Second order systems with nonlinear nonlocal boundary conditions

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. This paper is concerned with the second order differential equation with not necessarily linear nonlocal boundary condition. The existence of solutions is obtained using the properties of the Leray–Schauder degree. The results generalize and improve some known results with linear nonlocal boundary conditions.

Keywords: nonlinear boundary value problem, nonlinear and nonlocal boundary conditions, Leray–Schauder degree, Brouwer degree.

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1 Introduction

We consider the following problem

$$x'' = f(t, x, x'), \qquad x(0) = a, \qquad x'(1) = N(x'), \tag{1.1}$$

where $a \in \mathbb{R}^n$ is fixed, $t \in [0, 1]$, $f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $N : C([0, 1], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous and not necessarily linear application. Boundary value problems with nonlinear boundary conditions have been studied, using various methods, for instance in the following papers [2, 6, 17, 22].

In the nonlocal case, when N is a linear mapping given by a Riemann–Stieltjes integral, namely $N(x') = \int_0^1 x'(s) dg(s)$, the problem (1.1) was extensively studied. Results for the scalar nonresonant problem, i.e. $\int_0^1 dg(s) \neq 1$, and references to such multipoint problems are given in [3, 7, 9, 14–16]. The resonant scalar multipoint case was considered in [8] and existence results for resonant integral problem can be found in [18]. The nonlocal problem for systems

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has been less studied. Recently such problems were considered in [21] in the nonresonant case and in [20] at resonance. Example 3.2 shows that Theorem 2.1 covers situations escaping to those previous results.

In this paper, using the properties of the Leray–Schauder degree, we prove the existence of a solution to the problem (1.1). Under the elementary arguments of convex analysis inspired by the ones introduced in [10] and some suitable conditions imposed upon N, we obtain existence conditions for the problem (1.1) (see Theorems 2.1 and 2.2). The assumption imposed on the operator N in the first theorem is quite general, hence not only some estimates are needed but a topological assumption on the nontriviality of the Brouwer degree of $I - N$ on a set of constant functions as well. In the second theorem, the homotopy collapses the nonlinear term N too, and it can be applied to the linear case mentioned above.

As we use homotopy arguments, the main question is to find a priori bounds for solutions of a whole family of problems indexed by $\lambda \in [0,1]$, where, for $\lambda = 1$, we get the problem under consideration and, for $\lambda = 0$, a simpler one. Often, a priori bounds are first obtained for an unknown function and then for its derivative. This is the case for example in the vast literature devoted to lower and upper solutions arguments (see e.g. [4,6]) and its extensions to second order systems (see e.g. [1,6,11,19]). First, an a priori bound on the possible solutions is obtained through a maximum principle and the requested a priori bound on the derivative follows from Nagumo-like conditions. Here, like in other papers, the assumptions provide bounds for derivatives first and next provide a simple estimate for the function x.

In Section 3, special cases of the problem (1.1), where the convex set is a ball or a parallelotope, is studied (see Corollaries 3.1, 3.4 and 3.7). A concrete example is given for Corollary 3.1, and, in Corollaries 3.4 and 3.7, the abstract assumptions upon N are specialized to sign conditions of some inner products. Further applications of Theorem 2.2 to the nonlocal linear boundary conditions are also given (Corollaries 3.10 and 3.11). Corollary 3.10 improves some existence results for a nonresonant problem obtained in [21], where the sign condition was considered on a ball. In [20], the authors deal with a resonant nonlocal problem. Special cases of the main existence theorem were proved there under some monotonicity conditions upon the functions g_i, $i = 1, \ldots, n$. Here, we obtain a new existence result for the nonlocal resonant case (Corollary 3.11).

2   Existence results

Denote by $C([0,1], \mathbb{R}^n)$ the space of all continuous functions $y : [0,1] \to \mathbb{R}^n$ with the usual norm $\| \cdot \|$.

Let us consider the problem (1.1). The following assumptions upon f and N will be needed:

(F) $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function;

(N) $N : C([0,1], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous, not necessarily linear application taking bounded sets into bounded sets.

