Hopf bifurcation analysis of
scalar implicit neutral delay differential equation

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Abstract. Hopf bifurcation analysis is conducted on a scalar implicit Neutral Delay Differential Equation (NDDE) by means of the extension of two analytical methods: 1) center manifold reduction combined with normal form theory; 2) method of multiple scales. The modifications of the classical algorithms originally developed for explicit differential equations lead to the same algebraic results, which are further confirmed by numerical simulations. It is shown that the generalizations of these regular normal form calculation methods are useful for the local nonlinear analysis of implicit NDDEs where the explicit formalism is typically not accessible and the existence and uniqueness of solutions around the equilibrium are only assumed together with the existence of a smooth local center manifold.

Keywords: neutral delay differential equation, Hopf bifurcation, center manifold reduction, normal form, method of multiple scales, implicit differential equations.

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1 Introduction

Neutral Delay Differential Equations (NDDE) are a class of equations whose rate of change of state depends not only on the state at present and earlier instances, but also on the rate of change of the state at earlier instances [5]. We consider the simple nonlinear scalar NDDE

$$\dot{x}(t) = a\dot{x}(t-1) + bx(t-1) + c\dot{x}^3(t-1),\tag{1.1}$$

with non-zero parameters a, b, $c \in \mathbb{R}$, where the nonlinear term $c\dot{x}^3(t-1)$ indicates that equation (1.1) is an implicit differential equation as the highest derivative appears also with its 3rd power. What makes the problem even more intricate is the fact that the nonlinearity involves the delayed term of the first derivative. This means that however simple equation (1.1) is, it is not possible to express it in the NDDE form

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) - g(x(t-1))) = f(x(t), x(t-1)),\tag{1.2}$$

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as defined in [5] (see Section 12, Definition 1.1), which can be considered as the general explicit form of NDDEs. However, these implicit NDDEs do appear in some specific tasks, such as in a nonlinear model for human balancing subjected to saturated delayed proportional-derivative-acceleration (PDA) control [10]. There is a need to see whether normal form theories and the corresponding algorithms developed for NDDEs in [8, 11] are also valid for implicit NDDEs. We consider that the existence and uniqueness of the solutions of the implicit NDDEs around the trivial solution in question are true.

This paper carries out Hopf bifurcation analysis on the implicit NDDE (1.1) with two analytical methods: 1) center manifold reduction combined with normal form theory; 2) method of multiple scales. It is shown that the two analytical methods provide the same normal form and the result is also confirmed by numerical simulation. This indicates the applicability of the extension of these regular normal from calculation methods for the local bifurcation analysis of implicit NDDEs.

The rest of the paper is organized as follows. In Section 2, linear stability is analyzed and a stability chart is presented in the parameter plane (a, b). In Section 3, Hopf bifurcation analysis is carried out with the help of normal form theory and method of multiple scales together with numerical simulations, which leads to the conclusion in Section 4.

2   Linear analysis

The stability of the trivial solution $x \equiv 0$ of (1.1) is analyzed by means of the linearized system

$$\dot{x}(t) = a\dot{x}(t-1) + bx(t-1) \,. \tag{2.1}$$

The corresponding characteristic equation reads

$$\lambda(1 - a\mathrm{e}^{-\lambda}) - b\mathrm{e}^{-\lambda} = 0 \tag{2.2}$$

for the characteristic exponent $\lambda \in \mathbb{C}$. According to the D-subdivision method, substitution of $\lambda = \gamma + \mathrm{i}\omega$ into the characteristic equation (2.2) and then decomposition into real and imaginary parts yield

$$\mathrm{Re} : \gamma + ((-a\gamma - b)\cos\omega - a\omega\sin\omega)\mathrm{e}^{-\gamma} = 0 \,, \tag{2.3}$$

$$\mathrm{Im} : \omega + ((a\gamma + b)\sin\omega - a\omega\cos\omega)\mathrm{e}^{-\gamma} = 0 \,. \tag{2.4}$$

