior of this solution is also obtained.

Keywords: singular positive solution, Green's function, Karamata class, Kato class, blow-up.

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1 Introduction

Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ $(n \geq 3)$ containing 0. In [33], Zhang and Zhao proved the existence of infinitely many positive solutions for the following superlinear elliptic

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problem

$$\begin{cases} -\Delta u(x) = b(x)u^p(x),\ x \in \Omega\backslash\{0\} \text{ (in the distributional sense)}, \\ u > 0,\ \text{in } \Omega\backslash\{0\}, \\ u(x) = 0,\ x \in \partial\Omega, \\ u(x) \sim \frac{c}{|x|^{n-2}} \text{ near } x = 0,\ \text{for any sufficiently small } c > 0, \end{cases} \tag{1.1}$$

where $p > 1$ and b is a Borel measurable function such that the function $x \mapsto \frac{b(x)}{|x|^{(n-2)(p-1)}}$ is in the Kato class $K_n(\Omega)$ given by

$$K_n(\Omega) = \left\{ q \in \mathcal{B}(\Omega),\ \lim_{r \to 0} \left(\sup_{x \in \Omega} \int_{\Omega \cap B(x,r)} \frac{|q(y)|}{|x-y|^{n-2}} dy \right) = 0 \right\},$$

where $\mathcal{B}(\Omega)$ be the set of Borel measurable functions in Ω.

In [23], Mâagli and Zribi generalized the result of Zhang and Zhao [33] by considering the following class $K(\Omega)$:

$$K(\Omega) = \left\{ q \in \mathcal{B}(\Omega),\ \lim_{r \to 0} \left(\sup_{x \in \Omega} \int_{\Omega \cap B(x,r)} \frac{\delta(y)}{\delta(x)} G(x,y)|q(y)|dy \right) = 0 \right\},$$

where $G(x,y)$ is the Green's function of the Laplace operator in Ω and $\delta(x) = d(x, \partial\Omega)$ denotes the Euclidean distance from x to $\partial\Omega$. We recall that for $y \in \Omega$, the Green's function $G(x,y)$ of the Laplacian in Ω is defined as the solution of the following problem

$$\begin{cases} -\Delta G(\cdot, y)(x) = \delta_y(x),\ x \in \Omega, \\ G(x,y) = 0,\ x \in \partial\Omega, \end{cases}$$

where δ_y denotes the Dirac measure at y.

Note that the class $K(\Omega)$ properly contains $K_n(\Omega)$. Indeed, from [23, Remark 2 and Remark 4], we know that

$$x \mapsto |x|^{-\mu} (\delta(x))^{-\lambda} \in K(\Omega) \iff \mu < 2 \text{ and } \lambda < 2,$$

and for $1 \le \lambda < 2$, the function $x \mapsto (\delta(x))^{-\lambda} \notin K_n(\Omega)$.

For more results concerning the existence, uniqueness and asymptotic behavior of positive singular solutions associated with similar problems, we refer the reader to [1,3,7,8,12–15,20–22,27,28,30,31] and their references.

In the present paper, we are interested in the singular and sublinear case. More precisely, we are concerned with the existence and global behavior of positive continuous solutions to the following nonlinear problem:

$$\begin{cases} -\Delta u(x) = a(x)u^\sigma(x),\ x \in \Omega\backslash\{0\} \text{ (in the distributional sense)}, \\ u > 0,\ \text{in } \Omega\backslash\{0\}, \\ \lim\limits_{|x| \to 0} |x|^{n-2} u(x) = 0, \\ u(x) = 0,\ x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\sigma < 1$ and a is a positive continuous function in $\Omega\backslash\{0\}$ which may be singular at $x = 0$ and/or at the boundary $\partial\Omega$. The weight function $a(x)$ is required to satisfy suitable assumptions related to the following Karamata class $\mathcal{K}$.

Definition 1.1. Let $\eta > 0$ and L be a function defined on $(0, \eta)$. Then L belongs to the class $\mathcal{K}$ if

$$L(t) := c \exp\left(\int_t^\eta \frac{v(s)}{s} ds\right),$$

where $c > 0$ and $v \in C([0, \eta])$ with $v(0) = 0$.

Remark 1.2. This definition implies that the class $\mathcal{K}$ is given by

$$\mathcal{K} = \left\{L : (0, \eta) \longrightarrow (0, \infty),\ L \in C^1((0, \eta)) \text{ and } \lim_{t \to 0^+} \frac{tL'(t)}{L(t)} = 0\right\}.$$

We refer to [2,25,29] for examples of functions belonging to the class $\mathcal{K}$. A class of functions in this class is defined by

$$L(t) = \prod_{k=1}^{m} \left(\log_k \left(\frac{\omega}{t}\right)\right)^{\xi_k},$$

where $\log_k t = \log \circ \log \circ \cdots \circ \log t$ (k times), $\xi_k \in \mathbb{R}$, and ω is a sufficiently large positive real number such that L is defined and positive on $(0, \eta)$, and are frequently used as some weight functions (see, for example, [17] and [19]).

Observe that functions belonging to the class $\mathcal{K}$ are in particular slowly varying functions. The initial theory of such functions was developed by Karamata in [16].

In [7], Cîrstea and Rădulescu have proved that the Karamata theory is very useful to study the asymptotic analysis of solutions near the boundary for large classes of nonlinear elliptic problems.

Throughout this paper, we assume that

(**H**) a is a positive continuous function in $\Omega\backslash\{0\}$ satisfying

$$a(x) \approx |x|^{-\mu} L_1(|x|)(\delta(x))^{-\lambda} L_2(\delta(x)), \quad \text{for } x \in \Omega\backslash\{0\}, \tag{1.3}$$

where $\sigma < 1$, $\mu \leq n + (2-n)\sigma$, $\lambda \leq 2$ and L_1, $L_2 \in \mathcal{K}$ defined on $(0, \eta)$ (with $\eta > \operatorname{diam}(\Omega)$) such that

$$\int_0^\eta s^{n+(2-n)\sigma-\mu-1} L_1(s) ds < \infty, \qquad \int_0^\eta s^{1-\lambda} L_2(s) ds < \infty. \tag{1.4}$$