Since any $x \in C^1([0,1], \mathbb{R}^n)$ such that $x(0) = a$ can be written, with $y = x'$,

$$x(t) = a + \int_0^t y(s)\, ds, \tag{2.1}$$

the equation (1.1) is equivalent to the integro-differential system

$$y'(t) = f\left(t, a + \int_0^t y(s)\, ds, y(t)\right), \quad y(1) = N(y), \tag{2.2}$$

$t \in [0, 1]$.

Observe that solutions to the problem (2.2) are fixed points of the operator $T : C([0, 1], \mathbb{R}^n) \to C([0, 1], \mathbb{R}^n)$ given by

$$T(y)(t) := N(y) - \int_t^1 f\left(s, a + \int_0^s y, y(s)\right) ds. \tag{2.3}$$

It is standard, using the Arzelà–Ascoli theorem, to show that under assumptions (F)–(N) the operator T is completely continuous.

Denote by $\langle \cdot, \cdot \rangle$ the usual inner product in $\mathbb{R}^n$ corresponding to the Euclidean norm $|\cdot|$.

Let $C \subset \mathbb{R}^n$ be an open convex neighborhood of $0 \in \mathbb{R}^n$. Then, applying the Supporting Hyperplane Theorem [5,12], one gets that for each $y_0 \in \partial C$, there exists some $v(y_0) \in \mathbb{R}^n \setminus \{0\}$ such that $\langle v(y_0), y_0 \rangle > 0$ and $C \subset \{y \in \mathbb{R}^n : \langle v(y_0), y - y_0 \rangle < 0\}$. The vector $v(y_0)$ is called an outer normal to ∂C at y_0 and is orthogonal to a supporting hyperplane of C at y_0. Moreover, we have

$$\overline{C} \subset \{y \in \mathbb{R}^n : \langle v(y_0), y - y_0 \rangle \leq 0\}.$$

Denote by $B(0, |a|)$ the open ball in $\mathbb{R}^n$ of center 0 and radius $|a|$. For $a = 0$, $B(0, 0) = \overline{B}(0, 0) := \{0\}$.

Theorem 2.1. *Let the assumptions (F) and (N) be satisfied. Moreover, assume that there is an open, bounded, convex neighborhood C of $0 \in \mathbb{R}^n$ such that $\overline{B}(0, |a|) \subset C$ and the following conditions hold:*

(A) for every $y \in \partial C$ there is an outer normal $v(y)$ to ∂C at y such that

$$\langle v(y), f(t, x, y) \rangle \geq 0, \tag{2.4}$$

for all $t \in [0, 1]$ and $x - a \in \overline{C}$;

(B) for every $y \in C([0, 1], \mathbb{R}^n)$ such that $y(t) \in \overline{C}$ for each $t \in [0, 1]$ and $y(1) \in \partial C$, we have

$$y(1) \neq N(y);$$

(C) for the Brouwer degree of the map $I - N$ restricted to constant functions on the set C at the point 0, the following condition holds

$$\deg_B(I - N|_{\text{const}}, C, 0) \neq 0.$$

Then the problem (1.1) has a solution x such that $x(t) - a \in \overline{C}$ and $x'(t) \in \overline{C}$ for all $t \in [0, 1]$.

Proof. Let us consider a homotopy $H : [0, 1] \times C([0, 1], \mathbb{R}^n) \to C([0, 1], \mathbb{R}^n)$ given by

$$H(\lambda, y)(t) := y(t) - T_\lambda(y)(t),$$

where

$$T_\lambda(y)(t) = N(y) - \lambda \int_t^1 \left[f\left(s, a + \int_0^s y, y(s)\right) + (1 - \lambda)y(s) \right] ds, \tag{2.5}$$

in the open bounded set

$$\Omega = \{y \in C([0,1], \mathbb{R}^n) : y(t) \in C, \ \forall \, t \in [0,1]\}.$$

Observe that $T_1 = T$ and that the fixed points of (2.5) are solutions to the problem

$$y'(t) = \lambda f \left(t, a + \int_0^t y(s) \, ds, y(t) \right) + \lambda(1 - \lambda)y(t), \quad y(1) = N(y). \tag{2.6}$$

We shall show that the homotopy does not vanish on the boundary of Ω for $\lambda \in [0,1)$. First, notice that if $y \in \partial\Omega$, then $y(t) \in \overline{C}$ for all $t \in [0,1]$ and there is some $t_0 \in [0,1]$ such that $y(t_0) \in \partial C$ and, by (2.1), $x'(t) = y(t) \in \overline{C}$. Consequently, as

$$x(t) - a = \int_0^1 u(s) \, ds \quad \text{with} \quad u(s) = \begin{cases} x'(s) & \text{for } s \in [0,t] \\ 0 & \text{for } s \in (t,1] \end{cases}$$

such that $u(s) \in \overline{C}$ for all $s \in [0,1]$, $x(t) - a$ is the limit of convex combination of points $x'(\xi_i) \in \overline{C}$ and of $0 \in C$, and hence $x(t) - a \in \overline{C}$ for all $t \in [0,1]$.