If $|a| > 1$, the characteristic equation (2.2) has infinitely many characteristic exponents with positive real parts and the linear system is always unstable [6]. Therefore, only the case $|a| < 1$ is studied here. By substituting the critical characteristic exponent $\lambda_{\mathrm{c}} = \mathrm{i}\omega_{\mathrm{c}}$ into (2.2), that is, by substituting $\gamma = 0$ into equations (2.3) and (2.4), the D-curves in the parameter plane (a, b) are obtained as

$$b_{\mathrm{c}} = 0, \quad \text{if } \omega_{\mathrm{c}} = 0 \,, \tag{2.5}$$

and

$$a_{\mathrm{c}} = \cos\omega_{\mathrm{c}} \,, \qquad b_{\mathrm{c}} = -\omega_{\mathrm{c}}\sin\omega_{\mathrm{c}} \,, \quad \text{if } \omega_{\mathrm{c}} > 0 \,. \tag{2.6}$$

The root tendency along the D-curves is needed for determining the stability area and also the number of unstable exponents. By taking the partial derivative of (2.2) with respect to b, one gets the root tendency

$$\alpha := \left. \frac{\mathrm{d}\lambda(b)}{\mathrm{d}b} \right|_{b=b_{\mathrm{c}}} = \frac{1}{\mathrm{i}\omega_{\mathrm{c}}a_{\mathrm{c}} - a_{\mathrm{c}} + b_{\mathrm{c}} + \mathrm{e}^{\mathrm{i}\omega_{\mathrm{c}}}} \,. \tag{2.7}$$

Along the D-curve $b_c = 0$ when $\omega_c = 0$,

$$\alpha = \left.\frac{d\lambda(b)}{db}\right|_{b=0} = \frac{1}{1 - a_c} > 0, \tag{2.8}$$

which indicates that a real characteristic root crosses the imaginary axis from left to right through $\lambda = 0$ when b increases through $b_c = 0$, i.e. the number of unstable characteristic roots increases by 1. Along the D-curves (2.6) when $\omega_c \neq 0$,

$$\begin{aligned}
\operatorname{Re}\alpha = \operatorname{Re}\left.\frac{d\lambda(b)}{db}\right|_{b=b_c} &= -\frac{\omega_c \sin\omega_c}{b_c{}^2 + (a_c\omega_c + \sin\omega_c)^2} \\
&\begin{cases} < 0, & \omega_c \in (2k\pi, (2k+1)\pi) \\ > 0, & \omega_c \in ((2k+1)\pi, (2k+2)\pi), \quad k = 0, 1, 2, \ldots \end{cases}
\end{aligned} \tag{2.9}$$

which provides another clue for determining the number of unstable characteristic roots as $\operatorname{Re}\alpha > 0$ (< 0) suggests that a pair of complex characteristic roots crosses the imaginary axis from left to right (from right to left), i.e. the number of unstable characteristic roots increases (decreases) by 2.

For the linear NDDE (2.1), exponential stability only exists when

$$-\sqrt{1 - a^2}\arccos a < b < 0 \tag{2.10}$$

since the number of unstable exponents is 0, as shown in Fig. 2.1. At the stability boundary $b_c = -\sqrt{1 - a_c{}^2}\arccos a_c$ (the red line in Fig. 2.1), only a pair of imaginary roots $\pm i\omega_c$ ($0 < \omega_c < \pi$) exists while all other exponents have negative real parts and the root tendency is $\operatorname{Re}\alpha < 0$, which guarantee [4] the occurrence of Hopf bifurcation to be studied in the next section .
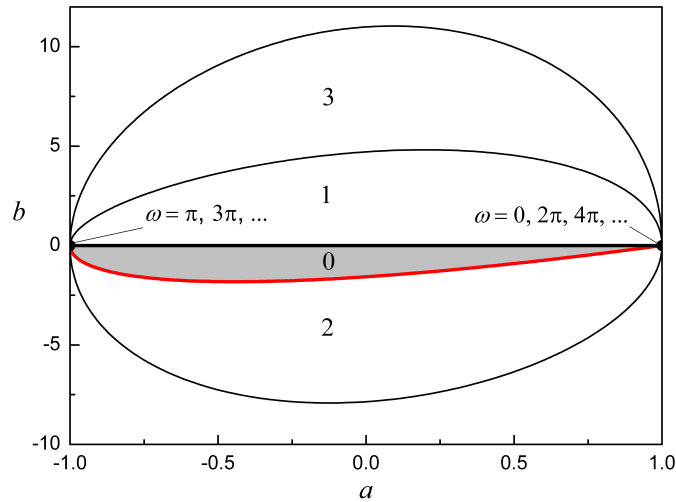

