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On the stability properties of a delay differential neoclassical model of economic growth

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. The main aim of this paper is to establish sharp global stability conditions for the positive equilibrium of a well-known model of economic growth when a delay is considered in the production function. In order to deal with a broad scenario, we establish some results of global attraction for a general family of differential equations with variable delay; for it, we use the notion of strong attractor, which allows us to simplify the proofs, as well as to generalize previous results. Our study reveals that sometimes production delays are not able to destabilize the positive equilibrium, even if they are large. In other cases, the stability properties of the equilibrium depend on the interaction between the delay and other relevant model parameters, leading sometimes to stability windows in the bifurcation diagram.

Keywords: delay differential equation, neoclassical growth model, global stability, stability switches, Lasota equation, gamma-Ricker map

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1 Introduction

The neoclassical aggregate growth model, also called Solow–Swan model [2], is an economic model that attempts to explain long-run economic growth based on capital accumulation and labor or population growth. The fundamental equation of this theory was proposed by Solow in 1956 [36], and it is based on the assumption that there is only one commodity and its rate of production is defined by a function $P = P(K, L)$, where K and L are the capital stock and labor's rate of input, respectively. Capital represents the durable physical inputs, such as machines, while labor represents the inputs associated with human body, such as

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the number of workers and the amount of time they work. It is also supposed that part of the instantaneous output is consumed and the rest is saved and invested. Introducing a new variable $x = K/L$ (the capital–labor rate, that is, the capital stock per unit of effective labor), and assuming constant returns to scale (which in mathematical terms means that the function P is homogeneous of degree one), Solow arrived at the ordinary differential equation

$$x'(t) = -\alpha x(t) + s(x(t))P(x(t), 1), \tag{1.1}$$

where α is the rate of growth of the labor force (population growth rate), and $s(x)$ is the instantaneous rate of saving. See [2,36] for more details about the derivation of (1.1). Roughly speaking, equation (1.1) states that the rate of change of the capital–labor ratio is the difference between the increment of capital and the increment of labor [36]. The simplest case of equation (1.1) assumes a constant saving ratio s, leading to equation

$$x'(t) = -\alpha x(t) + s\, p_1(x(t)), \tag{1.2}$$

where $p_1(x) = P(x, 1)$. The typical assumptions for the neoclassical production function $P(K, L)$ (see, e.g., [2, pp. 26–28]) ensure that (1.2) has a unique positive equilibrium x^*, which is globally asymptotically stable; this means that, regardless of the initial value of the capital–labor ratio, the system will develop towards a state of balanced growth at the natural rate [36]. See also [12,13] for more discussions on the global stability of neoclassical models.

Attempting to show how cyclic and complex behavior can emerge from a neoclassical model, Day [5] proposed to express the neoclassical growth model (1.1) as a difference equation

$$x_{n+1} = \frac{s(x_n)p_1(x_n)}{1-\lambda}, \tag{1.3}$$

where λ is the natural rate of population growth. In the same paper, an alternative model is suggested where s is constant but the productivity is reduced by a "pollution effect" caused by increasing concentrations of capital, thus leading to an equation

$$x_{n+1} = \frac{s\, p_1(x_n)p_2(x_n)}{1-\lambda}, \tag{1.4}$$

where the pollution function $p_2(x)$ is a decreasing function of x.

Recently, Matsumoto and Szidarovsky [29,30] suggested another way to introduce a production lag in Solow's model (1.1), using a delay differential equation. They consider constant saving rate, constant population growth rate, and a negative factor in the production function in the direction suggested by Day, thus arriving at the following generalization of equation (1.2):

$$x'(t) = -\alpha x(t) + sf(x(t-\tau)), \tag{1.5}$$

where $f(x) = p_1(x)p_2(x)$, and $\tau > 0$ represents the time delay inherent in the production process. A more general form of (1.5), allowing variable and instantaneous saving rate would lead to equation

$$x'(t) = -\alpha x(t) + s(x(t))f(x(t-\tau)). \tag{1.6}$$

The possibility of cycles and complex dynamics both in the discrete models (1.3), (1.4), and in the delay differential equation (1.5) for large values of the delay τ was shown in [5] and [29,30], respectively. Thus, it is important to find sufficient conditions to ensure the global attractivity of the balanced equilibrium in these models.

For differential equations with delay such as (1.6), the issue of global stability is considerably more complicated than for ordinary differential equations, and different approaches have been recently applied to some economic models (see, e.g., [1,3,23]). As in [23], in this paper we apply an approach based on the interplay between delay differential equations and maps, which goes back at least up to [15,27], and has been generalized and successfully employed to prove global stability results for many different models in the past fifteen years (e.g., see [14,21–26]).