We introduce the function θ defined in $\overline{\Omega}\backslash\{0\}$ by

$$\theta(x) := |x|^{\min(0, \frac{2-\mu}{1-\sigma})} \left(\widetilde{L}_1(|x|)\right)^{\frac{1}{1-\sigma}} (\delta(x))^{\min(1, \frac{2-\lambda}{1-\sigma})} \left(\widetilde{L}_2(\delta(x))\right)^{\frac{1}{1-\sigma}}, \tag{1.5}$$

where $\widetilde{L}_1$ and $\widetilde{L}_2$ are defined on $(0, \eta)$ by

$$\widetilde{L}_1(t) := \begin{cases} 1, & \text{if } \mu < 2, \\ \int_t^\eta \frac{L_1(s)}{s} ds, & \text{if } \mu = 2, \\ L_1(t), & \text{if } 2 < \mu < n + (2-n)\sigma, \\ \int_0^t \frac{L_1(s)}{s} ds, & \text{if } \mu = n + (2-n)\sigma, \end{cases}$$

and

$$\widetilde{L}_2(t) := \begin{cases} 1, & \text{if } \lambda < 1, \\ \int_t^\eta \frac{L_2(s)}{s} ds, & \text{if } \lambda = 1, \\ L_2(t), & \text{if } 1 < \lambda < 2, \\ \int_0^t \frac{L_2(s)}{s} ds, & \text{if } \lambda = 2. \end{cases}$$

Using Karamata's theory and the Schauder fixed point theorem, we prove the following qualitative property.

Theorem 1.3. *Let $\sigma < 1$ and assume that the function a satisfies (H). Then problem (1.2) has at least one positive continuous solution u on $\overline{\Omega}\backslash\{0\}$ such that*

$$\frac{1}{c}\theta(x) \leq u(x) \leq c\theta(x), \tag{1.6}$$

for $x \in \overline{\Omega}\backslash\{0\}$, where c is a positive constant.

Remark 1.4. For $\mu > 2$, it is important to note that $u(x) \to \infty$ as $|x| \to 0$.

From now on, we denote by $\mathcal{B}^+(\Omega)$ the collection of all nonnegative Borel measurable functions in Ω. We refer to the set $C(\overline{\Omega})$ of all continuous functions in $\overline{\Omega}$ and let $C_0(\overline{\Omega})$ be the subclass of $C(\overline{\Omega})$ consisting of functions which vanish continuously on $\partial\Omega$. For $f, g \in \mathcal{B}^+(\Omega)$, we say that $f \approx g$ in Ω, if there exists $c > 0$ such that $\frac{1}{c}f(x) \leq g(x) \leq cf(x)$, for all $x \in \Omega$. The letter c will denote a generic positive constant which may vary from line to line.

We define the potential kernel V on $\mathcal{B}^+(\Omega)$ by

$$Vf(x) = \int_\Omega G(x,y)f(y)dy.$$

We recall that for any function $f \in \mathcal{B}^+(\Omega)$ such that $f \in L^1_{\text{loc}}(\Omega)$ and $Vf \in L^1_{\text{loc}}(\Omega)$, we have

$$-\Delta(Vf) = f, \quad \text{in } \Omega \quad \text{(in the distributional sense)}. \tag{1.7}$$

Note that for any function $f \in \mathcal{B}^+(\Omega)$ such that $Vf(x_0) < \infty$ for some $x_0 \in \Omega$, we have $Vf \in L^1_{\text{loc}}(\Omega)$ (see [6, Lemma 2.9]).

2 Preliminaries and key tools

2.1 Green's function

In this section, we recall some basic properties on $G(x,y)$, the Green's function of the Laplace operator in Ω. By [32], we have

$$G(x,y) \approx \frac{1}{|x-y|^{n-2}} \min\left\{1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right\}, \qquad x, y \in \Omega. \tag{2.1}$$

Remark 2.1. Since for $a > 0$, $\min(1,a) \approx \dfrac{a}{1+a}$, we deduce from (2.1) that for $x, y \in \Omega$,

$$G(x,y) \approx \frac{\delta(x)\delta(y)}{|x-y|^{n-2}\left(|x-y|^2 + \delta(x)\delta(y)\right)}. \tag{2.2}$$

Lemma 2.2. *Let $t > 0$. Then for $x \in \mathbb{R}^n$, we have*

$$\int_{S^{n-1}} \frac{1}{|x - t\omega|^{n-2}} \sigma(d\omega) = \frac{1}{\max(|x|, t)^{n-2}},$$

where σ is the normalized measure on the unit sphere S^{n-1} of $\mathbb{R}^n$.

Proof. See [26, Proposition 1.7]. □

The next result is due to Mâagli and Zribi, see [22, Lemma 1].

Lemma 2.3. *Let $g \in \mathcal{B}^+(\Omega)$ and v be a nonnegative superharmonic function on Ω. Then for any $w \in \mathcal{B}(\Omega)$ such that $V(g|w|) < \infty$ and $w + V(gw) = v$, we have*

$$0 \leq w \leq v.$$

2.2 Kato class $K(\Omega)$

In this subsection, we recall and prove some properties concerning the class $K(\Omega)$.

Proposition 2.4. *Let $q \in K(\Omega)$, $x_0 \in \overline{\Omega}$ and h_0 be a positive superharmonic function in Ω. Then we have*

(i)
$$\lim_{r \to 0} \left(\sup_{x \in \Omega} \frac{1}{h_0(x)} \int_{\Omega \cap B(x_0, r)} G(x, y) h_0(y) |q(y)| dy \right) = 0.$$

(ii) The function $x \mapsto \delta(x) q(x)$ is in $L^1(\Omega)$.

Proof. See [23]. □

Proposition 2.5. *Let $q \in K(\Omega)$. Then the function*

$$v(x) := |x|^{n-2} \int_\Omega G(x, y) |y|^{2-n} q(y) dy$$

belongs to $C_0(\overline{\Omega})$.

Proof. Let $\varepsilon > 0$, $x_0 \in \overline{\Omega}$ and $q \in K(\Omega)$. Using Proposition 2.4 (i) with $h_0(x) = |x|^{2-n}$, there exists $r > 0$, such that

$$\sup_{\xi \in \Omega} |\xi|^{n-2} \int_{\Omega \cap B(0,r)} G(\xi, y) |y|^{2-n} |q(y)| dy \leq \frac{\varepsilon}{8}$$

and

$$\sup_{\xi \in \Omega} |\xi|^{n-2} \int_{\Omega \cap B^c(0,r) \cap B(x_0, r)} G(\xi, y) |y|^{2-n} |q(y)| dy \leq \frac{\varepsilon}{8}.$$

If $x_0 \in \Omega$ and $x \in B(x_0, \frac{r}{2}) \cap \Omega$, we have

$$\begin{aligned}
|v(x) - v(x_0)| &\leq 2 \sup_{\xi \in \Omega} |\xi|^{n-2} \int_{\Omega \cap B(0,r)} G(\xi, y) |y|^{2-n} |q(y)| dy \\
&\quad + 2 \sup_{\xi \in \Omega} |\xi|^{n-2} \int_{\Omega \cap B^c(0,r) \cap B(x_0, r)} G(\xi, y) |y|^{2-n} |q(y)| dy \\
&\quad + \int_{\Omega_0} \left| |x|^{n-2} G(x, y) - |x_0|^{n-2} G(x_0, y) \right| |y|^{2-n} |q(y)| dy, \\
&\leq \frac{\varepsilon}{2} + \int_{\Omega_0} \left| |x|^{n-2} G(x, y) - |x_0|^{n-2} G(x_0, y) \right| |y|^{2-n} |q(y)| dy,
\end{aligned}$$

where $\Omega_0 = \Omega \cap B^c(0,r) \cap B^c(x_0,r)$.