Figure 2.1: Stability chart of the linear NDDE (2.1) with the number of unstable characteristic exponents. Grey area indicates stable region. Red line is the stability boundary where Hopf bifurcations occur in the nonlinear NDDE (1.1).

3   Hopf bifurcation analysis

Suppose that there exists a smooth local center manifold embedded in the phase space of the implicit NDDE (1.1) at the zero solution. Hopf bifurcation analysis is carried out for the implicit NDDE (1.1) by centre manifold reduction combined with normal form calculation then also by the method of multiple scales. To analyze the effect of varying b on system dynamics, the bifurcation parameter μ is chosen as

$$b = b_c + \mu. \tag{3.1}$$

3.1   Centre manifold reduction combined with normal form calculation

Rewrite the scalar equation (1.1) into the following operator form:

$$\frac{\mathrm{d}}{\mathrm{d}t} Dx_t = L_0 x_t + (L x_t - L_0 x_t) + G(\dot{x}_t), \tag{3.2}$$

where $x_t(\theta) \equiv x(t + \theta)$, $-1 \leq \theta \leq 0$, $x_t \in C = C([-1, 0], \mathbb{R})$ is the Banach space of uniformly continuous functions from $[-1, 0]$ to $\mathbb{R}$ with the uniform norm, D, L_0 and L are bounded linear operators from C to $\mathbb{R}$, $Dx_t = x_t(0) - a_c x_t(-1)$, $L_0 x_t = b_c x(t-1)$, $L x_t - L_0 x_t = \mu x(t-1)$, and the nonlinear part $G(\dot{x}_t) = c \dot{x}_t^3(-1)$. The solution of the linearized equation

$$\frac{\mathrm{d}}{\mathrm{d}t} Dx_t = L_0 x_t \tag{3.3}$$

defines a C^0 semigroup $T(t)$ on $C([-1, 0], \mathbb{R})$, $T(t)\phi = x_t^\phi$, $t \geq 0$. The infinitesimal generator $\mathcal{A}$ associated with $T(t)$ is given by $\mathcal{A}\phi = \dot{\phi}$ and has domain

$$\mathrm{Dom}(\mathcal{A}) = \left\{ \phi \in C : \frac{\mathrm{d}\phi}{\mathrm{d}\theta} \in C, \, D\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = L\phi \right\}. \tag{3.4}$$

The spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ coincides with its point spectrum if and only if it satisfies the corresponding characteristic equation (2.2). Define $C^* = C([0, 1], \mathbb{R})$. Consider the adjoint bilinear form on $C^* \times C$:

$$(\psi, \phi) = (\psi(0) - a_c \psi(1))\phi(0) + a_c \int_{-1}^{0} \psi(s+1)\phi'(s)\mathrm{d}s + b_c \int_{-1}^{0} \psi(s+1)\phi(s)\mathrm{d}s, \tag{3.5}$$

which is the specific form deduced also in [9] from the general case given in [5]. Let $\mathcal{A}^*$ denote the adjoint operator of $\mathcal{A}$ with respect to the bilinear form defined in Equation (3.5), i.e., $\mathcal{A}^* : C^* \to C^*$, so that $(\psi, \mathcal{A}\phi) = (\mathcal{A}^*\psi, \phi)$ holds for all $\phi \in \mathrm{Dom}(\mathcal{A})$ and $\psi \in \mathrm{Dom}(\mathcal{A}^*)$. Let $\Lambda = \{i\omega_c, -i\omega_c\}$. There exist two subspaces P_Λ and Q_Λ splitting the space C, invariant under $T(t)$, such that $C = P_\Lambda \oplus Q_\Lambda$. A basis Φ of P_Λ is given by