It is worth mentioning here that stability theory for delayed functional differential equations has been one of the main research topics addressed by Professor Hatvani, who published very influential papers such as [4].

2 Neoclassical models with the Cobb–Douglas function

For the production function, a common choice is the well-known Cobb–Douglas function $P(K,L) = BK^\gamma L^{1-\gamma}$, where $B > 0$ refers to the level of labor-augmenting technology, and $\gamma \in (0,1)$ represents the output elasticity of capital, that is, the part of the output produced by the capital. Using this form in equation (1.6) leads to

$$x'(t) = -\alpha x(t) + s(x(t))Bx^\gamma(t-\tau)p_2(x(t-\tau)). \tag{2.1}$$

In his seminal paper [36], Solow suggested several examples involving the Cobb–Douglas function and some of its generalizations. For example, assuming constant saving ratio $s > 0$, equation (1.2) becomes

$$x'(t) = -\alpha x(t) + \beta x^\gamma(t), \tag{2.2}$$

with $\beta = sB$. It is easy to check that the positive balanced equilibrium $x^* = (\beta/\alpha)^{1/(1-\gamma)}$ is globally asymptotically stable [28,36].

In this paper we consider several examples of equation (2.1). Matsumoto and Szidarovszky [30] studied the local stability of the equilibrium in (2.1), for constant saving ratio s, and the "pollution effect" function $p_2(x) = e^{-\delta x}$, that is, they considered the delay differential equation

$$x'(t) = -\alpha x(t) + \beta x^\gamma(t-\tau)e^{-\delta x(t-\tau)}. \tag{2.3}$$

Equation (2.3) belongs to a well-known family of differential equations with instantaneous linear decay and delayed feedback (e.g., see [17]). Actually, for $\gamma = 1$, (2.3) becomes the celebrated Nicholson's blowflies equation [8]; for $\gamma = 0$, (2.3) is the equation for the red-blood cell system proposed by Wazewska-Czyzewska and Lasota [37]; Lasota himself proposed equation (2.3) with positive values of γ in a later paper [18]; finally, for $\gamma > 1$, (2.3) can be used as a model for populations subject to Allee effects [11,25]. However, a suitable formulation for (2.1) is the more general family of delay differential equations considered by Ivanov, Liz and Trofimchuk in [14], namely,

$$x'(t) = -g_1(x(t))f_2(x(t-\tau)) + f_1(x(t-\tau))g_2(x(t)), \tag{2.4}$$

where f_1, f_2, g_1, g_2 are positive functions.

Our main results in this paper are the following: first, we generalize a global stability result for equation (2.4), allowing for variable delays, and apply it to some examples of (2.1), including the possibility of variable saving ratio. Then we consider a variation of (2.3), replacing the pollution function $e^{-\delta x}$ with $p_2(x) = 1/(1+\delta x)$, $\delta > 0$; in this case, we prove that the positive equilibrium is globally attracting regardless the value of the delay, which highlights the key role of the pollution function in the stability properties of neoclassical models with delay. The rest of the paper is devoted to equation (2.3); we prove a sharp global result of absolute (delay-independent) stability for the positive equilibrium, and also a delay-dependent global stability result. Moreover, we analyze how the parameter γ influences the stability of the equilibrium in a subtle way, depending on the other model parameters, and leading in some cases to stability windows in the bifurcation diagram.

3 Global stability results

In this section, we consider equation (2.4) with a continuous variable delay $\tau : [0, \infty) \to [0, h]$, that is,

$$x'(t) = -g_1(x(t))f_2(x(t - \tau(t))) + f_1(x(t - \tau(t)))g_2(x(t)). \tag{3.1}$$

First we introduce some definitions and assumptions. If I is a real interval, we say that a fixed point $x^* \in I$ of a map $F : I \to I$ is a global attractor for F on (a, b) if

$$\lim_{n\to\infty} F^n(x) = x^*, \qquad \forall\, x \in (a, b),$$

where, as usual, $F^n = \overbrace{F \circ \cdots \circ F}^{n}$.

We assume the following hypotheses on functions f_i, g_i, $i = 1, 2$:

(H1) f_1, f_2, g_1 and g_2 are continuous and positive functions defined in a real interval (a, b), with $0 \le a < b \le \infty$, and $g := g_1/g_2$ is strictly increasing.

(H2) If $f := f_1/f_2$, then there is a unique solution $x^* \in (a, b)$ of equation $g(x) = f(x)$. Moreover, the function $F(x) := g^{-1}(f(x))$ maps (a, b) in (a, b), and x^* is a global attractor for F on (a, b).