Since for all $x \in B(x_0, \frac{r}{2}) \cap \Omega$ and $y \in \Omega_0$, we have $|x-y| \geq \frac{r}{2}$ and $|y| \geq r$, we deduce by (2.2) that

$$|x|^{n-2} G(x,y) |y|^{2-n} \leq c \frac{\delta(x)\delta(y)}{|x-y|^n} \leq \frac{\widetilde{c}}{r^n}\delta(y),$$

where c and $\widetilde{c}$ are some positive constants.

Since $(x,y) \mapsto |x|^{n-2} G(x,y)$ is continuous on $\left(B(x_0, \frac{r}{2}) \cap \Omega\right) \times \Omega_0$, we get by Proposition 2.4 (ii) and Lebesgue's dominated convergence theorem,

$$\int_{\Omega_0} \left| |x|^{n-2} G(x,y) - |x_0|^{n-2} G(x_0,y) \right| |y|^{2-n} |q(y)| dy \to 0 \quad \text{as } x \to x_0.$$

It follows that there exists $\delta > 0$ with $\delta < \frac{r}{2}$ such that if $x \in B(x_0, \delta) \cap \Omega$,

$$\int_{\Omega_0} \left| |x|^{n-2} G(x,y) - |x_0|^{n-2} G(x_0,y) \right| |y|^{2-n} |q(y)| dy \leq \frac{\varepsilon}{2}.$$

Hence for $x \in B(x_0, \delta) \cap \Omega$, we have

$$|v(x) - v(x_0)| \leq \varepsilon.$$

This implies that

$$\lim_{x \to x_0} v(x) = v(x_0).$$

If $x_0 \in \partial\Omega$ and $x \in B(x_0, \frac{r}{2}) \cap \Omega$, then we have

$$\begin{aligned} |v(x)| \leq & \sup_{\xi \in \Omega} |\xi|^{n-2} \int_{\Omega \cap B(0,r)} G(\xi,y) |y|^{2-n} |q(y)| dy \\ & + \sup_{\xi \in \Omega} |\xi|^{n-2} \int_{\Omega \cap B^c(0,r) \cap B(x_0,r)} G(\xi,y) |y|^{2-n} |q(y)| dy \\ & + \int_{\Omega_0} |x|^{n-2} G(x,y) |y|^{2-n} |q(y)| dy. \end{aligned}$$

Now, since $\lim_{x \to x_0} |x|^{n-2} G(x,y) |y|^{2-n} = 0$, for all $y \in \Omega_0$, we deduce by similar arguments as above that

$$\lim_{x \to x_0} v(x) = v(x_0).$$

So, we conclude that $v \in C_0(\overline{\Omega})$. □

2.3 Karamata class

In this section, we collect some properties of the Karamata functions, which will be used later.

Lemma 2.6 (See [25,29]). *Let $\gamma \in \mathbb{R}$ and $L \in \mathcal{K}$. We have*

(i) *If $\gamma > -1$, then $\int_0^\eta s^\gamma L(s) ds$ converges and $\int_0^t s^\gamma L(s) ds \underset{t \to 0^+}{\sim} \frac{t^{1+\gamma} L(t)}{1+\gamma}$.*

(ii) *If $\gamma < -1$, then $\int_0^\eta s^\gamma L(s) ds$ diverges and $\int_t^\eta s^\gamma L(s) ds \underset{t \to 0^+}{\sim} -\frac{t^{1+\gamma} L(t)}{1+\gamma}$.*

Lemma 2.7 (See [5,29]).

(i) For $L \in \mathcal{K}$ and $\varepsilon > 0$, we have

$$\lim_{t\to 0^+} t^{\varepsilon} L(t) = 0 \text{ and } \lim_{t\to 0^+} t^{-\varepsilon} L(t) = \infty.$$

(ii) Let $L_1, L_2 \in \mathcal{K}$ and $p \in \mathbb{R}$. Then we have

$$L_1 + L_2,\ L_1 L_2, \text{ and } L_1^p \text{ are in } \mathcal{K}.$$

(iii) For $L \in \mathcal{K}$, we have

$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s} ds} = 0.$$

In particular, we have

$$t \mapsto \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(s)}{s} ds < \infty$, then we have $\lim_{t\to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s} ds} = 0$.
In particular

$$t \to \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}.$$

Proposition 2.8. *For $i \in \{1,2\}$, let $M_i \in \mathcal{K}$ and $\lambda_i \in \mathbb{R}$. The following properties are equivalent.*

(i) The function $x \mapsto |x|^{-\lambda_1} M_1(|x|) (\delta(x))^{-\lambda_2} M_2(\delta(x))$ is in $K(\Omega)$.

(ii) $\int_0^\eta s^{1-\lambda_i} M_i(s) ds < \infty$, for $i \in \{1,2\}$.

(iii) $\lambda_i < 2$ or $\lambda_i = 2$ with $\int_0^\eta \frac{M_i(s)}{s} ds < \infty$, for $i \in \{1,2\}$.

Proof. The proof follows by similar arguments as in [24, Proposition 7]. □

Next, we recall the following lemma due to Lazer and McKenna [18, p. 726].

Lemma 2.9.

$$\int_\Omega (\delta(x))^r \, dx < \infty \quad \text{if and only if} \quad r > -1.$$

Following the proof of the previous lemma, we deduce the following property.

Remark 2.10. Let $\eta > 0$ and ψ be a nonnegative continuous monotone function on $(0,\eta)$ such that $\int_0^\eta \psi(t) dt < \infty$. Then

$$\int_\Omega \psi(\delta(x)) \, dx < \infty.$$

Proposition 2.11. *Let $\nu \leq 2$ and $L \in \mathcal{K}$ defined on $(0,\eta)$ ($\eta > \operatorname{diam}(\Omega)$) such that $\int_0^\eta t^{1-\nu} L(t) dt < \infty$. Then*

$$0 < \int_\Omega (\delta(x))^{1-\nu} L(\delta(x)) dx < \infty,$$

that is, $\int_\Omega (\delta(x))^{1-\nu} L(\delta(x)) dx \approx 1$.