$$\Phi(\theta) = \begin{bmatrix} \mathrm{e}^{i\omega_c\theta} & \mathrm{e}^{-i\omega_c\theta} \end{bmatrix} \tag{3.6}$$

and a basis Ψ for the dual space $P_{\Lambda^*}^*$ can be expressed as

$$\Psi(\xi) = \begin{bmatrix} \beta \mathrm{e}^{-i\omega_c\xi} \\ \bar{\beta} \mathrm{e}^{i\omega_c\xi} \end{bmatrix} \tag{3.7}$$

where $\beta = \alpha \mathrm{e}^{i\omega_c}$, so that $(\Psi, \Phi) = I$ is the unit matrix, and $T(t)\Phi = \Phi \mathrm{e}^{Bt}$ where

$$B = \begin{bmatrix} i\omega_c & 0 \\ 0 & -i\omega_c \end{bmatrix}. \tag{3.8}$$

To write equation (3.2) as an infinite dimensional ordinary differential equation (ODE), the infinitesimal generator $\mathcal{A}$ is extended to an operator $\tilde{\mathcal{A}}$ by

$$\tilde{\mathcal{A}}\phi = \mathcal{A}\phi + X_0[L_0\phi - D\phi'], \qquad \phi' = \frac{d\phi}{d\theta} \tag{3.9}$$

onto BC, the space of functions, which are uniformly continuous on $[-1, 0)$ and with a possible finite jump discontinuity at 0, where

$$X_0(\theta) = \begin{cases} 0, & -1 \le \theta < 0, \\ I, & \theta = 0. \end{cases} \tag{3.10}$$

Functions ψ in BC can be represented as $\psi = \phi + X_0\zeta$, where $\phi \in C$, $\zeta \in \mathbb{R}$. The bilinear form (3.5) can be extended to $C^* \times BC$ by setting $(\psi, X_0) = \psi(0)$. Let $\Pi : BC \to P$ be the projection defined as $\Pi(\phi + X_0\zeta) = \Phi[(\Psi, \phi) + \Psi(0)\zeta]$. Then $BC = P \oplus \ker \Pi$ and let $Q \subset \ker \Pi$. $\mathrm{Dom}(\tilde{\mathcal{A}}) = C^1([-1, 0], \mathbb{R})$, which makes sure that it is compatible with the domain of equation (3.2). The abstract ODE on the space BC is of the form:

$$\dot{x}_t = \tilde{\mathcal{A}}x_t + X_0\left((Lx_t - L_0x_t) + G(\dot{x}_t)\right), \tag{3.11}$$

Decompose $x_t(\theta) = \Phi(\theta)y(t) + z_t(\theta)$ where $y(t) \in \mathbb{R}^2$, $z_t \in Q$. One obtains the following decomposition of the neutral system

$$\begin{aligned}
\dot{y} &= By + \Psi(0)\left(\mu(\Phi(-1)y + z_t(-1)) + c(\Phi(-1)\dot{y} + \dot{z}_t(-1))^3\right), \\
\dot{z}_t &= \tilde{\mathcal{A}}z_t + (I - \Pi)X_0\left(\mu(\Phi(-1)y + z_t(-1)) + c(\Phi(-1)\dot{y} + \dot{z}_t(-1))^3\right), \\
\dot{\mu} &= 0.
\end{aligned} \tag{3.12}$$