To prove the main result in this section, which generalizes some previous results in [14,24], we need the following auxiliary result from [24], whose proof was provided by Víctor Jiménez López [16].

Lemma 3.1. *Let $F : (a, b) \to (a, b)$ be a continuous map, and let x^* be the global attractor of F on (a, b). Then x^* is a strong attractor of F on (a, b), that is, for every compact set $K \subset (a, b)$, there exists a family of non-degenerate intervals $\{I_n\} = \{[a_n, b_n]\}$, $n \ge 1$, such that the following properties hold:*

(A1) $K \subset \mathrm{Int}(I_1) \subset (a, b)$.

(A2) $F(I_n) \subset I_{n+1} \subset \mathrm{Int}(I_n)$, $\forall\, n \ge 1$.

(A3) $\displaystyle\bigcap_{n=1}^{\infty} I_n = \{x^*\}$.

We are now in a position to prove the main result of this section. We assume that for any continuous initial condition $\phi : [-h, 0] \to (a, b)$, the corresponding solution $x(t) = x(t, \phi)$ of (3.1) exists for all $t \ge 0$, and it is unique. For example, under conditions **(H1)** and **(H2)**, it is enough to assume that g_1 and g_2 are Lipschitz continuous (see [14]).

Theorem 3.2. *Assume that conditions* **(H1)** *and* **(H2)** *hold. Then, x^* is a global attractor of* (3.1) *on (a,b), that is, if $x(t) = x(t,\phi)$ is the solution of* (3.1) *with initial condition $\phi : [-h,0] \to (a,b)$, then*

$$\lim_{t\to\infty} x(t,\phi) = x^*.$$

Proof. For a given continuous function $\phi : [-h,0] \to (a,b)$, define the compact set

$$K = \left[\min_{t\in[-h,0]} \phi(t), \max_{t\in[-h,0]} \phi(t)\right] \subset (a,b),$$

and consider the family $\{I_n\} = \{[a_n, b_n]\}$ defined in Lemma 3.1. First we prove that $x(t) \in I_1 = [a_1, b_1]$ for all $t \geq 0$. Assume, by the contrary, that there exists a first instant $t_0 > 0$ such that $x(t_0) = b_1$ (the same argument applies if $x(t_0) = a_1$). Since $F(I_1) \subset \mathrm{Int}(I_1)$, it follows that $F(x(t_0 - \tau(t_0))) < b_1$, which is equivalent to $f(x(t_0 - \tau(t_0))) < g(b_1)$. Then, using (3.1), we have:

$$\frac{x'(t_0)}{f_2(x(t_0-\tau(t_0)))g_2(x(t_0))} = -g(x(t_0)) + f(x(t_0-\tau(t_0))) = -g(b_1) + f(x(t_0-\tau(t_0))) < 0,$$

which is a contradiction, because $x'(t_0) \geq 0$.

Next we show that there is $t_1 > 0$ such that $x(t) \in I_2$ for all $t \geq t_1$. Following the previous argument, and using that $F(I_1) \subset I_2$, it is easy to prove that if there is t_1 such that $x(t_1) \in I_2$, then $x(t) \in I_2$ for all $t \geq t_1$. Thus, we prove the existence of such a point t_1. Assume, by contradiction, that t_1 does not exist, and, to fix ideas, assume that $x(t) > b_2$ for all $t \geq 0$ (the same arguments apply to the case $x(t) < a_2$ for all $t \geq 0$). Then, using (3.1) and the increasing character of g, we have:

$$\frac{x'(t)}{f_2(x(t-\tau(t))g_2(x(t))} = -g(x(t)) + f(x(t-\tau(t))) < 0\,,\ \forall\, t > 0.$$

Indeed, notice that

$$x(t) > b_2 \Longrightarrow g(x(t)) > g(b_2),$$

$$x(t-\tau(t)) \in I_1 \Longrightarrow F(x(t-\tau(t))) \in I_2 \Longrightarrow f(x(t-\tau(t))) \leq g(b_2).$$

Hence, $x(t)$ is a decreasing function, and there is $\lim_{t\to\infty} x(t) = L \geq b_2$. This is a contradiction because the only possible limit of $x(t)$ is x^*, and $x^* < b_2$.

An inductive argument, using the same reasoning employed above, proves that, for each $n > 1$, there exists a point $t_n > 0$ such that $x(t_n) \in I_{n+1}$, for all $t \geq t_n$. Finally, it follows by **(A3)** that $\lim_{t\to\infty} x(t) = x^*$. □

We notice that Theorem 3.2 generalizes Theorem 3 in [14], and its proof is much simpler. For the case of constant functions $f_2 = g_2 = 1$, a multi-dimensional version of Theorem 3.2 has been proved in [24].