Proof. Case 1: $\nu < 2$.

Using Lemma 2.7 (i), we deduce that the function $x \mapsto (\delta(x))^{1-\frac{\nu}{2}} L(\delta(x))$ is positive and belongs to $C_0(\overline{\Omega})$. Thus, there exists $c > 0$ such that for $x \in \Omega$

$$0 < (\delta(x))^{1-\nu} L(\delta(x)) \leq c\,(\delta(x))^{-\frac{\nu}{2}}.$$

Therefore by Lemma 2.9, we deduce that

$$0 < \int_\Omega (\delta(x))^{1-\nu} L(\delta(x))dx < \infty.$$

Case 2: $\nu = 2$.

Let $\psi(t) = \frac{L(t)}{t}$. Since by Remark 1.2 $\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0$, we deduce that $\lim_{t\to 0^+} \frac{t\psi'(t)}{\psi(t)} = -1$. So the function ψ is nonincreasing in a neighborhood of zero. Hence the result follows from Remark 2.10. This ends the proof. □

The next result will play an important role in the proof of our main result.

Proposition 2.12. *Let $\gamma \leq n$, $\nu \leq 2$ and $L_3, L_4 \in \mathcal{K}$ such that*

$$\int_0^\eta s^{n-\gamma-1} L_3(s)ds < \infty \quad \text{and} \quad \int_0^\eta s^{1-\nu} L_4(s)ds < \infty, \quad \text{for } \eta > \operatorname{diam}(\Omega). \tag{2.3}$$

Set

$$b(x) = |x|^{-\gamma} L_3(|x|)(\delta(x))^{-\nu} L_4(\delta(x)), \quad \text{for } x \in \Omega\backslash\{0\}.$$

Then for $x \in \Omega\backslash\{0\}$,

$$Vb(x) \approx |x|^{\min(0,2-\gamma)} \widetilde{L}_3(|x|)(\delta(x))^{\min(1,2-\nu)} \widetilde{L}_4(\delta(x)),$$

where $\widetilde{L}_3$ and $\widetilde{L}_4$ are defined on $(0,\eta)$ by

$$\widetilde{L}_3(t) := \begin{cases} 1, & \text{if } \gamma < 2, \\ \int_t^\eta \frac{L_3(s)}{s} ds, & \text{if } \gamma = 2, \\ L_3(t), & \text{if } 2 < \gamma < n, \\ \int_0^t \frac{L_3(s)}{s} ds, & \text{if } \gamma = n, \end{cases}$$

and

$$\widetilde{L}_4(t) := \begin{cases} 1, & \text{if } \nu < 1, \\ \int_t^\eta \frac{L_4(s)}{s} ds, & \text{if } \nu = 1, \\ L_4(t), & \text{if } 1 < \nu < 2, \\ \int_0^t \frac{L_4(s)}{s} ds, & \text{if } \nu = 2. \end{cases}$$

Proof. Let $r > 0$ such that $B(0,3r) \subset \Omega$. For $x \in \Omega\backslash\{0\}$, we have

$$Vb(x) = \int_{B(0,2r)} G(x,y)b(y)dy + \int_{\Omega\backslash B(0,2r)} G(x,y)b(y)dy.$$

We distinguish two cases.

Case 1. $0 < |x| < r$. Using (2.2), we obtain that

$$Vb(x) \approx \int_{B(0,2r)} \frac{1}{|x-y|^{n-2}} |y|^{-\gamma} L_3(|y|)dy + \int_{\Omega\backslash B(0,2r)} (\delta(y))^{1-\nu} L_4(\delta(y))dy.$$

Now we have

$$0 < \int_{\Omega\backslash B(0,2r)} (\delta(y))^{1-\nu} L_4(\delta(y))dy \leq \int_{\Omega} (\delta(y))^{1-\nu} L_4(\delta(y))dy.$$

Using Lemma 2.2 and Proposition 2.11, we deduce that

$$\begin{aligned} Vb(x) &\approx \int_0^{2r} \frac{t^{n-1-\gamma}}{\max\left(|x|,t\right)^{n-2}} L_3(t)dt + 1 \\ &\approx |x|^{2-n} \int_0^{|x|} t^{n-1-\gamma} L_3(t)dt + \left(1 + \int_{|x|}^{2r} t^{1-\gamma} L_3(t)dt\right). \end{aligned}$$

Using (2.3) and Lemma 2.6, we deduce that

$$\int_0^{|x|} t^{n-\gamma-1} L_3(t)dt \approx \begin{cases} |x|^{n-\gamma} L_3(|x|), & \text{if } \gamma < n, \\ \int_0^{|x|} \frac{L_3(t)}{t} dt, & \text{if } \gamma = n \end{cases}$$

and

$$1 + \int_{|x|}^{2r} t^{1-\gamma} L_3(t)dt \approx \begin{cases} 1, & \text{if } \gamma < 2, \\ \int_{|x|}^{\eta} \frac{L_3(t)}{t} dt, & \text{if } \gamma = 2, \\ |x|^{2-\gamma} L_3(|x|), & \text{if } 2 < \gamma \leq n. \end{cases}$$

Hence, it follows by (2.3) and Lemmas 2.6 and 2.7 that for $0 < |x| < r$,

$$Vb(x) \approx \begin{cases} 1, & \text{if } \gamma < 2, \\ \int_{|x|}^{\eta} \frac{L_3(t)}{t} dt, & \text{if } \gamma = 2, \\ |x|^{2-\gamma} L_3(|x|), & \text{if } 2 < \gamma < n, \\ |x|^{2-\gamma} \int_0^{|x|} \frac{L_3(t)}{t} dt, & \text{if } \gamma = n. \end{cases}$$

It follows that

$$Vb(x) \approx |x|^{\min(0,2-\gamma)} \widetilde{L}_3(|x|), \text{ for } 0 < |x| < r. \tag{2.4}$$

Case 2. $x \in \Omega\backslash B(0,3r)$. Using (2.2), we have for $y \in B(0,2r)$,

$$G(x,y) \approx \delta(x) \quad \text{and} \quad (\delta(y))^{-\nu} L_4(\delta(y)) \approx 1.$$

Therefore

$$Vb(x) \approx \delta(x) \int_{B(0,2r)} |y|^{-\gamma} L_3(|y|)dy + \int_{\Omega\backslash B(0,2r)} G(x,y)(\delta(y))^{-\nu} L_4(\delta(y))dy.$$