The normal form analysis is based on a recursive sequence of nonlinear transformations. As non-resonance condition relative to Λ is satisfied [3, 8], there exists a formal nonlinear transformation

$$\begin{cases} y = \tilde{y} + V^1(\tilde{y}, \mu), \\ z_t = \tilde{z}_t + V^2(\tilde{y}, \mu,) \end{cases} \tag{3.13}$$

such that a local manifold satisfies $\tilde{z}_t = 0$ and the normal form on this invariant manifold yields the following two dimensional ODE

$$\dot{\tilde{y}} = B\tilde{y} + g(\tilde{y}, 0, \mu). \tag{3.14}$$

The third-order normal form is needed for the study of Hopf bifurcations. Based on similar deduction as in [11], the simpler form of the second-order nonlinear term $g_2(\tilde{y}, 0, \mu)$ and the third-order nonlinear term $g_3(\tilde{y}, 0, \mu)$ can be obtained by the following recursive formula:

$$g_j(\tilde{y}, 0, \mu) = \Psi(0)K_j - \sum_{k=2}^{j-1} D_{\tilde{y}}V_k^1(\tilde{y}, \mu)g_{j+1-k}(\tilde{y}, 0, \mu) - (D_{\tilde{y}}V_j^1(\tilde{y}, \mu)B\tilde{y} - BV_j^1(\tilde{y}, \mu)), \tag{3.15}$$

where

$$\begin{aligned}
K_2 &= (e^{-i\omega_c}\tilde{y}_1 + e^{i\omega_c}\tilde{y}_2)\mu, \\
K_3 &= c(ie^{-i\omega_c}\tilde{y}_1\omega_c - ie^{i\omega_c}\tilde{y}_2\omega_c)^3 + w(\tilde{y}_1\mu^2, \tilde{y}_2\mu^2),
\end{aligned}$$

with $w(\tilde{y}_1\mu^2, \tilde{y}_2\mu^2)$ representing terms related to μ^2 which are not expressed here in details as they are not necessary for the Hopf bifurcation analysis. The third-order normal form is then obtained as follows

$$\begin{bmatrix} \dot{\tilde{y}}_1 \\ \dot{\tilde{y}}_2 \end{bmatrix} \approx \begin{bmatrix} i\omega_c & 0 \\ 0 & -i\omega_c \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} + \begin{bmatrix} \alpha\tilde{y}_1\mu + \Delta\tilde{y}_1^2\tilde{y}_2 \\ \bar{\alpha}\tilde{y}_2\mu + \overline{\Delta}\tilde{y}_1\tilde{y}_2^2 \end{bmatrix} \tag{3.16}$$

where $\Delta = 3\,i\alpha\,c\omega_c{}^3$. Through the change of variables

$$\tilde{y}_1 = \frac{1}{2}(\rho\cos\varphi - i\rho\sin\varphi)\,, \quad \tilde{y}_2 = \frac{1}{2}(\rho\cos\varphi + i\rho\sin\varphi)\,, \tag{3.17}$$

equation (3.16) is transformed into the polar form

$$\dot{\rho} = \operatorname{Re}\alpha\,\rho\mu + \frac{1}{4}\operatorname{Re}\Delta\,\rho^3\,, \tag{3.18}$$

where $\operatorname{Re}\alpha > 0$ for $\omega_c \in (0, \pi)$ as given in equation (2.9) and

$$\operatorname{Re}\Delta = \frac{3c\omega_c{}^3(a_c\omega_c + \sin\omega_c)}{b_c^2 + (a_c\omega_c + \sin\omega_c)^2}\,. \tag{3.19}$$

If $\operatorname{Re}\Delta \neq 0$ and formula (2.6) of the critical parameter values are substituted, the amplitude of the bifurcated periodic motions of equation (1.1) yields

$$\rho = \left(-\frac{4\operatorname{Re}\alpha}{\operatorname{Re}\Delta}\mu\right)^{\frac{1}{2}} = 2\sqrt{\frac{\omega_c\sin\omega_c}{3c\omega_c{}^3(\omega_c\cos\omega_c + \sin\omega_c)}(b - b_c)}\,. \tag{3.20}$$

The sign of $\operatorname{Re}\Delta$ determines the sense of Hopf bifurcation: if $\operatorname{Re}\Delta < 0$ ($\operatorname{Re}\Delta > 0$), it is supercritical (subcritical), i.e. the bifurcated periodic solutions are stable (unstable). The sense of Hopf bifurcations for equations (1.1) is summarized in Table 3.1.

