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# Robustness with respect to exponents for nonautonomous reaction–diffusion equations

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**Abstract.** In this work we consider a family of nonautonomous problems with homogeneous Neumann boundary conditions and spatially variable exponents with equation of the form

$$\frac{\partial u_\lambda}{\partial_t}(t) - \operatorname{div}\Big(D(t)|\nabla u_\lambda(t)|^{p_\lambda(x)-2}\nabla u_\lambda(t)\Big) + |u_\lambda(t)|^{p_\lambda(x)-2}u_\lambda(t) = B(t, u_\lambda(t)).$$

We study the continuity of the flow and we study the behavior of attractors when $p_\lambda(\cdot) \to p(\cdot)$ in $L^\infty(\Omega)$ as $\lambda \to \infty$ where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$.

**Keywords:** $p(x)$-Laplacian, variable exponents, pullback attractor and nonautonomous asymptotic behavior.

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## 1 Introduction

In several physical, chemical and biological problems the reaction–diffusion systems can be a good model to describe the behavior of the problem, and in many of these problems may appear operators in which some exponent $p$ depends on the spatial variable, as in the case of the operator $p(x)$-Laplacian [3, 4, 9]. These require the use of function spaces with spatially dependent exponents and new mathematical techniques.

The asymptotic behavior of nonautonomous evolution problems has been investigated recently [6, 8]. An interesting problem is to investigate how is the asymptotic behavior of these problems with the variation of parameters, more specifically, try to establish the existence of attractors for each parameter and to study the continuity of these attractors with respect to the variation of the parameters, see [6, 12, 13].

In this paper we establish upper semicontinuity of pullback attractors for a nonautonomous evolution equation of the form

$$\begin{cases} \frac{\partial u_\lambda}{\partial_t}(t) - \operatorname{div}\Big(D(t)|\nabla u_\lambda(t)|^{p_\lambda(x)-2}\nabla u_\lambda(t)\Big) + |u_\lambda(t)|^{p_\lambda(x)-2}u_\lambda(t) = B(t, u_\lambda(t)), \\ u_\lambda(\tau) = u_{0\lambda} \end{cases} \tag{$P_\lambda$}$$

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with a homogeneous Neumann boundary condition, for $(t,x) \in (\tau,+\infty)\times\Omega$ where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ for some $N \geq 1$ and the initial condition $u_\lambda(\tau) \in H := L^2(\Omega)$. The terms $p_\lambda$, $B$ and $D$ are assumed to satisfy:

**Assumpition p** $p_\lambda(\cdot) \in C(\overline{\Omega},\mathbb{R})$, for each $\lambda \in [0,\infty)$, satisfies

(P1) there are $m, M \in \mathbb{R}$ such that

$$2 < m \leq p_\lambda^- := \min_{x\in\overline{\Omega}} p_\lambda(x) \leq p_\lambda^+ := \max_{x\in\overline{\Omega}} p_\lambda(x) \leq M;$$

(P2) $p_\lambda \to p$ in $L^\infty(\Omega)$ for some $p$ such that $p(\cdot) \in C(\overline{\Omega},\mathbb{R})$ and $m \leq p^- \leq p^+ \leq M$.

**Assumpition B** The mapping $B : [\tau,T]\times H \to H$ is such that

(B1) there exists $L \geq 0$ such that

$$\|B(t,x_1) - B(t,x_2)\|_H \leq L\|x_1 - x_2\|_H$$

for all $t \in [\tau,T]$ and $x_1, x_2 \in H$;

(B2) for all $x \in H$ the mapping $t \to B(t,x)$ belongs to $L^2(\tau,T;H)$;

(B3) the function $t \to \|B(t,0)\|_H$ is nondecreasing, absolutely continuous and bounded on compact subsets of $\mathbb{R}$.

**Assumpition D** $D : [\tau,T]\times\Omega \to \mathbb{R}$ is a function in $L^\infty([\tau,T]\times\Omega)$ such that

(D1) there are positive constants, $\beta$ and $\mathcal{M}$ such that $0 < \beta \leq D(t,x) \leq \mathcal{M}$ for almost all $(t,x) \in [\tau,T]\times\Omega$;

(D2) $D(t,x) \geq D(s,x)$ for each $x \in \Omega$ and $t \leq s$ in $[\tau,T]$.

The authors in [6] also considered the nonautonomous problem and proved the robustness with respect to the diffusion coefficient whereas in this work we study the robustness with respect to the exponents.

The paper is organized as follows. In Section 2 we present the time-dependent evolution operator and the results of [6] that ensure some properties of the operator and existence and uniqueness of solution for the problem. The work [6] also ensures the existence of a pullback attractor for the problem, these results are in Section 3. With the objective of guaranteeing the upper semicontinuity of attractors we need to develop some new uniform estimates for the problem, these estimates are established in Section 4. In Section 5, we ensure the continuity of the process associated with the problem and the upper semicontinuity of the pullback attractors.

## 2 Preliminaries

In this section we present some definitions about Lebesgue and Sobolev spaces with variable exponents, a general theory about this spaces can be found in [4,5,11]. Also we show some results about the operator associated with our problem and the results that ensure the existence and uniqueness of solution, following [6].

Let us recall the definitions of the Lebesgue and Sobolev spaces with variable exponents. Considering $p \in L^\infty_+(\Omega) := \{q \in L^\infty(\Omega) : \text{ess inf } q \geq 1\}$, then

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} : u \text{ is mensurable}, \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}$$

is a Banach space with the norm $\|u\|_{p(x)} := \inf\{\lambda > 0 : \rho(\frac{u}{\lambda}) \leq 1\}$, where $\rho(u) := \int_\Omega |u(x)|^{p(x)} dx$. Furthermore,

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

which is a Banach space with the norm $\|u\|_{W^{1,p(x)}(\Omega)} := \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$.