Since $\int_0^{\eta} s^{n-\gamma-1} L_3(s)ds < \infty$, we deduce that

$$\begin{aligned} Vb(x) &\approx \delta(x) + \int_{\Omega\backslash B(0,2r)} G(x,y)(\delta(y))^{-\nu} L_4(\delta(y))dy \\ &\approx \int_{B(0,2r)} G(x,y)(\delta(y))^{-\nu} L_4(\delta(y))dy + \int_{\Omega\backslash B(0,2r)} G(x,y)(\delta(y))^{-\nu} L_4(\delta(y))dy \\ &\approx \int_{\Omega} G(x,y)(\delta(y))^{-\nu} L_4(\delta(y))dy. \end{aligned}$$

Therefore by [21, Proposition 1], we deduce that

$$Vb(x) \approx (\delta(x))^{\min(1,2-\nu)}\widetilde{L}_4(\delta(x)), \quad \text{for } x \in \Omega\backslash B(0,3r). \tag{2.5}$$

Now, it is clear that the function

$$x \mapsto |x|^{\min(0,2-\gamma)}\,\widetilde{L}_3(|x|)(\delta(x))^{\min(1,2-\nu)}\widetilde{L}_4(\delta(x))$$

is positive and continuous on $\Omega\backslash\{0\}$.

On the other hand, by using (2.3) and Proposition 2.8, the function

$$x \mapsto q(x) := |x|^{n-2-\gamma}\,L_3(|x|)(\delta(x))^{-\nu}L_4(\delta(x))$$

belongs to the class $K(\Omega)$.

So, observing that $b(x) = |x|^{2-n}\,q(x)$, we deduce by Proposition 2.5 that the function Vb is positive and continuous on $\Omega\backslash\{0\}$.

Hence

$$Vb(x) \approx |x|^{\min(0,2-\gamma)}\,\widetilde{L}_3(|x|)(\delta(x))^{\min(1,2-\nu)}\widetilde{L}_4(\delta(x)), \quad \text{on } D, \tag{2.6}$$

where D is the compact set defined by $D := \{x \in \Omega, r \leq |x| \leq 3r\}$.

Combining (2.4), (2.5) and (2.6), we obtain for $x \in \Omega\backslash\{0\}$,

$$Vb(x) \approx |x|^{\min(0,2-\gamma)}\,\widetilde{L}_3(|x|)(\delta(x))^{\min(1,2-\nu)}\widetilde{L}_4(\delta(x)).$$

This completes the proof. □

Proposition 2.13. *Under condition (H), we have*

$$Vp(x) \approx \theta(x), \quad \textit{for } x \in \Omega\backslash\{0\},$$

where $p(x) := a(x)\theta^{\sigma}(x)$, $\sigma < 1$ and θ is defined in (1.5).

Proof. Let a be a function satisfying (H). Using (1.3) and (1.5), we obtain

$$p(x) \approx |x|^{-\gamma}\,L_1(|x|)\left(\widetilde{L}_1(|x|)\right)^{\frac{\sigma}{1-\sigma}}(\delta(x))^{-\nu}L_2(\delta(x))\left(\widetilde{L}_2(\delta(x))\right)^{\frac{\sigma}{1-\sigma}},$$

where $\gamma = \mu - \min\left(0, \frac{2-\mu}{1-\sigma}\right)\sigma$ and $\nu = \lambda - \min\left(1, \frac{2-\lambda}{1-\sigma}\right)\sigma$.

Since $\mu \leq n + (2-n)\sigma$ and $\lambda \leq 2$, then one can easy check that $\gamma \leq n$ and $\nu \leq 2$.

Now using Lemmas 2.6 and 2.7 and Proposition 2.12 with $L_3 = L_1\left(\widetilde{L}_1\right)^{\frac{\sigma}{1-\sigma}} \in \mathcal{K}$ and $L_4 = L_2\left(\widetilde{L}_2\right)^{\frac{\sigma}{1-\sigma}} \in \mathcal{K}$, we deduce that for $x \in \Omega\backslash\{0\}$,

$$Vp(x) \approx |x|^{\min(0,2-\gamma)}\,\widetilde{L}_3(|x|)(\delta(x))^{\min(1,2-\nu)}\widetilde{L}_4(\delta(x)).$$

Since $\min(0, 2-\gamma) = \min\left(0, \frac{2-\mu}{1-\sigma}\right)$ and $\min(1, 2-\nu) = \min\left(1, \frac{2-\lambda}{1-\sigma}\right)$, we deduce for $x \in \Omega\backslash\{0\}$,

$$Vp(x) \approx |x|^{\min(0,\frac{2-\mu}{1-\sigma})}\,\widetilde{L}_3(|x|)(\delta(x))^{\min(1,\frac{2-\lambda}{1-\sigma})}\widetilde{L}_4(\delta(x)) \approx \theta(x).$$

This completes the proof. □

3 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. So, we need to establish some preliminary results. Our approach is inspired from methods developed in [22, 24] with necessary modifications.

For $\alpha > 0$, we denote by (P_α) the following problem

$$\begin{cases} -\Delta u(x) = a(x)u^\sigma(x), \ x \in \Omega\backslash\{0\} \text{ (in the distributional sense)}, \\ u > 0 \text{ in } \Omega\backslash\{0\}, \\ \lim\limits_{|x|\to 0} |x|^{n-2} u(x) = \alpha, \\ \lim\limits_{x\to\partial\Omega} |x|^{n-2} u(x) = \alpha. \end{cases} \tag{P_α}$$

Proposition 3.1. *Let $\sigma < 0$ and assume that hypothesis (H) is satisfied. Then for each $\alpha > 0$, problem (P_α) has at least one positive solution $u_\alpha \in C(\overline{\Omega}\backslash\{0\})$ satisfying for $x \in \overline{\Omega}\backslash\{0\}$*

$$u_\alpha(x) = \alpha\,|x|^{2-n} + \int_\Omega G(x,y)a(y)u_\alpha^\sigma(y)dy. \tag{3.1}$$

Proof. Let $\sigma < 0$ and $\alpha > 0$. Using hypothesis (H) and Proposition 2.8, we deduce that the function $q(y) := |y|^{(2-n)(\sigma-1)}\,a(y)$ belongs to $K(\Omega)$.