$\omega_c \in$	$\operatorname{sgn}\operatorname{Re}(\Delta)$	Sense
$(0, 2.03)$	positive	subcritical
$(2.03, \pi)$	negative	supercritical

Table 3.1: Sense of Hopf bifurcations for $c > 0$.

3.2 Method of multiple scales

To study the small amplitude oscillation, let

$$x(t) = \sqrt{\varepsilon}\varphi(t)\,, \qquad \mu = \varepsilon\bar{\mu}\,, \tag{3.21}$$

where ε is a nondimensional bookkeeping parameter, $0 < \varepsilon \ll 1$. $\bar{\mu} = \operatorname{O}(1)$ is the detuning parameter. Then equation (1.1) is transformed into the form

$$\dot{\varphi}(t) - a_c\dot{\varphi}(t-1) - b_c\varphi(t-1) = \varepsilon(\bar{\mu}\varphi(t-1) + c\dot{\varphi}^3(t-1))\,. \tag{3.22}$$

The multiple time scales are defined as $T_k = \varepsilon^k t$, $k = 0, 1, 2, \ldots$ To study the Hopf bifurcation, a two scale expansion of the solution is assumed as

$$\varphi(t) = \varphi_0(T_0, T_1) + \varepsilon\varphi_1(T_0, T_1) + \operatorname{O}(\varepsilon^2)\,. \tag{3.23}$$

By using the differential operators [7]

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \mathrm{O}(\varepsilon^2) =: \mathrm{D}_0 + \varepsilon \mathrm{D}_1 + \mathrm{O}(\varepsilon^2),$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} =: \mathrm{D}_0{}^2 + 2\varepsilon \mathrm{D}_0 \mathrm{D}_1 + \mathrm{O}(\varepsilon^2),$$

(3.24)

the delayed terms can be expressed as

$$\varphi(t-1) = \varphi_0(T_0 - 1, T_1 - \varepsilon) + \varepsilon \varphi_1(T_0 - 1, T_1 - \varepsilon) + \cdots$$
$$= \varphi_0(T_0 - 1, T_1) + \varepsilon(\varphi_1(T_0 - 1, T_1) - \mathrm{D}_1 \varphi_0(T_0 - 1, T_1)) + \mathrm{O}(\varepsilon^2).$$

(3.25)

Substituting equations (3.23), (3.24) and (3.25) into equation (3.22) and equating the same powers of ε, a set of linear partial differential equations can be obtained in the form

$$\mathrm{D}_0 \varphi_0 - a_c \mathrm{D}_0 \varphi_{0\tau} - b_c \varphi_{0\tau} = 0,$$

(3.26)

$$\mathrm{D}_0 \varphi_1 - a_c \mathrm{D}_0 \varphi_{1\tau} - b_c \varphi_{1\tau} = (a_c - b_c) \mathrm{D}_1 \varphi_{0\tau} - a_c \mathrm{D}_0 \mathrm{D}_1 \varphi_{0\tau} + \bar{\mu} \varphi_{0\tau} - \mathrm{D}_1 \varphi_0 + c(\mathrm{D}_0 \varphi_{0\tau})^3,$$ (3.27)

where $\varphi_i = \varphi_i(T_0, T_1)$, $\varphi_{i\tau} = \varphi_i(T_0 - 1, T_1)$. At the stability boundary (the red line in Fig. 2.1), only one pair of pure imaginary characteristic roots $\pm \mathrm{i}\omega_c$ exists, while all other eigenvalues have negative real parts. All the solution terms related to eigenvalues with negative real parts decay with time. Thus, to study the long-time behavior of the system, the solution of equation (3.26) can be assumed as

$$\varphi_0 = R(T_1)\mathrm{e}^{\mathrm{i}\omega_c T_0} + \overline{R(T_1)}\mathrm{e}^{-\mathrm{i}\omega_c T_0}.$$