The authors in [6] considered, for each $\lambda \in [0, \infty)$ and $t \in \mathbb{R}$, the operator $A_\lambda(t) : X_\lambda \to X_\lambda^*$, where $X_\lambda := W^{1,p_\lambda(x)}(\Omega)$ with norm $\|\cdot\|_{X_\lambda} := \|\cdot\|_{W^{1,p_\lambda(x)}(\Omega)}$, defined by

$$A_\lambda(t)u(v) := \int_\Omega D(t,x)|\nabla u(x)|^{p_\lambda(x)-2}\nabla u(x) \cdot \nabla v(x) dx + \int_\Omega |u(x)|^{p_\lambda(x)-2} u(x)v(x) dx,$$

for each $u, v \in X_\lambda$ and they have proved that the operator $A_\lambda(t)$ is monotone, hemicontinuous and coercive for each $t \in [\tau, T]$ and $\lambda \in [0, \infty)$. Then, they concluded that the operator is maximal monotone and the realization operator of $A_\lambda(t)$ at $H = L^2(\Omega)$ is maximal monotone in $H$, for each $t \in [\tau, T]$ and $\lambda \in [0, \infty)$. With this it is possible to show that the operator $A_\lambda(t)$ is the subdifferential $\partial\varphi_\lambda(t)$ of the convex, proper and lower semicontinuous map $\varphi_\lambda(t) : H \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi_\lambda(t)(u) := \begin{cases} \int_\Omega \frac{D(t,x)}{p_\lambda(x)} |\nabla u|^{p_\lambda(x)} dx + \int_\Omega \frac{1}{p_\lambda(x)} |u|^{p_\lambda(x)} dx, & \text{if } u \in X_\lambda \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.1}$$

Note that, defining $A(t)$ as $A_\lambda(t)$ with function $p$ in place of function $p_\lambda$ we have that all properties described above holds for operator $A(t)$, for all $t \in [\tau, T]$, we will denote the space $X := W^{1,p(x)}(\Omega)$.

We will present some estimates that will be useful in the course of the work.

**Theorem 2.1** ([5]). *If $u \in L^{p(x)}(\Omega)$. Then*

*i)* $\|u\|_{p(x)} < 1 (= 1; > 1)$ *if and only if* $\rho(u) < 1 (= 1; > 1)$;

*ii) if* $\|u\|_{p(x)} > 1$, *then* $\|u\|_{p(x)}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}^{p^+}$;

*iii) if* $\|u\|_{p(x)} < 1$, *then* $\|u\|_{p(x)}^{p^+} \leq \rho(u) \leq \|u\|_{p(x)}^{p^-}$.

From Lemmas 2.2 and 2.3 in [6] and Proposition 3.1 in [1] we can conclude the following result.

**Lemma 2.2.** *For $t \in [\tau, T]$, we have that for every $u \in X$*

$$\langle A(t)u, u \rangle_{X^*,X} \geq \begin{cases} \frac{\min\{1,\beta\}}{2^{p^+}} \|u\|_X^{p^+}, & \text{if } \|u\|_X < 1 \\ \frac{\min\{1,\beta\}}{2^{p^+}} \|u\|_X^{p^-}, & \text{if } \|u\|_X \geq 1 \end{cases}$$

**Remark 2.3.** It is obvious that both of the above results are satisfied if instead of function $p$ we take function $p_\lambda$ for each $\lambda \in [0, \infty)$ and with their respective spaces $X_\lambda$ and bounds $p_\lambda^-$ and $p_\lambda^+$.

We recall the result of existence of solution from [6].

**Theorem 2.4.** *If $B : [\tau, T] \times H \to H$ satisfies Assumptions (B1) and (B2) and $u_{0\lambda} \in H$, then for each $\lambda \in [0, \infty)$ there exists a unique strong global solution of the problem ($P_\lambda$), i.e., there exists $u_\lambda \in C([\tau, T]; H)$, with $u_\lambda(\tau) = u_{0\lambda}$ such that*

$$\frac{du_\lambda}{dt}(t) + A_\lambda(t)u_\lambda(t) = B(t, u_\lambda(t)) \quad \text{a.e. on } [\tau, T].$$

## 3 Pullback attractor

The theory about pullback attractors can be found in [2]. The existence of pullback attractor for the Problem ($P_\lambda$) was ensured in [6], for each $\lambda \in [0, \infty)$.

**Definition 3.1.** An evolution process in a metric space $X$ is a family $\{U(t, \tau) : X \to X;\ t \geq \tau \in \mathbb{R}\}$ satisfying:

i) $U(\tau, \tau) = Id_X$;

ii) $U(t, \tau) = U(t, s)U(s, \tau), \tau \leq s \leq t$.

**Definition 3.2.** Let $\{U(t, \tau);\ t \geq \tau \in \mathbb{R}\}$ be an evolution process in a metric space $X$. Given $A$ and $B$ subsets of $X$, we say that $A$ pullback attracts $B$ at time $t$ if

$$\lim_{\tau \to -\infty} \operatorname{dist}_H(U(t, \tau)B, A) = 0,$$

where $\operatorname{dist}_H$ denote the Hausdorff semi-distance.

**Definition 3.3.** A family of subsets $\{A(t) : t \in \mathbb{R}\}$ of $X$ is called a pullback attractor for the evolution process $\{U(t, \tau);\ t \geq \tau \in \mathbb{R}\}$ if, for each $t \in \mathbb{R}$, $A(t)$ is compact, $A(t)$ pullback attracts all bounded subsets of $X$ at time $t$ and the family is invariant, i.e., $U(t, \tau)A(\tau) = A(t)$ for any $t \geq \tau$.

Note that, for each $\lambda \in [0, \infty)$, Theorem 2.4 defines an evolution process $\{U_\lambda(t, \tau) : t \geq \tau\}$ in the space $H$ associated with problem ($P_\lambda$). Indeed, given $u_{0\lambda} \in H$ define $U_\lambda(t, \tau)u_{0\lambda} := u_\lambda(t)$ where $u_\lambda$ is a solution of problem ($P_\lambda$) with initial condition $u_\lambda(\tau) = u_{0\lambda}$, see [6] for details. Denote by $\{U(t, \tau) : t \geq \tau\}$ the evolution process associated with a problem like Problem ($P_\lambda$) but with the function $p$ in place of $p_\lambda$.

The next result was developed in [6] and ensures, for each $\lambda \in [0, \infty)$, the existence of pullback attractors.

**Theorem 3.4.** *The evolution process associated with Problem ($P_\lambda$) has a pullback attractor $\mathcal{A}_\lambda = \{\mathcal{A}_\lambda(t) : t \in \mathbb{R}\}$.*

This theorem also ensure the existence of a pullback attractor $\mathcal{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ for the problem with function $p$.

## 4 Estimates

Our objective in this work is to show the upper semicontinuity of the pullback attractors, for this, we will need develop some estimates uniform in $\lambda$.