By the assumption (B), $H(0,y) \neq 0$ for $y \in \partial\Omega$, since in this case $y(t) = N(y)$ for each $t \in [0,1]$. Now, suppose that there exists $\lambda \in (0,1)$ and $y \in \partial\Omega$ such that $y = T_\lambda(y)$.

Assume that $y(t_0) \in \partial C$ with $t_0 \in [0,1)$ and define

$$\varphi(t) := \langle \nu(y(t_0)), y(t) - y(t_0) \rangle.$$

Observe that $\varphi(t) \leq 0$ for $t \in [0,1]$, since $y(t) \in \overline{C}$ for each $t \in [0,1]$, and φ reaches its maximum 0 at t_0. By (2.6) and the assumption (A), we reach a contradiction with

$$0 \geq \varphi'(t_0) = \langle \nu(y(t_0)), y'(t_0) \rangle$$
$$= \lambda \langle \nu(y(t_0)), f(t_0, x(t_0), y(t_0)) \rangle + \lambda(1 - \lambda) \langle \nu(y(t_0)), y(t_0) \rangle > 0.$$

By the above, it remains to exclude only functions y such that $y(t) \in C$ for $t \in [0,1)$ and $y(1) \in \partial C$. In this case, since y is a solution to (2.6), we reach a contradiction with the assumption (B). Finally, if $H(1,y) = 0$ for some $y \in \partial\Omega$, the result is proved. If $H(1,y) \neq 0$ for all $y \in \partial\Omega$, it follows from the above reasoning that $H(\lambda, y) \neq 0$ for all $(\lambda, y) \in [0,1] \times \partial\Omega$, and hence, by the homotopy invariance of the Leray–Schauder degree

$$\deg_{LS}(I - T, \Omega, 0) = \deg_{LS}(I - N, \Omega, 0). \tag{2.7}$$

But, as N sends $C([0,1], \mathbb{R}^n)$ to its subspace of constant mappings isomorphic to $\mathbb{R}^n$, we have

$$\deg_{LS}(I - N, \Omega, 0) = \deg_B(I - N|_{\text{const}}, C, 0),$$

where in the second term we have the Brouwer degree of a map from $\mathbb{R}^n$ into $\mathbb{R}^n$.

By the assumptions (B), (C) and the existence property of degrees, T has a fixed point y in Ω. Furthermore, the corresponding functions (2.1) are solutions to the problem (1.1). $\square$

Theorem 2.2. *Let the assumptions (F), (N) and (A) hold. Moreover, assume that there is an open, bounded, convex neighborhood C of $0 \in \mathbb{R}^n$ such that $\overline{B}(0, |a|) \subset C$ and the following condition is fulfilled*

(B') *for every $\lambda \in [0,1]$ and $y \in C([0,1], \mathbb{R}^n)$ such that $y(t) \in C$ for each $t \in [0,1)$ and $y(1) \in \partial C$, one has*

$$y(1) \neq \lambda N(y).$$

Then the problem (1.1) has a solution x such that such that $x(t) - a \in \overline{C}$ and $x'(t) \in \overline{C}$ for all $t \in [0, 1]$.

Proof. Define a homotopy $H : [0, 1] \times C([0, 1], \mathbb{R}^n) \to C([0, 1], \mathbb{R}^n)$ by

$$H(\lambda, y)(t) := y(t) - \lambda T(y)(t) + \lambda(1 - \lambda) \int_t^1 y(s) \, ds,$$

with T given in (2.3). Now, using the homotopy and proceeding in the same way as in the proof of Theorem 2.1, we obtain that either $I - T$ has a zero in $\partial \Omega$, and the result is proved, or that

$$\deg_{LS}(I - T, \Omega, 0) = \deg_{LS}(I, \Omega, 0) \neq 0. \qquad \square$$

3   Special cases and examples

Let $C := B(0, M)$ be the open ball in $\mathbb{R}^n$ of center 0 and radius $M > |a|$. Taking $v(y) = y$, for each $y \in \partial B(0, M)$, and applying Theorem 2.1, we obtain immediately the following existence result.