By Proposition 2.5, we conclude that the function

$$x \mapsto h(x) := |x|^{n-2} \int_\Omega G(x,y)a(y)\,|y|^{(2-n)\sigma}\,dy \quad \text{is in } C_0\left(\overline{\Omega}\right). \tag{3.2}$$

Let $\beta := \alpha + \alpha^\sigma\,\|h\|_\infty$. In order to apply a fixed point argument, we consider the convex set Λ given by

$$\Lambda = \{v \in C\left(\overline{\Omega}\right) : \alpha \le v \le \beta\}.$$

Define the operator T on Λ by

$$Tv(x) = \alpha + |x|^{n-2} \int_\Omega G(x,y)a(y)\,|y|^{(2-n)\sigma}\,v^\sigma(y)dy.$$

We claim that $T\Lambda$ is equicontinuous at each point of $\overline{\Omega}$.

Indeed, let $x_0 \in \overline{\Omega}$. Since for all $v \in \Lambda$, $v^\sigma \le \alpha^\sigma$, we have for each $v \in \Lambda$ and all $x \in \overline{\Omega}$,

$$|Tv(x) - Tv(x_0)| \le \alpha^\sigma \int_\Omega \left| |x|^{n-2}\,G(x,y) - |x_0|^{n-2}\,G(x_0,y)\right| |y|^{2-n}\,q(y)dy,$$

where $q(y) = |y|^{(2-n)(\sigma-1)}\,a(y) \in K(\Omega)$.
Now, by the proof of Proposition 2.5, we have for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } x \in \overline{\Omega} \text{ and } |x - x_0| < \delta \Longrightarrow \alpha^\sigma \int_\Omega \left| |x|^{n-2}\,G(x,y) - |x_0|^{n-2}\,G(x_0,y)\right| |y|^{2-n}\,q(y)dy \le \varepsilon.$$

This implies that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } x \in \overline{\Omega} \text{ and } |x - x_0| < \delta \Longrightarrow |Tv(x) - Tv(x_0)| \le \varepsilon, \text{ for all } v \in \Lambda.$$

So the family $T\Lambda$ is equicontinuous at each point of $\overline{\Omega}$ and the claim is proved. In particular, for all $v \in \Lambda$, $Tv \in C\left(\overline{\Omega}\right)$ and therefore $T\Lambda \subset \Lambda$.

Moreover, since the family $\{Tv(x),\, v \in \Lambda\}$ is uniformly bounded in $\overline{\Omega}$, then by Arzelà–Ascoli theorem (see, for example [4, Theorem 2.3]) the set $T(\Lambda)$ becomes relatively compact in $C\left(\overline{\Omega}\right)$. Next, we prove the continuity of T in Λ. Let $(v_k)_k \subset \Lambda$ and $v \in \Lambda$ such that $\|v_k - v\|_\infty \to 0$ as $k \to \infty$. Then we have

$$|Tv_k(x) - Tv(x)| \leq |x|^{n-2} \int_\Omega G(x,y)a(y)\, |y|^{(2-n)\sigma}\, |v_k^\sigma(y) - v^\sigma(y)|\, dr.$$

Now, since

$$|v_k^\sigma(y) - v^\sigma(y)| \leq 2\alpha^\sigma,$$

we deduce by (3.2) and the convergence dominated theorem that

$$\forall x \in \overline{\Omega}, \quad Tv_k(x) \to Tv(x) \quad \text{as } k \to \infty.$$

Since $T(\Lambda)$ is relatively compact in $C\left(\overline{\Omega}\right)$, we obtain

$$\|Tv_k - Tv\|_\infty \to 0 \quad \text{as } k \to \infty.$$

So T is a compact mapping of Λ to itself. Therefore, by the Schauder fixed point theorem, there exists $v_\alpha \in \Lambda$ such that for each $x \in \overline{\Omega}$

$$v_\alpha(x) = \alpha + |x|^{n-2} \int_\Omega G(x,y)a(y)\, |y|^{(2-n)\sigma}\, v_\alpha^\sigma(y)dy. \tag{3.3}$$

Since $v_\alpha^\sigma \leq \alpha^\sigma$, we deduce from (3.3) and (3.2) that

$$\lim_{x \to \partial\Omega} v_\alpha(x) = \alpha. \tag{3.4}$$

We claim that

$$\lim_{|x| \to 0} v_\alpha(x) = \alpha. \tag{3.5}$$

Indeed, let $r > 0$ such that $B(0,3r) \subset \Omega$ and $x \in B(0,r)\backslash\{0\}$. Since $\alpha \leq v_\alpha \leq \beta$, by using (3.3), hypothesis (H) and similar arguments as in the proof of Proposition 2.12, we obtain

$$(v_\alpha(x) - \alpha) \approx |x|^{n-2} \left(\int_0^{2r} \frac{t^{n+(2-n)\sigma-1-\mu}}{\max\left(|x|,t\right)^{n-2}} L_1(t)dt + 1 \right). \tag{3.6}$$

Now since $n \geq 3$, and for $x \in B(0,r)\backslash\{0\}$ and $t \in (0,2r)$, we have

$$|x|^{n-2} \frac{t^{n+(2-n)\sigma-1-\mu}}{\max\left(|x|,t\right)^{n-2}} L_1(t) \leq t^{n+(2-n)\sigma-1-\mu} L_1(t) =: \psi(t) \in L^1(0,\eta),$$

and

$$\lim_{|x| \to 0} |x|^{n-2} \frac{t^{n+(2-n)\sigma-1-\mu}}{\max\left(|x|,t\right)^{n-2}} L_1(t) = 0,$$

we deduce from (3.6) and the dominated convergence theorem that

$$\lim_{|x| \to 0} (v_\alpha(x) - \alpha) = 0.$$

Put $u_\alpha(x) = |x|^{2-n}\, v_\alpha(x)$, for $x \in \overline{\Omega}\backslash\{0\}$. Then $u_\alpha \in C\left(\overline{\Omega}\backslash\{0\}\right)$ and we have

$$u_\alpha(x) = \alpha\, |x|^{2-n} + \int_\Omega G(x,y)a(y)u_\alpha^\sigma(y)dy, \tag{3.7}$$

and
$$\alpha\,|x|^{2-n} \le u_\alpha(x) \le \beta\,|x|^{2-n}. \tag{3.8}$$
Now, since the function $y \mapsto a(y)u_\alpha^\sigma(y) \in L^1_{loc}(\Omega\backslash\{0\})$ and from (3.7) the function $x \mapsto \int_\Omega G(x,y)a(y)u_\alpha^\sigma(y)dy \in L^1_{loc}(\Omega\backslash\{0\})$, we deduce by (1.7) that u_α satisfies
$$-\Delta u_\alpha(x) = a(x)u_\alpha^\sigma(x), \quad x \in \Omega\backslash\{0\}, \quad \text{(in the distributional sense).}$$
By (3.4), we have
$$\lim_{|x|\to 0} |x|^{n-2}\,u_\alpha(x) = \lim_{x\to\partial\Omega} |x|^{n-2}\,u_\alpha(x) = \alpha.$$
This completes the proof. □