**Theorem 4.1.** *Let $u_\lambda$ be a solution of Problem ($P_\lambda$). Then there exist a constant $T_1 \geq 0$ and a non decreasing function $B_1 : \mathbb{R} \to \mathbb{R}$ such that*

$$\|u_\lambda(t)\|_H \leq B_1(t), \qquad \forall\, t \geq T_1 + \tau$$

*and $\lambda \in [0, \infty)$.*

*Proof.* Multiplying the equation of the Problem ($P_\lambda$) by $u_\lambda(t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_\lambda(t)\|_H^2 + \langle A_\lambda(t)u_\lambda(t), u_\lambda(t)\rangle = \langle B(t, u_\lambda(t)), u_\lambda(t)\rangle .$$

It is easy to see that $\|u_\lambda(t)\|_H \leq 4(|\Omega|+1)^2\|u_\lambda(t)\|_{X_\lambda}$. Without loss of generality assume that $\|u_\lambda(t)\|_{X_\lambda} \geq 1$, if not the theorem is already proved. Then by, Lemma 2.2 and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_\lambda(t)\|_H^2 &\leq -\frac{\min\{1,\beta\}}{2^{p_\lambda^+}}\|u_\lambda(t)\|_{X_\lambda}^{p_\lambda^-} + \langle B(t,u_\lambda(t)) - B(t,0), u_\lambda(t)\rangle + \langle B(t,0), u_\lambda(t)\rangle \\ &\leq -\frac{\min\{1,\beta\}}{2^M}\|u_\lambda(t)\|_{X_\lambda}^m + C_1\|u_\lambda(t)\|_{X_\lambda}^2 + C_2(t)\|u_\lambda(t)\|_{X_\lambda}\end{aligned}$$

where $C_1 := L\left[4(|\Omega|+1)^2\right]^2$ and $C_2(t) := 4(|\Omega|+1)^2\|B(t,0)\|_H$.

If $\theta := \frac{m}{2}$, $\frac{1}{\theta} + \frac{1}{\theta'} = 1$ and $\frac{1}{m} + \frac{1}{m'} = 1$. Then from Young's inequality with $\varepsilon > 0$, we obtain

$$\begin{aligned}C_1\|u_\lambda(t)\|_{X_\lambda}^2 + C_2(t)\|u_\lambda(t)\|_{X_\lambda} &= \frac{C_1\varepsilon}{\varepsilon}\|u_\lambda(t)\|_{X_\lambda}^2 + \frac{C_2(t)\varepsilon}{\varepsilon}\|u_\lambda(t)\|_{X_\lambda} \\ &\leq \frac{1}{\theta'}\left(\frac{C_1}{\varepsilon}\right)^{\theta'} + \frac{1}{\theta}\varepsilon^\theta\|u_\lambda(t)\|_{X_\lambda}^m + \frac{1}{m'}\left(\frac{C_2(t)}{\varepsilon}\right)^{m'} + \frac{1}{m}\varepsilon^m\|u_\lambda(t)\|_{X_\lambda}^m.\end{aligned}$$

Choose $\varepsilon_0 > 0$ such that

$$\gamma := \frac{\min\{1,\beta\}}{2^M} - \frac{1}{\theta}\varepsilon_0^\theta - \frac{1}{m}\varepsilon_0^m > 0,$$

we have

$$\frac{1}{2}\frac{d}{dt}\|u_\lambda(t)\|_H^2 + \gamma\|u_\lambda(t)\|_{X_\lambda}^m \leq \frac{1}{\theta'}\left(\frac{C_1}{\varepsilon_0}\right)^{\theta'} + \frac{1}{m'}\left(\frac{C_2(t)}{\varepsilon_0}\right)^{m'}.$$

Let $\delta(t) := \frac{2}{\theta'}\left(\frac{C_1}{\varepsilon_0}\right)^{\theta'} + \frac{2}{m'}\left(\frac{C_2(t)}{\varepsilon_0}\right)^{m'}$, $\tilde{\gamma} := \frac{2\gamma}{[4(|\Omega|+1)^2]^m}$ and $y_\lambda(t) := \|u_\lambda(t)\|_H^2$. Then

$$y_\lambda'(t) + \tilde{\gamma}y_\lambda(t)^{\frac{m}{2}} \leq \delta(t), \qquad \forall t \geq \tau.$$

From a slight generalization of Lemma 5.1 in [14], we obtain

$$y_\lambda(t) \leq \left(\frac{\delta(t)}{\tilde{\gamma}}\right)^{\frac{2}{m}} + \left(\tilde{\gamma}\left(\frac{m-2}{2}\right)(t-\tau)\right)^{-\frac{2}{m-2}}.$$

Let $T_1 > 0$ such that $\left[\tilde{\gamma}\left(\frac{m-2}{2}\right)T_1\right]^{-\frac{2}{m-2}} \leq 1$. Then

$$\|u_\lambda(t)\|_H \leq \left(\frac{\delta(t)}{\tilde{\gamma}}\right)^{\frac{1}{m}} + 1 =: K_1(t),$$

for all $t \geq T_1 + \tau$. Observe that $K_1(t)$ is nondecreasing by Assumption (B3).

Taking $B_1(t) := \max\{K_1(t), 4(|\Omega|+1)^2\}$ the theorem follows. $\square$

**Theorem 4.2.** *Given $T > \tau$ and a bounded set $B \subset H$, there exists $D_1(T) > 0$ such that $\|u_\lambda(t)\|_H \leq D_1(T)$, for all $\tau \leq t \leq T$ and $\lambda \in [0, \infty)$ such that $u_{0\lambda} \in B$.*

*Proof.* Without loss of generality assume that $\|u_\lambda(t)\|_{X_\lambda} \geq 1$, if not the theorem is already proved. Proceeding as in the first lines of the proof of the previous theorem above, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_\lambda(t)\|_H^2 &\leq -\gamma\|u_\lambda(t)\|_{X_\lambda}^m + \frac{1}{\theta'}\left(\frac{C_1}{\varepsilon_0}\right)^{\theta'} + \frac{1}{m'}\left(\frac{C_2(t)}{\varepsilon_0}\right)^{m'} \\ &\leq \frac{1}{\theta'}\left(\frac{C_1}{\varepsilon_0}\right)^{\theta'} + \frac{1}{m'}\left(\frac{C_2(t)}{\varepsilon_0}\right)^{m'}\end{aligned}$$

where $\varepsilon_0 > 0$ was given in the proof of the previous theorem. Integrating from $\tau$ to $t \leq T$, we obtain

$$\|u_\lambda(t)\|_H^2 \leq \|u_\lambda(\tau)\|_H^2 + (t-\tau)\left[\frac{1}{\theta'}\left(\frac{C_1}{\varepsilon_0}\right)^{\theta'}\right] + K(t)$$

where $K(t) := \frac{1}{m'\varepsilon_0^{m'}}\int_\tau^t C_2(s)^{m'}ds < \infty$ is bounded.

Indeed, as $m > 2$ we have $\frac{1}{m} < \frac{1}{2}$, therefore $\frac{1}{2} < \frac{1}{m'}$, and then $1 < m' < 2$, because $\frac{1}{m} + \frac{1}{m'} = 1$. Remembering the definition of $C_2(t)$ and the assumption (B2), we can see that $K(T)$ is bounded.