In order to check condition **(H2)** in applications, we recall that the Schwarzian derivative of a $\mathcal{C}^3$ map $F : (a,b) \to (a,b)$ is defined for every $x \in (a,b)$ such that $F'(x) \neq 0$, by the expression

$$(SF)(x) = \left(\frac{F'''(x)}{F'(x)}\right) - \frac{3}{2}\left(\frac{F''(x)}{F'(x)}\right)^2.$$

Corollary 3.3. *Assume that* **(H1)** *holds and* $F : (a,b) \to (a,b)$ *satisfies condition* **(C0)** *and at least one of conditions* **(C1)–(C3)** *below.*

(C0) *There is* $x^* \in (a,b)$ *such that* $(F(x) - x)(x - x^*) < 0$ *for every* $x \neq x^*$.

(C1) F *is increasing.*

(C2) F *is a* C^3 *unimodal map, with a unique critical point* c*, which is a local maximum. Moreover,* F *has a unique fixed point* x^* *in* (a,b), $-1 \leq F'(x^*) < 1$, *and* $(SF)(x) < 0$ *for all* $x > c$.

(C3) *Equation* $F^2(x) = x$ *has no solutions on* (a,b) *different from* x^*.

Then x^* *is a global attractor of (3.1) on* (a,b).

Proof. The result follows from a combination of Theorem 3.2 and global stability conditions for difference equations. When **(C1)** or **(C2)** hold, it is a consequence of Corollaries 2.9 and 2.10 from [6]. When **(C3)** holds, the result follows from a classical result (see, e.g., [35, Section 9.3]). □

Remark 3.4. It is easy to check that **(C0)** is equivalent to say that $(f(x) - g(x))(x - x^*) < 0$ for all $x \neq x^*$, and, assuming **(H1)** is satisfied, **(C1)** holds if f is increasing.

Next we state a delay-dependent stability result for delay differential equations with constant delay τ of the form (1.5). It is a straightforward consequence of Corollary 17 in [10] and Corollary 2.10 in [6].

Theorem 3.5. *Denote by* $F(x) = (1/\alpha)f(x)$. *Assume that* F *is a* C^3 *unimodal map, with a unique critical point* c*, which is a local maximum. Moreover,* F *has a unique fixed point* x^* *in* (a,b)*, and* $(SF)(x) < 0$ *for all* $x > c$. *If* $-1 \leq F'(x^*) < 1$, *or* $F'(x^*) < -1$ *and*

$$e^{-\alpha\tau} \geq 1 + \frac{1}{F'(x^*)} \tag{3.2}$$

then x^* *is a global attractor on* (a,b) *for equation*

$$x'(t) = -\alpha x(t) + f(x(t-\tau)). \tag{3.3}$$

4 Applications

4.1 Constant saving ratio and no pollution effects

The first application of the results in Section 3 is that if we consider equation (2.2) with continuous variable delay $\tau(t) \in [0,h]$, that is,

$$x'(t) = -\alpha x(t) + \beta x^{\gamma}(t - \tau(t)), \tag{4.1}$$

then the positive equilibrium $x^* = (\beta/\alpha)^{1/(1-\gamma)}$ is still a global attractor. This result is a consequence of Corollary 3.3, because **(H1)** holds with $f_1(x) = \beta x^{\gamma}$, $f_2(x) = 1$, $g(x) = g_1(x) = \alpha x$, and $g_2(x) = 1$, while conditions **(C0)** and **(C1)** are satisfied with $F(x) = (\beta/\alpha)x^{\gamma}$ and $(a,b) = (0,\infty)$.

4.2 Variable saving ratio and no pollution effects

Next we introduce a variable saving ratio $s(x(t))$ in equation (4.1). As it had been argued by Solow in [36, p. 88], it is natural to assume that the savings ratio depends inversely on the capital–labor ratio x, and that for sufficiently large x, $s(x)$ approaches zero . Thus, we assume that $s(x)$ is decreasing and tends to 0 as x tends to infinity. We prove the following result:

Theorem 4.1. *Assume that* $s : (0, \infty) \to (0, \infty)$ *is decreasing,* $\lim_{x\to\infty} s(x) = 0$*, and* $x/s(x)$ *is convex. Then the equation*

$$x'(t) = -\alpha x(t) + s(x(t))Bx^{\gamma}(t - \tau(t)) \tag{4.2}$$

has a unique positive equilibrium x^**, and it is a global attractor on* $(0, \infty)$.