Corollary 3.2. *Let $\sigma < 0$, and assume that hypothesis (H) is satisfied. For $0 < \alpha_1 \le \alpha_2$, we denote by $u_{\alpha_i} \in C(\overline{\Omega}\backslash\{0\})$ the solution of problem (P_α) given by (3.1). Then we have*
$$0 \le u_{\alpha_2}(x) - u_{\alpha_1}(x) \le (\alpha_2-\alpha_1)\,|x|^{2-n}, \quad \text{for } x \in \overline{\Omega}\backslash\{0\}. \tag{3.9}$$

Proof. Let g be the function defined on $\Omega\backslash\{0\}$ by
$$g(x) = \begin{cases} a(x)\frac{u_{\alpha_2}^\sigma(x)-u_{\alpha_1}^\sigma(x)}{u_{\alpha_1}(x)-u_{\alpha_2}(x)}, & \text{if } u_{\alpha_1}(x) \ne u_{\alpha_2}(x)\\ 0, & \text{if } u_{\alpha_1}(x) = u_{\alpha_2}(x).\end{cases}$$
Since $\sigma < 0$, then $g \in \mathcal{B}^+(\Omega\backslash\{0\})$ and we have
$$u_{\alpha_2} - u_{\alpha_1} + V\left(g\left(u_{\alpha_2}-u_{\alpha_1}\right)\right) = (\alpha_2-\alpha_1)\,|x|^{2-n}. \tag{3.10}$$
On the other hand, by using (3.1), (3.2) and (3.8), we obtain for $x \in \overline{\Omega}\backslash\{0\}$,
$$V(g\left|u_{\alpha_2}-u_{\alpha_1}\right|)(x) \le (\alpha_1^\sigma + \alpha_2^\sigma)\int_\Omega G(x,y)a(y)\,|y|^{(2-n)\sigma}\,dy < \infty.$$
Therefore inequalities in (3.9) follow from (3.10) and Lemma 2.3. □

Proposition 3.3. *Let $\sigma < 0$. Under hypothesis (H), problem (1.2) has at least one positive solution $u \in C(\overline{\Omega}\backslash\{0\})$ satisfying for $x \in \overline{\Omega}\backslash\{0\}$*
$$u(x) = \int_\Omega G(x,y)a(y)u^\sigma(y)dy. \tag{3.11}$$

Proof. Let $(\alpha_k)_k$ be a positive sequence decreasing to zero. Let $u_k \in C(\overline{\Omega}\backslash\{0\})$ be the solution of problem (P_{α_k}) given by (3.1). By Corollary 3.2, the sequence $(u_k)_k$ decreases to a function u, and since $\sigma < 0$ the sequence $\left(u_k - \alpha_k\,|x|^{2-n}\right)_k$ increases to u. Therefore, by using (3.1), (3.8) and the fact that $\sigma < 0$, we obtain for each $x \in \overline{\Omega}\backslash\{0\}$,
$$\begin{aligned} u(x) \ge u_k(x) - \alpha_k\,|x|^{2-n} &= \int_\Omega G(x,y)a(y)u_k^\sigma(y)dy\\ &\ge \beta_k^\sigma \int_\Omega G(x,y)a(y)\,|y|^{(2-n)\sigma}\,dy > 0,\end{aligned}$$
where $\beta_k := \alpha_k + \alpha_k^\sigma\,\|h\|_\infty$ and h is given by (3.2).

By the monotone convergence theorem, we obtain
$$u(x) = \int_\Omega G(x,y)a(y)u^\sigma(y)dy.$$

Since for each $x \in \overline{\Omega}\backslash\{0\}$, $u(x) = \inf_k u_k(x) = \sup_k \left(u_k(x) - \alpha_k |x|^{2-n}\right)$, then u is upper and lower semi-continuous function on $\overline{\Omega}\backslash\{0\}$ and so $u \in C(\overline{\Omega}\backslash\{0\})$.

Since the function $y \mapsto a(y)u^\sigma(y)$ is in $L^1_{\text{loc}}(\Omega\backslash\{0\})$ and from (3.11) the function $x \mapsto \int_\Omega G(x,y)a(y)u^\sigma(y)dy$ is also in $L^1_{\text{loc}}(\Omega\backslash\{0\})$, we deduce by (1.7) that

$$-\Delta u(x) = a(x)u^\sigma(x), \quad x \in \Omega\backslash\{0\}, \quad \text{(in the distributional sense)}.$$

Finally, using the fact that for all $x \in \Omega\backslash\{0\}$, $0 < u(x) \le u_k(x)$ and that u_k is a solution of problem (P_{α_k}), we deduce that

$$\lim_{|x|\to 0} |x|^{n-2} u(x) = 0 \text{ and } \lim_{x\to\partial\Omega} u(x) = 0.$$

Hence u is a solution of problem (1.2). □

Proof of Theorem 1.3

Assume that the function a satisfies hypothesis (H). By Proposition 2.13, there exists $M \ge 1$ such that for each $\Omega\backslash\{0\}$,

$$\frac{1}{M}\theta(x) \le Vp(x) \le M\theta(x), \tag{3.12}$$

where θ is the function defined in (1.5) and $p(y) := a(y)\theta^\sigma(y)$.

To prove Theorem 1.3, we discuss the following two cases.

Case 1: $\sigma < 0$.

By Proposition 3.3 problem (1.2) has a positive continuous solution u satisfying (3.11). We claim that u satisfies (1.6).

By (3.12), we have

$$M^\sigma (Vp)^\sigma (x) \le \theta^\sigma(x) \le M^{-\sigma} (Vp)^\sigma(x), \tag{3.13}$$

Let $c = M^{-\frac{\sigma}{1-\sigma}}$. Then by elementary calculus we have

$$cVp = V\left(a\left(cVp\right)^\sigma\right) + Vf, \tag{3.14}$$

where $f(x) := ca(x)[\theta^\sigma(x) - M^\sigma (Vp)^\sigma(x)]$, for $x \in \Omega\backslash\{0\}$.