Consequently

$$\|u_\lambda(t)\|_H^2 \leq \|u_\lambda(\tau)\|_H^2 + (T-\tau)\left[\frac{1}{\theta'}\left(\frac{C_1}{\varepsilon_0}\right)^{\theta'}\right] + K(T). \qquad \square$$

**Theorem 4.3.** *Let $u_\lambda \in C([\tau, \infty); H)$ be the global solution of Problem ($P_\lambda$). Then there exist a constant $T_2 > 0$ and a nondecreasing function $B_2 : \mathbb{R} \to \mathbb{R}$ such that*

$$\|u_\lambda(t)\|_{X_\lambda} \leq B_2(t), \qquad \forall t \geq T_2 + \tau, \ \lambda \in [0, \infty).$$

*Proof.* Let $u_\lambda$ be the global solution of ($P_\lambda$). Using the identity

$$\begin{aligned}\frac{d}{dt}\varphi_{p_\lambda}(t)(u_\lambda(t)) &= \left\langle \partial\varphi_{p_\lambda}(t)(u_\lambda(t)), \frac{du_\lambda}{dt}(t)\right\rangle \\ &= \left\langle A_\lambda(t)u_\lambda(t), \frac{du_\lambda}{dt}(t)\right\rangle \\ &= \left\langle B(t,u_\lambda(t)) - \frac{du_\lambda}{dt}(t), \frac{du_\lambda}{dt}(t)\right\rangle \\ &= -\left\|B(t,u_\lambda(t)) - \frac{du_\lambda}{dt}(t)\right\|_H^2 + \left\langle B(t,u_\lambda(t)) - \frac{du_\lambda}{dt}(t), B(t,u_\lambda(t))\right\rangle \\ &\leq -\left\|B(t,u_\lambda(t)) - \frac{du_\lambda}{dt}(t)\right\|_H^2 + \frac{1}{2}\left\|B(t,u_\lambda(t)) - \frac{du_\lambda}{dt}(t)\right\|_H^2 \\ &\quad + \frac{1}{2}\|B(t,u_\lambda(t))\|_H^2.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\varphi_{p_\lambda}(t)(u_\lambda(t)) + \frac{1}{2}\left\|B(t,u_\lambda(t)) - \frac{du_\lambda}{dt}(t)\right\|_H^2 \leq \frac{1}{2}\|B(t,u_\lambda(t))\|_H^2,$$

and thus,

$$\begin{aligned}\frac{d}{dt}\varphi_{p_\lambda}(t)(u_\lambda(t)) &\le \frac{1}{2}\Big[\,\|B(t,u_\lambda(t))-B(t,0)\|_H+\|B(t,0)\|_H\Big]^2\\ &\le \frac{1}{2}\Big[L\,\|u_\lambda(t)\|_H+\|B(t,0)\|_H\Big]^2.\end{aligned}$$

From Theorem 4.1, we obtain

$$\frac{d}{dt}\varphi_{p_\lambda}(t)(u_\lambda(t)) \le M_1(t), \qquad \forall t \ge T_1+\tau,$$

where $M_1(t) := \frac{1}{2}\left[LB_1(t)+\|B(t,0)\|_H\right]^2$.

From the definition of subdifferential, we have

$$\varphi_{p_\lambda}(t)(u_\lambda(t)) \le \left\langle \partial\varphi_{p_\lambda}(t)(u_\lambda(t)), u_\lambda(t)\right\rangle.$$

Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_\lambda(t)\|_H^2+\varphi_{p_\lambda}(t)(u_\lambda(t)) &\le \left\langle \frac{du_\lambda}{dt}(t),u_\lambda(t)\right\rangle+\left\langle \partial\varphi_{p_\lambda}(t)(u_\lambda(t)),u_\lambda(t)\right\rangle\\ &=\left\langle \frac{du_\lambda}{dt}(t)+\partial\varphi_{p_\lambda}(t)(u_\lambda(t)),u_\lambda(t)\right\rangle\\ &=\langle B(t,u_\lambda(t)),u_\lambda(t)\rangle\\ &\le \|B(t,u_\lambda(t))\|_H\|u_\lambda(t)\|_H\\ &\le \frac{1}{2}\|B(t,u_\lambda(t))\|_H^2+\frac{1}{2}\|u_\lambda(t)\|_H^2\\ &\le M_1(t)+\frac{1}{2}B_1(t)^2, \qquad \forall t\ge T_1+\tau.\end{aligned} \tag{4.1}$$

Fixing $r>0$ and integrating both sides of (4.1) over $(t,t+r)$ for $t\ge T_1+\tau$,

$$\begin{aligned}\int_t^{t+r}\varphi_{p_\lambda}(s)(u_\lambda(s))ds &\le \frac{1}{2}\|u_\lambda(t)\|_H^2+\int_t^{t+r}M_1(s)+\frac{1}{2}B_1(s)^2ds\\ &\le \frac{1}{2}B_1(t)^2+\int_t^{t+r}M_1(s)+\frac{1}{2}B_1(s)^2ds =: a_3(t),\end{aligned}$$

Let $y_\lambda(s)=\varphi_{p_\lambda}(s)(u_\lambda(s))$, $g:=0$ and $h(s):=M_1(s)$. Then

$$\int_t^{t+r}g(s)ds=0=:a_1(t), \qquad \int_t^{t+r}h(s)ds=:a_2(t), \qquad \int_t^{t+r}y_\lambda(s)ds\le a_3(t),$$

from a slight generalization of the uniform Gronwall lemma [14], we obtain

$$y_\lambda(t+r)\le\left(\frac{a_3(t)}{r}+a_2(t)\right)e^0=:\tilde r_1(t), \qquad \forall\, t\ge T_1+\tau. \tag{4.2}$$

Therefore,

$$\int_\Omega\frac{D(t,x)}{p_\lambda(x)}\,|\nabla u_\lambda(\ell,x)|^{p_\lambda(x)}\,dx+\int_\Omega\frac{1}{p_\lambda(x)}\,|u_\lambda(\ell,x)|^{p_\lambda(x)}\,dx\le\tilde r_1(t),$$

for all $\ell \geq T_1 + \tau + r$ and $\lambda \in [0, \infty)$. Then

$$\frac{\min\{1, \beta\}}{M} \left[\rho_\lambda(\nabla u_\lambda(\ell)) + \rho_\lambda(u_\lambda(\ell))\right] \leq \tilde{r}_1(t)$$

for all $\ell \geq T_1 + \tau + r$ and $\lambda \in [0, \infty)$, and hence,

$$\rho_\lambda(\nabla u_\lambda(\ell)) + \rho_\lambda(u_\lambda(\ell)) \leq \frac{M}{\min\{1, \beta\}} \tilde{r}_1(t) \tag{4.3}$$

for all $\ell \geq T_1 + \tau + r$ and $\lambda \in [0, \infty)$.