Corollary 3.1. *Let the assumptions (F) and (N) hold. Moreover, assume that the following conditions are fulfilled.*

(A1) there exists $M > |a|$ such that

$$\langle y, f(t, x, y) \rangle \geq 0,$$

for all $t \in [0, 1]$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ with $|x - a| \leq M$ and $|y| = M$;

(B1) for every $y \in C([0, 1], \mathbb{R}^n)$ such that $|y(t)| < M$ for every $t \in [0, 1)$ and $|y(1)| = M$, one has

$$y(1) \neq N(y);$$

(C1) for the Brouwer degree of the map $I - N$ restricted to constant functions on the set $B(0, M)$ at the point 0, the following condition holds

$$\deg_B(I - N|_{\text{const}}, B(0, M), 0) \neq 0.$$

Then the problem (1.1) has a solution x such that $\|x\| \leq |a| + M$ and $\|x'\| \leq M$.

Example 3.2. Let us identify $\mathbb{R}^2$ with $\mathbb{C}$, use complex notation with z instead of x and consider the boundary value problem for the Rayleigh-type system

$$z'' = \frac{z}{1 + |z|} + \varphi_p(z') + e(t), \qquad z(0) = a, \qquad z'(1)^2 - \int_0^1 z'(s) \, dg(s) = b, \qquad (3.1)$$

where $\varphi_p(y) = |y|^{p-2} y$ for $y \neq 0$, $\varphi_p(0) = 0$, $p > 1$, $a, b \in \mathbb{C}$, $e \in C([0, 1], \mathbb{C})$ and $g : [0, 1] \to \mathbb{R}$ is of bounded variation. This is a special case of problem (1.1) with $n = 2$,

$$f(t, z, y) = \frac{z}{1 + |z|} + \varphi_p(y) + e(t),$$

and $N : C([0, 1], \mathbb{C}) \to \mathbb{C}$ defined by

$$N(y) = y(1) - y(1)^2 + \int_0^1 y(s) \, dg(s) + b.$$

The assumptions (F) and (N) of Corollary 3.1 are trivially satisfied. Furthermore, for $M > |a|$ to be fixed and $\langle u, v \rangle = \Re(u\bar{v})$ the inner product in $\mathbb{C} \simeq \mathbb{R}^2$, we have, when $z \in \mathbb{C}$, $|y| = M$ and $t \in \mathbb{R}$,

$$\left\langle \frac{z}{1+|z|} + \varphi_p(y) + e(t), y \right\rangle = \frac{\langle z, y \rangle}{1+|z|} + |y|^p + \langle e(t), y \rangle$$
$$\geq |y|^p - (\|e\| + 1)|y| = |y| \left[|y|^{p-1} - (\|e\| + 1) \right] \geq 0,$$

if $M > (\|e\| + 1)^{\frac{1}{p-1}}$.

On the other hand, if $y \in C([0,1], \mathbb{C})$ is such that $|y(t)| \leq M$ for all $t \in [0,1]$ and $|y(1)| = M$, then

$$\left| \int_0^1 y(s)\, dg(s) + b \right| \leq \int_0^1 |y(s)|\, d|g|(s) + |b| \leq M \operatorname{Var}(g) + |b| < M^2, \tag{3.2}$$

if $M > M_0(|b|, \operatorname{Var}(g))$ where $M_0(|b|, \operatorname{Var}(g))$ denotes the unique positive root of equation

$$r^2 - (\operatorname{Var}(g))r - |b| = 0,$$

where we apply Jordan's decomposition of function g as the difference of two nondecreasing functions $g = g_1 - g_2$ and the integral with respect to $|g|$ is the same as w.r.t. $g_1 + g_2$. Consequently, for $M > \max\left\{ |a|, (\|e\| + 1)^{\frac{1}{p-1}}, M_0(|b|, \operatorname{Var}(g)) \right\}$ and C the open ball of center 0 and radius M, both assumptions $(A1)$ and $(B1)$ of Corollary 3.1 are satisfied.