Proof. Equation (4.2) is of the form of (3.1), with $g_1(x) = \alpha x$, $f_2(x) = 1$, $f_1(x) = Bx^{\gamma}$, and $g_2(x) = s(x)$. Since s is decreasing and $\gamma > 0$, it follows that $g(x) = g_1(x)/g_2(x) = \alpha x/s(x)$ and $f(x) = f_1(x)/f_2(x) = Bx^{\gamma}$ are increasing; thus **(C1)** holds. Next, since f is concave, $f'(0^+) = \infty$, and g is convex, it is clear that **(C0)** also holds (see Remark 3.4). An application of Corollary 3.3 provides the desired result. □

Remark 4.2. Conditions of Theorem 4.1 are satisfied, for example, if we choose the saving functions $s(x) = s_0 e^{-\delta x}$ or $s(x) = s_0/(1 + \delta x)$, with $s_0 > 0$, $\delta > 0$.

4.3 Constant saving ratio and pollution effects

In this subsection, we consider equation (2.1) with constant s, which includes as a particular case the Matsumoto–Szidarovszky equation (2.3). These authors considered a pollution effect given by a function $p_2(x) = e^{-\delta x}$, in such a way that the production function $f(x) = \beta x^{\gamma} e^{-\delta x}$ is unimodal, has zero value at $x = 0$, and converges to zero as x tends to infinity. They showed that a sufficiently large value of the delay τ is able to destabilize the positive equilibrium of (2.3).

To show that not any pollution function p_2 such that $f(x) = \beta x^{\gamma} p_2(x)$ satisfies the above mentioned conditions leads to destabilization, we first consider a simpler example in which $p_2(x) = 1/(1 + \delta x)$. That is, we consider equation

$$x'(t) = -\alpha x(t) + \frac{\beta x^{\gamma}(t - \tau(t))}{1 + \delta x(t - \tau(t))} , \tag{4.3}$$

where $\tau : [0, \infty) \to [0, h]$ is continuous.

We need the following result which can be derived from Proposition 1 and Theorem 3.1 in [20].

Proposition 4.3. *The map* $F(x) = (\beta/\alpha)x^{\gamma}/(1 + \delta x)$ $(\beta, \alpha, \delta > 0,\ 0 < \gamma < 1)$ *is of class* C^{∞} *in* $(0, \infty)$*, and the following properties hold:*

(i) $F(0) = 0$, $F(x) > 0$ *for all* $x > 0$, $\lim_{x\to\infty} F(x) = 0$, *and* $\lim_{x\to 0^+} F'(x) = \infty$.

(ii) *F is unimodal, with a unique critical point at*

$$c = \frac{\gamma}{\delta(1 - \gamma)} ,$$

where F attains its global maximum.

(iii) *There is a unique* $x^* \in (0,\infty)$ *such that* $F(x^*) = x^*$*. Moreover,* $(F(x)-x)(x-x^*) < 0$ *for every* $x \neq x^*$*, and equation* $F^2(x) = x$ *has no positive solutions different from* x^**.*

Our next result follows as a direct consequence of Proposition 4.3 and Corollary 3.3.

Theorem 4.4. *Equation* (4.3) *has a unique positive equilibrium* x^**, and it is a global attractor on* $(0,\infty)$*.*

The rest of the paper is devoted to the differential equation with constant delay (2.3), which we write again for convenience of the reader:

$$x'(t) = -\alpha x(t) + \beta x^\gamma(t-\tau)e^{-\delta x(t-\tau)}. \tag{4.4}$$

As before, we first list some basic properties of the production map $f(x) = \beta x^\gamma e^{-\delta x}$, which is referred to as the gamma-Ricker map. See Propositions 1 and 2, and Theorem 1 in [19].

Proposition 4.5. *The map* $F(x) = (\beta/\alpha)x^\gamma e^{-\delta x}$ $(\alpha,\beta,\delta > 0,\ 0 < \gamma < 1)$ *satisfies the following properties:*

(i) $F(0) = 0$*,* $F(x) > 0$ *for all* $x > 0$*,* $\lim_{x\to\infty} F(x) = 0$*, and* $\lim_{x\to 0^+} F'(x) = \infty$*.*

(ii) F *is a* $\mathcal{C}^\infty$*-unimodal map, with a unique critical point at* $c = \gamma/\delta$*, where* F *attains its global maximum.*

(iii) *There is a unique* $x^* \in (0,\infty)$ *such that* $F(x) = x$*. Moreover,* $(F(x)-x)(x-x^*) < 0$ *for every* $x \neq x^*$*.*

(iv) $(SF)(x) < 0$ *for all* $x > c$*.*

(v) $F'(x^*) = \gamma - \delta x^*$*.*

(vi) *Inequalities* $-1 \leq F'(x^*) < 1$ *hold if and only if the following inequality is satisfied:*

$$\frac{\beta}{\alpha} \leq e^{\gamma+1}\left(\frac{\gamma+1}{\delta}\right)^{1-\gamma}. \tag{4.5}$$

The main result for equation (4.4) is the following.