If $\ell \geq T_1 + \tau + r$ and $\|u_\lambda(\ell)\|_{X_\lambda} \geq 1$ there are four cases to analyze.

**Case 1:** If $\|\nabla u_\lambda(\ell)\|_{p_\lambda(x)} \geq 1$ and $\|u_\lambda(\ell)\|_{p_\lambda(x)} \geq 1$ we know that

$$\|\nabla u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^-} \leq \rho_\lambda(\nabla u_\lambda(\ell)) \leq \|\nabla u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^+}$$

and

$$\|u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^-} \leq \rho_\lambda(u_\lambda(\ell)) \leq \|u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^+}.$$

Since $m \leq p_\lambda^- \leq p_\lambda^+ \leq M$ and using (4.3), we have

$$\|u_\lambda(\ell)\|_{X_\lambda} \leq R_1(t), \quad t \geq T_2 + \tau, \ \lambda \in [0, \infty),$$

where $R_1(t) := 2\left[\frac{M}{\min\{1,\beta\}}\tilde{r}_1(t)\right]^{\frac{1}{m}}$ and $T_2 := T_1 + r$.

**Case 2:** If $\|\nabla u_\lambda(\ell)\|_{p_\lambda(x)} \geq 1$ and $\|u_\lambda(\ell)\|_{p_\lambda(x)} \leq 1$ we know that

$$\|\nabla u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^-} \leq \rho_\lambda(\nabla u_\lambda(\ell)) \leq \|\nabla u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^+}$$

and

$$\|u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^+} \leq \rho_\lambda(u_\lambda(\ell)) \leq \|u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^-}.$$

Since $m \leq p_\lambda^- \leq p_\lambda^+ \leq M$ and using (4.3), we have

$$\|u_\lambda(\ell)\|_{X_\lambda} \leq R_2(t), \qquad t \geq T_2 + \tau, \ \lambda \in [0, \infty),$$

where $R_2(t) := \left[\frac{M}{\min\{1,\beta\}}\tilde{r}_1(t)\right]^{\frac{1}{m}} + \left[\frac{M}{\min\{1,\beta\}}\tilde{r}_1(t)\right]^{\frac{1}{M}}$.

**Case 3:** If $\|\nabla u_\lambda(\ell)\|_{p_\lambda(x)} \leq 1$ and $\|u_\lambda(\ell)\|_{p_\lambda(x)} \geq 1$ we know that

$$\|\nabla u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^+} \leq \rho_\lambda(\nabla u_\lambda(\ell)) \leq \|\nabla u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^-}$$

and

$$\|u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^-} \leq \rho_\lambda(u_\lambda(\ell)) \leq \|u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^+}.$$

Since $m \leq p_\lambda^- \leq p_\lambda^+ \leq M$ and using (4.3), we have

$$\|u_\lambda(\ell)\|_{X_\lambda} \leq R_3(t), \qquad t \geq T_2 + \tau, \ \lambda \in [0, \infty),$$

where $R_3(t) := R_2(t)$.

**Case 4:** If $\|\nabla u_\lambda(\ell)\|_{p_\lambda(x)} \leq 1$ and $\|u_\lambda(\ell)\|_{p_\lambda(x)} \leq 1$ we know that

$$\|\nabla u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^+} \leq \rho_\lambda(\nabla u_\lambda(\ell)) \leq \|\nabla u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^-}$$

and
$$\|u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^+} \leq \rho_\lambda(u_\lambda(\ell)) \leq \|u_\lambda(\ell)\|_{p_\lambda(x)}^{p_\lambda^-}.$$
Since $m \leq p_\lambda^- \leq p_\lambda^+ \leq M$ and using (4.3), we have
$$\|u_\lambda(\ell)\|_{X_\lambda} \leq R_4(t), \qquad t \geq T_2 + \tau,\ \lambda \in [0, \infty),$$
where $R_4(t) := 2\left[\frac{M}{\min\{1,\beta\}}\tilde{r}_1(t)\right]^{\frac{1}{M}}$.

In summary, defining
$$B_2(t) := \max\left\{1, 2\left[\left(\frac{M}{\min\{1,\beta\}}\tilde{r}_1(t)\right)^{\frac{1}{m}} + \left(\frac{M}{\min\{1,\beta\}}\tilde{r}_1(t)\right)^{\frac{1}{M}}\right]\right\}$$
we have
$$\|u_\lambda(t)\|_{X_\lambda} \leq B_2(t), \qquad t \geq T_2 + \tau,\ \lambda \in [0, \infty). \qquad \square$$

**Corollary 4.4.** *Let $T_2 > 0$ obtained in the Theorem 4.3. The following statements are satisfied.*

*a) Let $u_\lambda$ be a solution of the Problem $(P_\lambda)$ in $[\tau, \infty)$. There exist a nondecreasing function $B_3 : \mathbb{R} \to \mathbb{R}$ such that*
$$\|u_\lambda(t)\|_{X_m} \leq B_3(t), \qquad \forall\, t \geq T_2 + \tau \text{ and } \lambda \in [0, \infty),$$
*where $X_m = W^{1,m}(\Omega)$;*

*b) There exist a family of bounded sets $D := D(t)_{t\in\mathbb{R}}$ in $X_m$ such that $\mathcal{A}_\lambda(t) \subset D(t)$ for each $t$ and $\lambda \in [0, \infty)$, where $\mathcal{A}_\lambda$ is the pullback attractor for the evolution process associated with Problem $(P_\lambda)$;*

*c) $\overline{\cup_{\lambda\in[0,\infty)}\mathcal{A}_\lambda(t)}$ is compact in $H$ for each $t \in \mathbb{R}$.*

*Proof.* The item a) follows from Theorem 4.3.

The item b) follows from item a).