Finally, if $h : \mathbb{C} \to \mathbb{C}$ is defined by $h(w) = w^2 - (g(1) - g(0))w - b$, then, by standard properties of Leray–Schauder and Brouwer degrees,

$$\deg_B(I - N|_{\text{const}}, C, 0) = \deg_B(h, B(0, M), 0) = 2.$$

Hence assumption $(C1)$ is satisfied, and problem (3.1) has at least one solution.

Remark 3.3. Example 3.2 corresponds to a problem whose equivalent fixed point formulation has Leray–Schauder degree equal to 2. This shows that Theorem 2.1 deals with situations distinct from those covered by the existence results in [20,21], which correspond to fixed point problems having Leray–Schauder degree equal to 1. When N is linear, the assumptions of Theorem 2.1 imply that the problem is non-resonant.

The following result is a special case of Theorem 2.2.

Corollary 3.4. *Let the assumptions (F), (N) and (A1) hold. Moreover, assume that the following condition is fulfilled*

$(B'1)$ *for every $y \in C([0,1], \mathbb{R}^n)$ such that $|y(t)| < M$ for all $t \in [0,1)$ and $|y(1)| = M$, one has*

$$\langle y(1), N(y) \rangle \leq 0.$$

Then the problem (1.1) *has at least one solution x such that $\|x\| \leq |a| + M$ and $\|x'\| \leq M$.*

Proof. Observe that, by the assumption $(B'1)$, we have, for every $y \in C([0,1], \mathbb{R}^n)$ such that $|y(t)| < M$ for all $t \in [0,1)$ and $|y(1)| = M$, and every $\lambda \in [0,1]$,

$$\langle y(1) - \lambda N(y), y(1) \rangle = |y(1)|^2 - \lambda \langle N(y), y(1) \rangle \geq |y(1)|^2 = M^2 > 0,$$

so that Assumption (B') of Theorem 2.2 is satisfied. $\square$

Example 3.5. Let us consider the problem (1.1) with $N : C([0,1], \mathbb{R}^n) \to \mathbb{R}^n$ given by

$$N(y) := S_1(y(1)) + S_2(y(\eta_1), \ldots, y(\eta_m)), \tag{3.3}$$

where $S_1 : \mathbb{R}^n \to \mathbb{R}^n$, $S_2 : \mathbb{R}^{mn} \to \mathbb{R}^n$ are continuous and $\eta_1, \ldots, \eta_m \in [0,1]$. Assume that there is $M > |a|$ such that the condition (A1) holds,

$$\langle y_0, S_1(y_0) \rangle < 0,$$

for $|y_0| = M$ and set

$$L := \max_{|y_0| = M} \langle y_0, S_1(y_0) \rangle.$$

Moreover, assume that

$$|S_2(y_1, \ldots, y_m)| < -L/M,$$

for any $|y_1|, \ldots, |y_m| \leq M$.

It is easy to observe that the assumption (B'1) is satisfied. Consequently, the problem (1.1) with N defined in (3.3) has at least one solution.

Example 3.6. Consider the problem (1.1) for which the assumption (A1) is fulfilled. Define

$$N(y) := S_1(y(1)) + S_2 \left(\int_0^1 y(s) \, dg(s) \right), \tag{3.4}$$

where $g : [0,1] \to \mathbb{R}^n$, $g = (g_1, \ldots, g_n)$ with $g_i : [0,1] \to \mathbb{R}$, i.e.,

$$\int_0^1 y(s) dg(s) = \left(\int_0^1 y_1(s) dg_1(s), \ldots, \int_0^1 y_n(s) dg_n(s) \right),$$

and the variation of g on the interval $[0,1]$ verifies

$$\text{Var}(g) := \left[\sum_{i=1}^n \left(\int_0^1 d|g_i| \right)^2 \right]^{\frac{1}{2}} = \left\{ \sum_{i=1}^n [\text{Var}(g_i)]^2 \right\}^{\frac{1}{2}} \leq 1. \tag{3.5}$$

Moreover, let $S_1, S_2 : \mathbb{R}^n \to \mathbb{R}^n$ be continuous, S_1 satisfy the assumption from Example 3.5 and

$$|S_2(y_1)| < -L/M,$$

when $|y_1| \leq M$.