(3.28)

Substituting equation (3.28) into equation (3.27), the secular term can be found

$$(a_c - b_c - \mathrm{i}a\omega_c - \mathrm{e}^{\mathrm{i}\omega_c})\dot{R}(T_1) + 3\mathrm{i}c\omega_c^2 R^2(T_1)\overline{R(T_1)} + \bar{\mu}R(T_1) = 0.$$

(3.29)

$\dot{R}(T_1)$ is determined by

$$\dot{R}(T_1) = \frac{R(T_1)\bar{\mu} + 3\mathrm{i}c\omega_c^3 R^2(T_1)\overline{R(T_1)}}{\mathrm{i}a_c\omega_c + \mathrm{e}^{\mathrm{i}\omega_c} - a_c + b_c}.$$

(3.30)

With

$$\tilde{y}_1 = \frac{1}{\sqrt{\varepsilon}} R(\varepsilon t)\mathrm{e}^{\mathrm{i}\omega_c t},$$

(3.31)

the normal form equation (3.30) is transformed to the exact same formula as the normal form equation (3.16) derived by using the centre manifold reduction and normal form theory.

3.3 Case study $a_c = 1/\sqrt{2}$, $c = 0.1$

To see whether the analytical results agree with numerical results, numerical analysis is also presented for the case $a_c = 1/\sqrt{2}$, $c = 0.1$ with the help of collocation method combined with path-following techniques [1,2]. Based on equation (2.6), one can have

$$b_c = -\frac{\pi}{4\sqrt{2}}, \qquad \omega_c = \frac{\pi}{4}.$$

According to Table 3.1, the analysis indicates that a subcritical Hopf bifurcation occurs since $\omega_c \in (0, 2.03)$, which is in good agreement with the numerical result shown in Figure 3.1. This illustrates the validity of the two normal form calculation methods for implicit NDDEs. Moreover, two time domain simulations with $b = -0.5$ and different initial conditions ($x_0(t) \equiv 0.1$; $x_0(t) \equiv 1$, $t \in [-1, 0]$) are presented in Figure 3.2 to demonstrate the characteristic of the subcritical Hopf bifurcation.
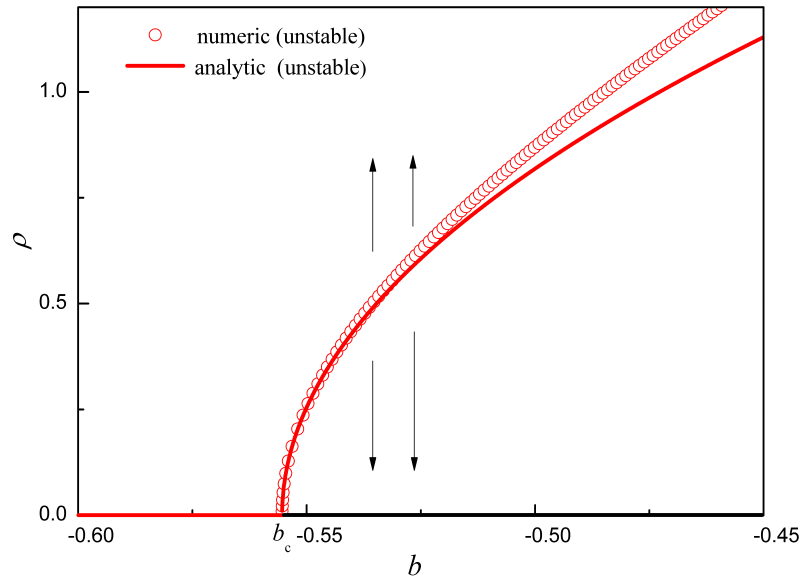
Figure 3.1: Bifurcation diagram with respect to b. $a = 1/\sqrt{2}$. Continuous lines refer to analytical results, series of circles refer to numerical results obtained by collocation method and path following. Red color represents unstable branches.
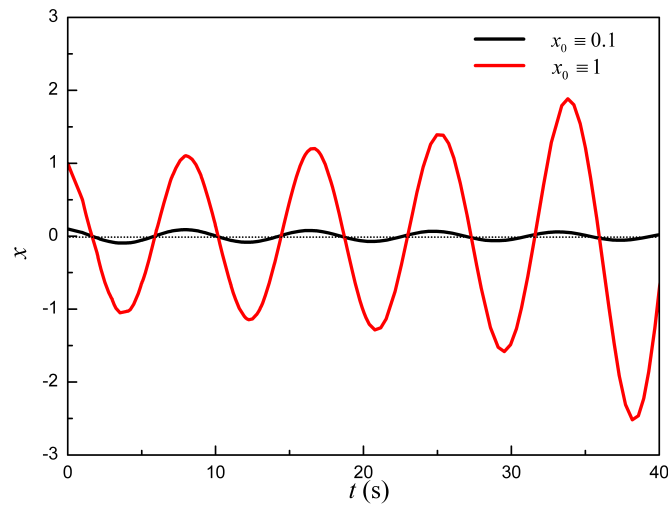
Figure 3.2: Time history with different initial conditions; $x_0(t) \equiv 0.1$ (black line), $x_0(t) \equiv 1$ (red line), $t \in [-1, 0]$, $a = 1/\sqrt{2}$, $b = -0.5$.