The item c) follows from compact embedding of $X_m$ in $H$. $\square$

**Theorem 4.5.** *Let $u_\lambda$ be a solution of Problem $(P_\lambda)$ such that $u_\lambda(\tau) = u_{0\lambda} \in X_\lambda$ and suppose that there is $C > 0$ such that $\|u_{0\lambda}\|_{X_\lambda} \leq C$ for all $\lambda \in [0, \infty)$. Given $T > \tau$, then we have that there exists $D_2(T) > 0$ such that $\|u_\lambda(t)\|_{X_\lambda} \leq D_2(T)$, for all $\tau \leq t \leq T$ and $\lambda \in [\tau, \infty)$.*

*Proof.* Proceeding as in the first lines of the proof of theorem above we obtain by Theorem 4.1 that
$$\begin{aligned}\frac{d}{dt}\varphi_{p_\lambda}(t)(u_\lambda(t)) &\leq \frac{1}{2}\Big[\|B(t,u_\lambda(t)) - B(t,0)\|_H + \|B(t,0)\|_H\Big]^2 \\ &\leq \frac{1}{2}\Big[L\,\|u_\lambda(t)\|_H + \|B(t,0)\|_H\Big]^2 \leq \frac{1}{2}\Big[LD_1(T) + \|B(t,0)\|_H\Big]^2.\end{aligned}$$

Integrating in $(\tau, t)$, $t \leq T$, we have
$$\begin{aligned}\varphi_{p_\lambda}(t)(u_\lambda(t)) \leq \varphi_{p_\lambda}(\tau)(u_\lambda(\tau)) &+ \frac{L^2}{2}D_1(T)^2(T-\tau) \\ &+ \frac{1}{2}\int_\tau^t \|B(s,0)\|_H^2\,ds + LD_1(T)\int_\tau^t \|B(s,0)\|_H ds.\end{aligned}$$

As $\|u_\lambda(\tau)\|_{X_\lambda} \leq C$ we have that $\varphi_{p_\lambda}(\tau)(u_\lambda(\tau)) \leq \tilde{C}$. Since

$$\frac{\min(1,\beta)}{M}\left[\rho(\nabla u_\lambda(t)) + \rho(u_\lambda(t))\right] \leq \varphi_{p_\lambda}(t)(u_\lambda(t)),$$

we can use assumptions (B2) and (B3) and divide in cases as in the proof of the above theorem and conclude the result. □

## 5 Upper semicontinuity of pullback attractors

Finally we will show the upper semicontinuity of the pullback attractors but first we established in the next theorem the continuity of the process, this proofs the robustness of the problem and help us to proof our main result which is the upper semicontinuity of the pullback attractors.

**Theorem 5.1.** *Let $\{U_\lambda(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be the evolution process generated by the problem ($P_\lambda$). If each $\|u_{0\lambda}\|_{X_\lambda} \leq C$ and $u_{0\lambda} \to u_0$ in $H$ when $\lambda \to \infty$, then $U_\lambda(t,\tau)u_{0\lambda} \to U(t,\tau)u_0$ in $H$ as $\lambda \to \infty$, uniformly for $t$ in compact subsets of $\mathbb{R}$.*

*Proof.* Subtracting equation ($P_\lambda$) from the limit equation gives

$$\frac{d}{dt}\left(u_\lambda(t) - u(t)\right) + A_\lambda(t)u_\lambda(t) - A(t)u(t) = B(t,u_\lambda(t)) - B(t,u(t))$$

for a.e. $t \in [\tau, T]$. Then multiplying by $u_\lambda(t) - u(t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_\lambda(t) - u(t)\|_H^2 + \langle A_\lambda(t)u_\lambda(t) - A(t)u(t), u_\lambda(t) - u(t)\rangle \\
&\quad = \langle B(t,u_\lambda(t)) - B(t,u(t)), u_\lambda(t) - u(t)\rangle \\
&\quad \leq \|B(t,u_\lambda(t)) - B(t,u(t))\|_H \|u_\lambda(t) - u(t)\|_H \leq L\|u_\lambda(t) - u(t)\|_H^2
\end{aligned}$$

Moreover, for any $\xi, \eta \in \mathbb{R}^N$ we have the following inequality for a constant $p \geq 2$ (see [5]):

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi - \eta) \geq \left(\frac{1}{2}\right)^p |\xi - \eta|^p. \tag{5.1}$$

Using (5.1), after some computation we obtain

$$\begin{aligned}
&\langle A_\lambda(t)u_\lambda(t) - A(t)u(t), u_\lambda(t) - u(t)\rangle \\
&\geq \beta\int_\Omega \left(|\nabla u_\lambda(t)|^{p_\lambda(x)-2}\nabla u_\lambda(t) - |\nabla u(t)|^{p_\lambda(x)-2}\nabla u(t)\right)(\nabla u_\lambda(t) - \nabla u(t))\,dx \\
&\quad + \int_\Omega \left(|u_\lambda(t)|^{p_\lambda(x)-2}u_\lambda(t) - |u(t)|^{p_\lambda(x)-2}u(t)\right)(u_\lambda(t) - u(t))\,dx \\
&\quad + \beta\int_\Omega \left(|\nabla u(t)|^{p_\lambda(x)-2}\nabla u(t) - |\nabla u(t)|^{p(x)-2}\nabla u(t)\right)(\nabla u_\lambda(t) - \nabla u(t))\,dx \\
&\quad + \int_\Omega \left(|u(t)|^{p_\lambda(x)-2}u(t) - |u(t)|^{p(x)-2}u(t)\right)(u_\lambda(t) - u(t))\,dx \\
&\geq \beta\left(\frac{1}{2}\right)^M \int_\Omega |\nabla u_\lambda(t) - \nabla u(t)|^{p_\lambda(x)}\,dx + \left(\frac{1}{2}\right)^M \int_\Omega |u_\lambda(t) - u(t)|^{p_\lambda(x)}\,dx \\
&\quad + \beta\int_\Omega \left(|\nabla u(t)|^{p_\lambda(x)-2}\nabla u(t) - |\nabla u(t)|^{p(x)-2}\nabla u(t)\right)(\nabla u_\lambda(t) - \nabla u(t))\,dx \\
&\quad + \int_\Omega \left(|u(t)|^{p_\lambda(x)-2}u(t) - |u(t)|^{p(x)-2}u(t)\right)(u_\lambda(t) - u(t))\,dx.
\end{aligned}$$

Then

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|u_\lambda(t)-u(t)\right\|_H^2 \\
&\quad \le L\left\|u_\lambda(t)-u(t)\right\|_H^2 \\
&\qquad -\beta\int_\Omega \left(|\nabla u(t)|^{p_\lambda(x)-2}\nabla u(t)-|\nabla u(t)|^{p(x)-2}\nabla u(t)\right)\left(\nabla u_\lambda(t)-\nabla u(t)\right)dx \\
&\qquad -\int_\Omega \left(|u(t)|^{p_\lambda(x)-2}u(t)-|u(t)|^{p(x)-2}u(t)\right)\left(u_\lambda(t)-u(t)\right)dx \\
&\quad = L\left\|u_\lambda(t)-u(t)\right\|_H^2 \\
&\qquad -\beta\int_\Omega \left(|\nabla u(t)|^{p_\lambda(x)-2}-|\nabla u(t)|^{p(x)-2}\right)\nabla u(t)\left(\nabla u_\lambda(t)-\nabla u(t)\right)dx \\
&\qquad -\int_\Omega \left(|u(t)|^{p_\lambda(x)-2}-|u(t)|^{p(x)-2}\right)u(t)\left(u_\lambda(t)-u(t)\right)dx \\
&\quad \le L\left\|u_\lambda(t)-u(t)\right\|_H^2 \\
&\qquad +\beta\int_\Omega \left||\nabla u(t)|^{p_\lambda(x)-1}-|\nabla u(t)|^{p(x)-1}\right|\left|\nabla u_\lambda(t)-\nabla u(t)\right|dx \\
&\qquad +\int_\Omega \left||u(t)|^{p_\lambda(x)-1}+|u(t)|^{p(x)-1}\right|\left|u_\lambda(t)-u(t)\right|dx,
\end{aligned}
$$

a.e. in $(\tau, T)$.