By (3.5) and the Cauchy–Schwarz inequality, for $|y(t)| \leq M$, $t \in [0,1]$, we obtain the following estimates

$$\left| \int_0^1 y(s) \, dg(s) \right|^2 = \sum_{i=1}^n \left(\int_0^1 y_i(s) \, dg_i(s) \right)^2 \leq \sum_{i=1}^n \left(\int_0^1 |y_i(s)| \, d|g_i|(s) \right)^2$$

$$\leq \sum_{i=1}^n \left[\left(\int_0^1 |y_i(s)|^2 \, d|g_i|(s) \right) \left(\int_0^1 d|g_i|(s) \right) \right]$$

$$\leq M^2 \sum_{i=1}^n \left(\int_0^1 d|g_i|(s) \right)^2 = M^2 [\text{Var}(g)]^2 \leq M^2.$$

Now, one can easily check that the assumption (B'1) holds. Consequently, the problem (1.1) with N defined in (3.4) has at least one solution.

We now consider situations where the convex set C is a product of intervals. Set $C = \prod_{i=1}^{n}(-M_i, M_i)$ for some $M_i > |a_i|$. Then, for each $y \in \partial C$, one can take $v(y) = y_i e_i$, where only i^{th} coordinate is such that $|y_i| = M_i$. If y belongs to more that one faces of C then i can be chosen arbitrarily among all j such that $|y_j| = M_j$. Here e_i is the ith element of the canonical basis of $\mathbb{R}^n$ ($i = 1, \ldots, n$). The following result follows from Theorem 2.2.

Corollary 3.7. *Let the assumptions (F) and (N) hold. Moreover, assume that the following conditions are fulfilled.*

(A2) *there exist $M_j > |a_j|$, $j = 1, \ldots, n$, such that, for every $t \in [0,1]$ and $i = 1, \ldots, n$, if $|y_i| = M_i$, $|y_j| < M_j$ for $j = 1, \ldots, n$ and $j \neq i$, and $x_i - a_i \in [-M_i, M_i]$ for any $i = 1, \ldots, n$, then we have*

$$y_i f_i(t, x, y) \geq 0; \tag{3.6}$$

(B2) *for every $y \in C([0,1], \mathbb{R}^n)$ such that $|y_i(t)| < M_i$ for $t \in [0,1)$ and $|y_i(1)| = M_i$, and such that $|y_j(t)| < M_j$ for each $j \neq i$ and all $t \in [0,1]$, we have*

$$y_i(1) N_i(y) \leq 0.$$

Then the problem (1.1) has at least one solution x such that $|x_i(t)| \leq |a_i| + M_i$ and $|x_i'(t)| \leq M_i$, where $t \in [0,1]$ and $i = 1, \ldots, n$.

Remark 3.8. Observe that the condition (3.6) is set only for y belonging to "open" faces of the cube. For points belonging to more than one face, the inequalities are fulfilled for all indices numerating these faces by continuity of f. Similar remark for (B2), using the continuity of N.

Example 3.9. Let $n = 2$. Consider the problem (1.1) with

$$f_1(t, x_1, x_2, y_1, y_2) := \frac{x_2}{2} + \sin \frac{\pi y_1}{2} + \frac{t}{2}, \qquad f_2(t, x_1, x_2, y_1, y_2) := \frac{x_1}{2} + \sin \frac{\pi y_2}{2} + \frac{t}{2},$$

and the following nonlinear boundary conditions

$$x_1(0) = 0, \qquad x_2(0) = 0, \qquad x_1'(1) = -x_1'^3(1) + \alpha_1 x_1'(\eta_1), \qquad x_2'(1) = -x_1'^5(1) + \alpha_2 x_2'(\eta_2),$$

where $|\alpha_j| \leq 1$, $\eta_j \in [0,1]$ ($j = 1, 2$). Let $C = (-1, 1) \times (-1, 1)$. Setting

$$v(y_1, y_2) = \begin{cases} (1,0) & \text{if } (y_1, y_2) \in \{1\} \times [-1,1], \\ (0,1) & \text{if } (y_1, y_2) \in (-1,1) \times \{1\}, \\ (-1,0) & \text{if } (y_1, y_2) \in \{-1\} \times [-1,1], \\ (0,-1) & \text{if } (y_1, y_2) \in (-1,1) \times \{-1\}, \end{cases}$$

one can easily check that Corollary 3.7 implies the existence of a solution to the problem (1.1) with $a = 0$.