Theorem 4.6. *The unique positive equilibrium* x^* *of* (4.4) *is a global attractor of all solutions* $x(t,\phi)$*, with* $\phi \in \mathcal{C}([-\tau,0],(0,\infty))$*, if the following inequality is satisfied:*

$$\frac{\beta}{\alpha} \leq e^{\gamma+\frac{1}{1-e^{-\alpha\tau}}}\left(\frac{1}{\delta}\left(\gamma+\frac{1}{1-e^{-\alpha\tau}}\right)\right)^{1-\gamma}. \tag{4.6}$$

Proof. It is easy to prove that (4.6) is equivalent to

$$e^{-\alpha\tau} \geq 1 + \frac{1}{F'(x^*)} = 1 + \frac{1}{\gamma - \delta x^*}$$

and therefore the conclusion of the theorem is a consequence of Theorem 3.5 and Proposition 4.5. □

Remark 4.7. Some remarks are in order:

(1) It is clear that (4.5) implies (4.6). Actually, inequality (4.5) is a sharp delay-independent (absolute) global stability condition for the positive equilibrium of (4.4). Indeed, if (4.5) does not hold, then there is $\tau^* > 0$ such that x^* is asymptotically stable for $0 < \tau < \tau^*$ and unstable if $\tau > \tau^*$. Taking into account Proposition 4.5 (v) and [33, Theorem 4.7], the value of τ^* can be calculated as

$$\tau^* = \frac{\arccos\left(\frac{1}{\gamma - \delta x^*}\right)}{\alpha\sqrt{-1 + (\gamma - \delta x^*)^2}}. \tag{4.7}$$

(2) As a consequence of Corollary 3.3, condition (4.5) ensures that the positive equilibrium x^* is a global attractor of (4.4) on $(0, \infty)$ for all values of the delay τ, even if we consider continuous variable delays $\tau(t) \in [0, h]$ instead of a constant delay τ in (4.4).

(3) The limit form of (4.5) as $\gamma \to 1$ provides the well-known absolute global stability condition $\beta \leq \alpha e^2$ for the Nicholson's blowflies equation (see, e.g., [10]). For $\gamma = 0$, (4.5) gives the global stability condition $\beta\delta \leq \alpha e$ for the Wazewska-Czyzewska and Lasota equation (see, e.g., [9]).

(4) The delay-dependent global stability condition (4.6) proves that, for a sufficiently small value of the delay, the positive equilibrium x^* of (4.4) is globally asymptotically stable.

From Remark 4.7 (1), it is clear that for fixed values of the parameters $\alpha, \beta, \gamma, \delta$ for which (4.5) does not hold, an increasing value of τ destabilizes the positive equilibrium of (4.4); this property has already been observed in [30].

It is interesting to study the role of the other parameters on the stability properties of (4.4). For example, inequalities (4.5) and (4.6) suggest that the equilibrium is also destabilized by increasing either β or δ, keeping constant the value of the other involved parameters. In [30], δ is assumed to reflect the strength of the negative effect caused by increasing concentration of capital.

The role of parameter γ is subtler. From (4.5), it is easy to prove that x^* is absolutely globally asymptotically stable for all $\gamma \in (0, 1)$ if the ratio β/α is small enough; specifically, a sufficient condition is $\beta/\alpha \leq \min\{e/\delta, e^2\}$. However, for other values of $\beta, \gamma, \delta, \tau$, increasing γ can be stabilizing, destabilizing or even can give rise to a pair of stability switches (see, e.g., [19, Proposition 3]). In Figure 4.1, we show the three possible situations via stability diagrams in the parameter plane (γ, τ) for $\alpha = 1$ and different values of β and δ.

- Figure 4.1 (a) corresponds to $\beta = 2$, $\delta = 2$, for which $e^2 > \beta/\alpha > e/\delta$. The positive equilibrium of (4.4) is globally asymptotically stable for $\gamma \geq \gamma_*$, with $\gamma_* \approx 0.156$, and it is asymptotically stable for $\tau < \tau^* \approx 3.826$. For larger values of τ, increasing γ is stabilizing.