Now, let us estimate the term

$$\int_\Omega \left||\nabla u(t)|^{p_\lambda(x)-1}-|\nabla u(t)|^{p(x)-1}\right|\left|\nabla u_\lambda(t)-\nabla u(t)\right|dx.$$

From Theorems 4.3 and 4.5 there exists a constant $K := K(T)$, which is independent of $\lambda$, satisfying $|\nabla u(t)| \le K$ for all $t \in [\tau, T]$ and a.e. for $x \in \Omega$. By the mean value theorem, for each $x \in \Omega$ and $\lambda \in [0, \infty)$ there is $q \in (p(x), p_\lambda(x))$, if $p(x) \le p_\lambda(x)$ (or $q \in (p_\lambda(x), p(x))$, if $p_\lambda(x) \le p(x)$) such that

$$\left||\nabla u(t)|^{p_\lambda(x)-1}-|\nabla u(t)|^{p(x)-1}\right| = \left||\nabla u(t)|^{q-1}\ln|\nabla u(t)|\right|\,|p_\lambda(x)-p(x)|$$

provided that $u(t) \neq 0$. From the bound of $|\nabla u(t)|$ and $|\nabla u_\lambda(t)|$ we have that there is $\kappa_1$ such that

$$\left||\nabla u(t)|^{q-1}\ln|\nabla u(t)|\right|\,|\nabla u_\lambda(t)-\nabla u(t)| \le \kappa_1,$$

for all $t \in [\tau, T]$ with $u(t) \neq 0$ and a.e. $x \in \Omega$. Thus,

$$\left||\nabla u(t)|^{p_\lambda(x)-1}-|\nabla u(t)|^{p(x)-1}\right|\,|\nabla u_\lambda(t)-\nabla u(t)| \le \kappa_1|p_\lambda(x)-p(x)|$$

for all $t \in [\tau, T]$ and a.e. $x \in \Omega$.

Analogously, we can get $\kappa_2$ such that

$$\left||u(t)|^{p_\lambda(x)-1}-|u(t)|^{p(x)-1}\right|\,|u_\lambda(t)-u(t)| \le \kappa_2|p_\lambda(x)-p(x)|$$

for all $t \in [\tau, T]$ and a.e. $x \in \Omega$.

Therefore,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_\lambda(t)-u(t)\|_H^2 \le\ & L\,\|u_\lambda(t)-u(t)\|_H^2 \\
&+\beta\int_\Omega \left||\nabla u(t)|^{p_\lambda(x)-1}-|\nabla u(t)|^{p(x)-1}\right| |\nabla u_\lambda(t)-\nabla u(t)|\,dx \\
&+\int_\Omega \left||u(t)|^{p_\lambda(x)-1}+|u(t)|^{p(x)-1}\right| |u_\lambda(t)-u(t)|\,dx, \\
\le\ & L\,\|u_\lambda(t)-u(t)\|_H^2 \\
&+\kappa_1\|p_\lambda(x)-p(x)\|_{L^\infty(\Omega)}\int_\Omega |\nabla u_\lambda(t)-\nabla u(t)|\,dx \\
&+\kappa_2\|p_\lambda(x)-p(x)\|_{L^\infty(\Omega)}\int_\Omega |u_\lambda(t)-u(t)|\,dx \\
\le\ & L\,\|u_\lambda(t)-u(t)\|_H^2 \\
&+\kappa_1\|p_\lambda(x)-p(x)\|_{L^\infty(\Omega)}\left[\frac{1}{2}|\Omega|+\frac{1}{2}\|\nabla u_\lambda(t)-\nabla u(t)\|_H^2\right] \\
&+\kappa_2\|p_\lambda(x)-p(x)\|_{L^\infty(\Omega)}\left[\frac{1}{2}|\Omega|+\frac{1}{2}\|u_\lambda(t)-u(t)\|_H^2\right] \\
\le\ & L\,\|u_\lambda(t)-u(t)\|_H^2 \\
&+\kappa_1\|p_\lambda(x)-p(x)\|_{L^\infty(\Omega)}\left[\frac{1}{2}|\Omega|+\frac{1}{2}\tilde{K}(T)\right] \\
&+\kappa_2\|p_\lambda(x)-p(x)\|_{L^\infty(\Omega)}\left[\frac{1}{2}|\Omega|+\frac{1}{2}\|u_\lambda(t)-u(t)\|_H^2\right]
\end{aligned}$$

a.e. in $[\tau, T]$, by Theorems 4.3 and 4.5.

Take $\kappa := 2\kappa_1 + 2\kappa_2 + 2|\Omega| + \tilde{K}(T)$. Integrating from $\tau$ to $t$, $t \le T$, we obtain

$$\begin{aligned}
\|u_\lambda(t)-u(t)\|_H^2 \le\ & \|u_{0\lambda}-u_0\|_H^2 + (t-\tau)\kappa\|p_\lambda(x)-p(x)\|_{L^\infty(\Omega)} \\
&\int_\tau^t \left(2L+\kappa_2\|p_\lambda(x)-p(x)\|_{L^\infty(\Omega)}\right)\|u_\lambda(s)-u(s)\|_H^2\,ds
\end{aligned}$$

Then, from the Gronwall–Bellman lemma, we obtain

$$\|u_\lambda(t)-u(t)\|_H^2 \le \left(\|u_{0\lambda}-u_0\|_H^2 + (T-\tau)\kappa\|p_\lambda(x)-p(x)\|_{L^\infty(\Omega)}\right) e^{\mathcal{L}(\lambda)(T-\tau)} \tag{5.2}$$

for all $t \in [\tau, T]$, where $\mathcal{L}(\lambda) := 2L + \kappa_2\|p_\lambda(x)-p(x)\|_{L^\infty(\Omega)}$ is bounded. Therefore, $u_\lambda \to u$ in $C([\tau, T]; H)$ as $\lambda \to \infty$. □

**Theorem 5.2.** *The family of pullback attractors $\{\mathcal{A}_\lambda(t) : t \in \mathbb{R}\}$, $\lambda \in [0, \infty)$ is upper semicontinuous.*

*Proof.* We will prove that for each $t \in \mathbb{R}$,

$$\operatorname{dist}(\mathcal{A}_\lambda(t), \mathcal{A}(t)) \to 0, \quad \text{as } \lambda \to \infty.$$

Given $t \in \mathbb{R}$ and $\varepsilon > 0$, let $\tau \in \mathbb{R}$ be such that

$$\operatorname{dist}(U(t,\tau)D(\tau), \mathcal{A}(t)) < \frac{\varepsilon}{3},$$

where $\cup_{\lambda\in[0,\infty)}\mathcal{A}_\lambda(\tau) \subset D(\tau)$ and $D(\tau)$ is a bounded set in $H$ (see Corollary 4.4).