Now, let us consider the problem (1.1) with linear boundary condition, i.e.

$$N(y) := \int_0^1 y(s) \, dg(s), \tag{3.7}$$

where $g = (g_1, \ldots, g_n)$ and $g_i : [0, 1] \to \mathbb{R}$, $i = 1, \ldots, n$. Then the problem (1.1) takes the form

$$x'' = f(t, x, x'), \qquad x(0) = a, \qquad x'(1) = \int_0^1 x'(s) dg(s). \tag{3.8}$$

Similar problems have been considered under different assumptions in [13].

The following assumptions upon the function g are introduced alternatively:

(G1) $\text{Var}(g) < 1$, where $\text{Var}(g)$ is defined in (3.5);

(G2) $\text{Var}(g_i) \leq 1$, $i = 1, \ldots, n$, and, if $\text{Var}(g_i) = 1$ for each $i = 1, \ldots, n$, then there is $i_0 \in \{1, \ldots, n\}$ such that g_{i_0} is not constant on $[0, 1)$.

Corollary 3.10. *Let the assumptions (F), (A1) and (G1) hold. Then the problem* (3.8) *with $C = B(0, M)$ has at least one solution.*

Proof. Let $\lambda \in [0, 1]$, $y \in C([0, 1], \mathbb{R}^n)$ be such that $y(t) \in \overline{B}(0, M)$ for each $t \in [0, 1)$ and $y(1) \in \partial B(0, M)$. By the assumption (G1) and the Cauchy–Schwarz inequality, one gets

$$M^2 = |y(1)|^2 = \lambda^2 N(y)|^2 \leq |N(y)|^2 = \sum_{i=1}^n \left(\int_0^1 y_i(s) \, dg_i(s) \right)^2 \leq M^2 [\text{Var}(g)]^2 < M^2,$$

a contradiction. Consequently, the conclusion (B') of Theorem 2.2 with $C = B(0, M)$ is satisfied, and the result follows. □

Corollary 3.11. *Let the assumptions (F), (A2) and (G2) be fulfilled. Then the problem* (3.8) *with $C = \prod_{j=1}^n (-M_j, M_j)$ has at least one solution.*

Proof. Let $\lambda \in [0, 1]$, $y \in C([0, 1], \mathbb{R}^n)$ be such that $y(t) \in \prod_{i=j}^n (-M_j, M_j)$ for each $t \in [0, 1)$ and $y(1) \in \partial \prod_{i=j}^n (-M_j, M_j)$. Then $|y_i(1)| = 1$ for some $i \in \{1, \ldots, n\}$. If $\text{Var}(g_i) < 1$, then

$$\lambda |N_i(y)| \leq |N_i(y)| = \left| \int_0^1 y_i(s) \, dg_i(s) \right| \leq \int_0^1 |y_i(s)| \, d|g_i|(s)$$
$$\leq |y_i(1)| \text{Var}(g_i) < |y_i(1)|,$$

so that Assumption (B2) holds. If $\text{Var}(g_i) = 1$, then by definition of the Riemann–Stieltjes integral, we obtain

$$\lambda |N_i(y)| \leq |N_i(y)| \leq \sup \sum_j |y_i(s_j)| |g_i(t_j) - g_i(t_{j-1})|$$
$$< M_i \cdot \sup \sum_j |g_i(t_j) - g_i(t_{j-1})| \leq M_i,$$

where supremum is taken over all subdivisions $0 = t_0 < t_1 < \cdots < t_n = 1$ and $s_j \in [t_{j-1}, t_j]$, $j = 1, \ldots, n$. The third inequality is sharp for any function y with values in the open set C for $t < 1$ since at least one summand does not vanish for each subdivision. Consequently, the assumptions of Theorem 2.2 with $C = \prod_{j=1}^n (-M_j, M_j)$ are fulfilled. □

Remark 3.12. Corollary 3.11 slightly generalizes Theorem 3.1 from [20], since here we do not assume that $\int_0^1 e^t \, dg(t) \neq e$; this is the additional assumptions in [20].

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