4   Conclusion

It is shown that both center manifold reduction combined with normal form theory and the method of multiple scales can be extended and applied for Hopf bifurcation analysis of implicit NDDEs.

Acknowledgements

References

[1] D. A. W. Barton, B. Krauskopf, R. E. Wilson, Collocation schemes for periodic solutions of neutral delay differential equations, *J. Differ. Equ. Appl.* **12**(2006), No. 11, 1087–1101. https://doi.org/10.1080/10236190601045663; MR2271819

[2] K. Engelborghs, T. Luzyanina, G. Samaey, *DDE-BIFTOOL v. 2.00: A Matlab package for bifurcation analysis of delay differential equations*, 2001.

[3] T. Faria, L. T. Magalhães, Normal forms for retarded functional-differential equations with parameters and applications to Hopf bifurcation, *J. Differential Equations* **122**(1995), No. 2, 181–200. https://doi.org/10.1006/jdeq.1995.1144; MR1355888

[4] J. Guckenheimer, P. Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Springer-Verlag, New York, 1983. https://doi.org/10.1007/978-1-4612-1140-2; MR1139515

[5] J. K. Hale, *Theory of functional differential equations*, Applied Mathematical Sciences, Vol. 3, Springer-Verlag, New York, 1977. https://doi.org/10.1007/978-1-4612-9892-2; MR0508721

[6] W. Michiels, S. Niculescu, *Stability and stabilization of time-delay systems: an eigenvalue-based approach*, SIAM, Philadelphia, 2007. https://doi.org/10.1137/1.9780898718645; MR2384531

[7] A. H. Nayfeh, D. T. Mook, *Nonlinear oscillations*, Wiley, New York, 1979. MR549322

[8] M. Weedermann, Normal forms for neutral functional differential equations, in: *Topics in functional differential and difference equations (Lisbon, 1999)*, Fields Inst. Commun., Vol. 29, 2001, pp. 361–368. MR1821791

[9] M. Weedermann, Hopf bifurcation calculations for scalar neutral delay differential equations, *Nonlinearity* **19**(2006), 2091–2102. https://doi.org/10.1088/0951-7715/19/9/005 MR2256652

[10] L. Zhang, G. Stépán, T, Insperger, Saturation limits the contribution of acceleration feedback to balancing against reaction delay, *J. R. Soc. Interface* **15**(2018), No. 138. https://doi.org/10.1098/rsif.2017.0771

[11] L. Zhang, H. L. Wang, H. Y. Hu, Symbolic computation of normal form for Hopf bifurcation in a neutral delay differential equation and an application to a controlled crane, *Nonlinear Dyn.* **70**(2012), No. 1, 463–473. https://doi.org/10.1006/jmaa.1993.1212; MR2991286