Given $\psi_0 \in \mathcal{A}_{\lambda_0}(\tau)$ from invariance of the attractors there are $\tau_1 \in \mathbb{R}$, with $\tau \geq T_2 + \tau_1$, and $\tilde{\psi}_0 \in \mathcal{A}_{\lambda_0}(\tau_1)$ such that, from Theorem 4.3, we have that

$$\|\psi_0\|_{X_{\lambda_0}} = \|U_{\lambda_0}(\tau,\tau_1)\tilde{\psi}_0\|_{X_{\lambda_0}} \leq B_2(\tau).$$

Note that for each $\lambda \in [0,\infty)$ and $\psi_\lambda \in \mathcal{A}_\lambda(\tau)$ we can found $\tau_\lambda \in \mathbb{R}$ and $\tilde{\psi}_\lambda \in \mathcal{A}_\lambda(\tau_\lambda)$ such that $\tau \geq T_2 + \tau_\lambda$, and then, like above, ensuring that

$$\|\psi_\lambda\|_{X_\lambda} \leq B_2(\tau).$$

Thus, from Theorem 5.1, more precisely by the expression (5.2), there is $\lambda_0 = \lambda_0(\varepsilon) > 0$ such that if $\lambda \geq \lambda_0$ we have that

$$\|U_\lambda(t,\tau)\psi_\lambda - U(t,\tau)\psi_\lambda\|_H < \frac{\varepsilon}{3}.$$

for all $\psi_\lambda \in \mathcal{A}_\lambda(\tau)$.

Then, we obtain

$$\sup_{\psi_\lambda \in \mathcal{A}_\lambda(\tau)} \|U_\lambda(t,\tau)\psi_\lambda - U(t,\tau)\psi_\lambda\| \leq \frac{\varepsilon}{3}$$

for all $\lambda \geq \lambda_0$.

Therefore,

$$\begin{aligned}
\operatorname{dist}(\mathcal{A}_\lambda(t),\mathcal{A}(t)) &= \operatorname{dist}(U_\lambda(t,\tau)\mathcal{A}_\lambda(\tau),\mathcal{A}(t)) \\
&= \sup_{\psi_\lambda \in \mathcal{A}_\lambda(\tau)} \operatorname{dist}(U_\lambda(t,\tau)\psi_\lambda,\mathcal{A}(t)) \\
&\leq \sup_{\psi_\lambda \in \mathcal{A}_\lambda(\tau)} \{\operatorname{dist}(U_\lambda(t,\tau)\psi_\lambda, U(t,\tau)\psi_\lambda) + \operatorname{dist}(U(t,\tau)\psi_\lambda,\mathcal{A}(t))\} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon,
\end{aligned}$$

for all $\lambda \geq \lambda_0$, showing the upper semicontinuity desired. □

Note that, in all this work we could have asked only that $p_\lambda, p \in C(\overline{\Omega},\mathbb{R})$ with $2 < m \leq p^- \leq p^+ \leq M$ and $p_\lambda \to p$ in $L^\infty(\Omega)$ as $\lambda \to \infty$, and then, for $\lambda$ large enough, all results would be satisfied.

# 6 Final remarks

In this work we have ensured the upper semicontinuity of the pullback attractors, in other words it means that for each $t \in \mathbb{R}$,

$$\mathcal{A}_\lambda(t) \to \mathcal{A}(t), \quad \text{as } \lambda \to \infty. \tag{6.1}$$

In [7] the authors considered $B(t,u(t)) \equiv B(u)$ and $D(t,\cdot) \to D^*(\cdot)$ in $L^\infty(\Omega)$ when $t \to \infty$, and ensured that

$$\mathcal{A}(t) \to \mathcal{A}_\infty, \quad \text{as } t \to \infty,$$

where $\mathcal{A}_\infty$ is the global attractor for the autonomous version of problem ($P_\lambda$) with $B(u)$, $D^*$ and $p(x)$. Of course for each $\lambda \in [0,\infty)$, using the same work cited above, we can ensure the existence of a global attractor $\mathcal{A}_\infty^\lambda$ associated with a autonomous version of problem as ($P_\lambda$) with $B(u)$, $D^*$ and $p_\lambda(x)$, and

$$\mathcal{A}_\lambda(t) \to \mathcal{A}_\infty^\lambda, \quad \text{as } t \to \infty. \tag{6.2}$$

Furthermore, with the work [13] we can ensure that

$$\mathcal{A}_\infty^\lambda \to \mathcal{A}_\infty, \quad \text{as } \lambda \to \infty.$$

Observing all this behavior we can construct the following diagrams for the case $B(t,u) \equiv B(u)$

$$\begin{tikzcd} \mathcal{A}_\lambda(t) \arrow[d, "\lambda\to\infty"] & \\ \mathcal{A}(t) \arrow[r, "t\to\infty"] & \mathcal{A}_\infty \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{A}_\lambda(t) \arrow[r, "t\to\infty"] & \mathcal{A}_\infty^\lambda \arrow[d, "\lambda\to\infty"] \\ & \mathcal{A}_\infty \end{tikzcd}.$$

This behavior draws our attention, because even for real functions this behavior does not always happen. A result for real functions says that we can ensure that

$$\lim_{t\to+\infty}\left(\lim_{n\to\infty} f_n(t)\right) = \lim_{n\to\infty}\left(\lim_{t\to+\infty} f_n(t)\right)$$

when $f_n \to f$ uniformly in $t$ (see Theorem 7.11 in [10]) otherwise there are examples where these two limits are not the same.

With these ideas in mind it is reasonable to wonder whether it is possible to show the double limit, i.e., $\lim_{\substack{\lambda\to\infty \\ t\to\infty}} \mathcal{A}_\lambda(t) = \mathcal{A}_\infty$. It seems that some uniformity on $t$ would be necessary in (6.1) or some uniformity on $\lambda$ in (6.2).

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