Clearly, we have $f \in \mathcal{B}^+(\Omega\backslash\{0\})$ and by using (3.11) and (3.14), we obtain

$$cVp - u + V\left(a\left(u^\sigma - (cVp)^\sigma\right)\right) = Vf. \tag{3.15}$$

Let g be the function defined on $\Omega\backslash\{0\}$ by

$$g(x) = \begin{cases} a(x)\frac{u^\sigma(x)-(cVp)^\sigma(x)}{(cVp)(x)-u(x)}, & \text{if } u(x) \ne (cVp)(x), \\ 0, & \text{if } u(x) = (cVp)(x). \end{cases}$$

Since $\sigma < 0$, then $g \in \mathcal{B}^+(\Omega\backslash\{0\})$ and we have

$$a\left(u^\sigma - (cVp)^\sigma\right) = g\left(cVp - u\right). \tag{3.16}$$

Therefore the relation (3.15) becomes

$$cVp - u + V\left(g\left(cVp - u\right)\right) = Vf.$$

Now since $f \in \mathcal{B}^+(\Omega\backslash\{0\})$ by using (3.16), (3.11), (3.14) and (3.12), we obtain

$$\begin{aligned} V\left(g\left|cVp-u\right|\right) &\leq V\left(au^{\sigma}\right) + V\left(a\left(cVp\right)^{\sigma}\right) \\ &\leq u + cVp \\ &\leq u + cM\theta < \infty. \end{aligned}$$

Hence by Lemma 2.3, we obtain

$$u \leq cVp.$$

Similarly, we prove that

$$\frac{1}{c}Vp \leq u.$$

Thus, by (3.12) u satisfies (1.6).

Case 2: $0 \leq \sigma < 1$.

Let $\varphi(x) = |x|^{n-2}\theta(x)$, for $x \in \Omega$. By (3.12), we have

$$\frac{1}{M}\varphi\left(x\right) \leq |x|^{n-2}\, Vp\left(x\right) \leq M\varphi\left(x\right). \tag{3.17}$$

Put $c = M^{\frac{1}{1-\sigma}}$ and consider the closed convex set given by

$$A = \left\{v \in C_0(\overline{\Omega}), \frac{1}{c}\varphi \leq v \leq c\varphi\right\}.$$

Clearly $\varphi \in A$.

We define the operator $\mathcal{T}$ on A by

$$\mathcal{T}v(x) := |x|^{n-2}\int_{\Omega} G(x,y)a(y)\,|y|^{(2-n)\sigma}\, v^{\sigma}(y)dy, \qquad x \in \Omega.$$

By using (3.17), we obtain for all $v \in A$,

$$\frac{1}{c}\varphi \leq \mathcal{T}v \leq c\varphi.$$

Since for all $v \in A$, we have

$$|v^{\sigma}(y)| \leq c^{\sigma}\left\|\varphi^{\sigma}\right\|_{\infty}, \quad \text{for all } y \in \Omega,$$

we deduce as in the proof of Proposition 3.1 that

$$\mathcal{T}v \in C_0(\overline{\Omega}), \quad \text{for all } v \in A.$$

So, $\mathcal{T}\left(A\right) \subset A$.

Let $(v_k)_k \subset C_0(\overline{\Omega})$ defined by

$$v_0 = \frac{1}{c}\varphi \quad \text{and} \quad v_{k+1} = \mathcal{T}v_k, \quad \text{for } k \in \mathbb{N}.$$

Since the operator $\mathcal{T}$ is nondecreasing and $\mathcal{T}\left(A\right) \subset A$, we deduce that

$$\frac{1}{c}\varphi = v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_k \leq v_{k+1} \leq c\varphi.$$

Therefore, the sequence $(v_k)_k$ converges by the convergence monotone theorem to a function v satisfying for each $x \in \overline{\Omega}$

$$\frac{1}{c}\varphi(x) \leq v(x) \leq c\varphi(x) \quad \text{and} \quad v(x) = |x|^{n-2} \int_\Omega G(x,y)a(y)\,|y|^{(2-n)\sigma}\, v^\sigma(y)dy.$$

Since v is bounded, we prove by similar arguments as in the proof of Proposition 3.1 that $v \in C_0(\overline{\Omega})$.

Put $u(x) = |x|^{2-n}\, v(x)$. Then $u \in C(\overline{\Omega}\backslash\{0\})$ and satisfies the equation

$$u(x) = V(au^\sigma)(x), \quad \text{for } x \in \overline{\Omega}\backslash\{0\}. \tag{3.18}$$

Finally, since the function $y \mapsto a(y)u^\sigma(y)$ is in $L^1_{\mathrm{loc}}(\Omega\backslash\{0\})$ and from (3.18) the function $x \mapsto V(au^\sigma)(x)$ is also in $L^1_{\mathrm{loc}}(\Omega\backslash\{0\})$, we deduce by (1.7) that u is a solution of problem (1.2). The proof of Theorem 1.3 is completed.

Example 3.4. Let $\sigma < 1$ and $a \in C(\Omega\backslash\{0\})$, such that

$$a(x) \approx |x|^{-\mu} \left(\log\left(\frac{3d}{|x|}\right)\right)^{-\beta} (\delta(x))^{-2} \left(\log\left(\frac{3d}{\delta(x)}\right)\right)^{-2},$$

where $d := \operatorname{diam}(\Omega)$, $\mu < n + (2-n)\,\sigma$ and $\beta \in \mathbb{R}$. Then, by Theorem 1.3, problem (1.2) has at least one positive solution $u \in C\left(\overline{\Omega}\backslash\{0\}\right)$ satisfying the following estimates.

(i) If $2 < \mu < n + (2-n)\,\sigma$, then for $x \in \Omega\backslash\{0\}$,

$$u(x) \approx |x|^{\frac{2-\mu}{1-\sigma}} \left(\log\left(\frac{3d}{|x|}\right)\right)^{\frac{-\beta}{1-\sigma}} \left(\log\left(\frac{3d}{\delta(x)}\right)\right)^{\frac{-1}{1-\sigma}}.$$

(ii) If $\mu = 2$ and $\beta > 1$ or $\mu < 2$, then for $x \in \Omega\backslash\{0\}$,

$$u(x) \approx \left(\log\left(\frac{3d}{\delta(x)}\right)\right)^{\frac{-1}{1-\sigma}}.$$

(iii) If $\mu = 2$ and $\beta = 1$, then for $x \in \Omega\backslash\{0\}$,

$$u(x) \approx \left(\log_2\left(\frac{3d}{|x|}\right)\right)^{\frac{1}{1-\sigma}} \left(\log\left(\frac{3d}{\delta(x)}\right)\right)^{\frac{-1}{1-\sigma}}.$$

(iv) If $\mu = 2$ and $\beta < 1$, then for $x \in \Omega\backslash\{0\}$,

$$u(x) \approx \left(\log\left(\frac{3d}{|x|}\right)\right)^{\frac{1-\beta}{1-\sigma}} \left(\log\left(\frac{3d}{\delta(x)}\right)\right)^{\frac{-1}{1-\sigma}}.$$

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