- Figure 4.1 (b) corresponds to $\beta = 7.7$, $\delta = 0.5$, for which $\beta/\alpha > \max\{e/\delta, e^2\}$. The positive equilibrium of (4.4) is globally asymptotically stable for $\gamma \in [\gamma_*, \gamma^*]$, with $\gamma_* \approx 0.469$ and $\gamma^* \approx 0.871$, and it is asymptotically stable for $\tau < \tau^* \approx 4.102$. For values of $\tau > 9.857$, increasing γ leads to a pair of stability switches.

- Figure 4.1 (c) corresponds to $\beta = 10$, $\delta = 0.13$, for which $e/\delta > \beta/\alpha > e^2$. The positive equilibrium of (4.4) is globally asymptotically stable for $\gamma \leq \gamma^*$, with $\gamma^* \approx 0.815$, and it is asymptotically stable for $\tau < \tau^* \approx 2.93$. For larger values of τ, increasing γ is destabilizing.

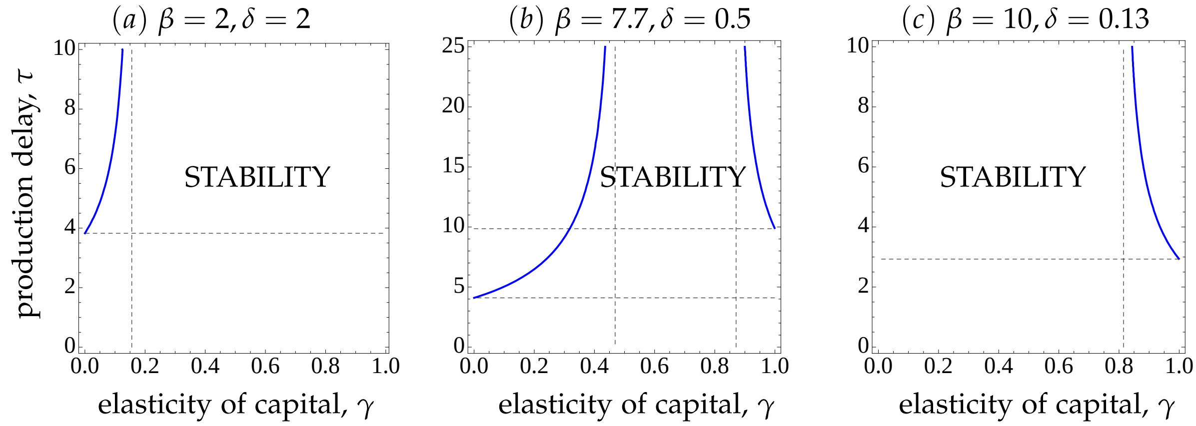

Figure 4.1: Stability diagrams for equation (4.4) in the plane (γ, τ), with $\alpha = 1$ and different values of β and δ. The blue solid lines represent the boundaries of asymptotic stability, where supercritical Hopf bifurcations occur. Dashed lines correspond to threshold values of γ and τ (see the text).

In Figure 4.2, we plot some numerical simulations for the solutions of (4.4) in the case (b) ($\alpha = 1$, $\beta = 7.7$, $\delta = 0.5$), with $\tau = 15$ and different values of γ, illustrating the stability switches.

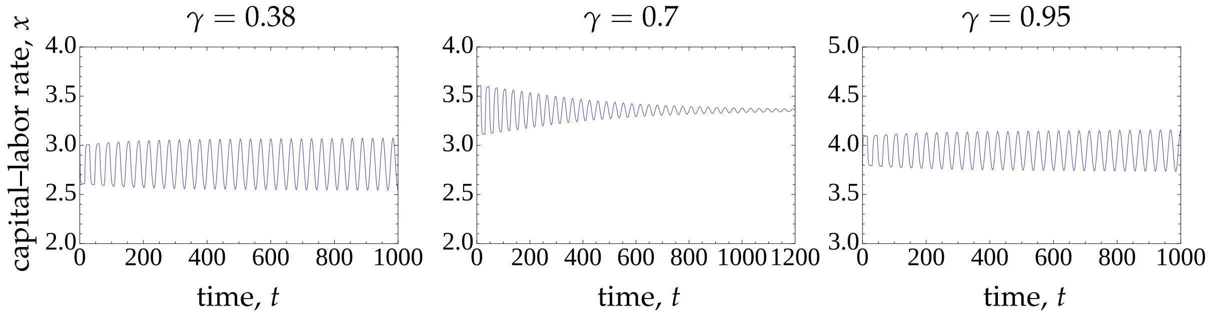

Figure 4.2: Numerical simulations of the solutions of equation (4.4), with $\alpha = 1$, $\beta = 7.7$, $\delta = 0.5$, $\tau = 15$, and different values of γ. For $\gamma = 0.38$ and $\gamma = 0.95$, the equilibrium x^* is unstable, and there are sustained oscillations; for $\gamma = 0.7$, x^* is globally asymptotically stable.

5 Discussion

Solow's paper [36] is one of the most influential works in economic theory [2, 28]. Although the original model is defined by an ordinary differential equation, the role of time delays in the production processes has been discussed both for the one-sector Solow's model by Matsumoto and Szidarovszky [30], and by Gori *et al.* [7] for the two-sector generalization proposed by Mankiw *et al.* [28]. These papers focus on a local stability analysis of the positive equilibrium, and Hopf bifurcations leading to sustained oscillations as time delay increases.

In this paper, we have carried out a deeper analysis about the role of time delays in the original Solow equation, allowing the possibility of variable saving rate, and a delayed production function with two factors: a positive factor x^γ coming from the Cobb–Douglas production function, and a negative factor $p_2(x)$, which represents a "pollution effect" due to increasing concentrations of capital [5]. These considerations lead to a general form of Solow's model given by equation (2.1). With this model, we showed that the influence of time delays on the stability properties of the positive equilibrium depends in an essential way on several factors, including the choice of the decreasing function $p_2(x)$, and the interplay among the different parameters involved in the model. It is of particular interest the role of parameter γ, which measures the responsiveness of output to a change in the level of capital used in production. Two remarkable consequences of our results are the following: first, under mild assumptions, variable saving rates and variable delays in the production function are not able to destabilize the positive equilibrium if a pollution effect is not considered, so the generalization of Solow's equation given by (4.2) still predicts convergence to the steady-state capital–labor ratio. Second, even if a negative factor is introduced in the production function, the particular form of this factor may prevent instabilities due to large delays. Roughly speaking, the "pollution function" $p_2(x)$ needs to have a fast rate of convergence to zero as x tends to infinity.

In contrast with other papers, our stability analysis focus on global results and allow for variable delays. In particular, we establish sharp delay-independent global stability conditions, and also prove that small delays cannot destroy the global stability of the equilibrium. The mathematical approach we use to get global stability results for (2.1) is not new; however, the use of the notion of strong attractor introduced in [24] has some advantages: on the one hand, the proofs are much simpler than in previous papers [14, 15]; on the other hand, it is easy to consider variable delays, providing more general results.

Another interesting novelty of our results is that we use a generalization of Singer's theory for maps with negative Schwarzian derivative [34], due to El-Morshedy and Jiménez-López [6]. This result is crucial in the analysis of equation (2.3) because, in contrast with other cases ($\gamma = 0$, $\gamma \geq 1$), the nonlinearity $f(x) = \beta x^\gamma e^{-\delta x}$ does not have negative Schwarzian derivative everywhere if $0 < \gamma < 1$. In this way, our results fill a gap in the stability theory of equation (2.3). As far as we know, equation (2.3) has been introduced for the first time by A. Lasota in 1977 [18], to model blood cell production (erythropoiesis). Lasota formulated a conjecture concerning ergodic properties of (2.3), see [32]. An interesting biological interpretation of parameter γ in the model has been given by P. J. Mitkowski in his Ph.D. thesis [31]. It is related to disturbed erythropoiesis (dyserythropoiesis), when the feedback loop that regulates the production of cells in the red bone marrow does not work properly. Roughly speaking, γ represents the degree of disturbance of the normal erythropoietic response. When $\gamma = 0$, the answer is correct, but when $\gamma > 0$, the response is inhibited and the greater is the inhibition, the greater is the value of γ.

It is remarkable that the same equation with different values of γ has been used for different mathematical models governed by delay differential equations:

- For $\gamma = 0$, it is a model for blood-cell production, proposed by Wazewska-Czyzewska and Lasota [37], and later modified by Lasota [18], allowing for positive values of γ.

- For $0 < \gamma < 1$, it is a model in economics, proposed by Matsumoto and Szidarovszky [30], as a generalization of the fundamental Solow's equation [36].

- For $\gamma = 1$, it is the famous equation introduced by Gurney, Blythe and Nisbet [8] to explain some qualitative aspects of Nicholson's classic experiments on laboratory cultures of sheep blowflies.

- For $\gamma > 1$, equation (2.3) has been proposed to study the dynamics of single-species populations subject to Allee effects [11,25].

Finally, it is worth mentioning that the Lasota equation (2.3) can be seen as a continuous version of the gamma-Ricker map, which has been used as a flexible discrete model for animal populations, and in the context of cooperative interaction in a group of individuals [19].

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