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Constant sign and nodal solutions for nonhomogeneous Robin boundary value problems with asymmetric reactions

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Abstract. We study a nonlinear, nonhomogeneous elliptic equation with an asymmetric reaction term depending on a positive parameter, coupled with Robin boundary conditions. Under appropriate hypotheses on both the leading differential operator and the reaction, we prove that, if the parameter is small enough, the problem admits at least four nontrivial solutions: two of such solutions are positive, one is negative, and one is sign-changing. Our approach is variational, based on critical point theory, Morse theory, and truncation techniques.

Keywords: non-linear elliptic equations, asymmetric reactions, variational methods.

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1 Introduction

We study the following nonlinear, nonhomogeneous Robin problem:

$$\begin{cases} -\operatorname{div} a(\nabla u) + \xi(x)|u|^{p-2}u = \lambda g(x,u) + f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_a} + \beta(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded domain with a C^2-boundary $\partial\Omega$, $p > 1$, and $a : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous, monotone mapping (hence maximal monotone too) which satisfies certain growth and regularity conditions (see hypotheses $\mathbf{H}_a$ below). These conditions are mild enough to include in our framework many non-linear operators of interest, such as the p-Laplacian, the (p,q)-Laplacian, and the generalized mean curvature operator. The potential function $\xi \in L^\infty(\Omega)$ is indefinite (i.e., sign-changing, see hypothesis $\mathbf{H}_\xi$).

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On the right-hand side, $\lambda > 0$ is a parameter and $g, f : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. We assume that for a.a. $x \in \Omega$ the mapping $g(x, \cdot)$ is strictly $(p-1)$-sublinear at $\pm\infty$ (concave nonlinearity), while $f(x, \cdot)$ exhibits an asymmetric behavior, being $(p-1)$-superlinear at $+\infty$ and asymptotically $(p-1)$-linear at $-\infty$ (see hypotheses $\mathbf{H}_g$, $\mathbf{H}_f$ below). So, in the positive semiaxis we have a competition phenomenon between a concave and a convex nonlinearity, while in the negative semiaxis and in the particular case of the p-Laplacian the equation may be resonant with respect to the first eigenvalue.

In the boundary condition, $\partial u/\partial n_a$ denotes the generalized normal derivative corresponding to the mapping a, namely the extension of

$$u \mapsto \langle a(\nabla u), n \rangle, \qquad u \in C^1(\overline{\Omega})$$

to $W^{1,p}(\Omega)$ (n denotes the outward unit normal to $\partial\Omega$). The boundary coefficient $\beta \in C^{0,\alpha}(\partial\Omega)$ is non-negative, and the special case $\beta = 0$ corresponds to the Neumann problem (see hypothesis $\mathbf{H}_\beta$ below).

In this paper, using variational methods based on the critical point theory, together with suitable truncation/perturbation techniques and Morse theory, we prove that, for $\lambda > 0$ small enough, problem (1.1) has at least four nontrivial solutions: two positive, one negative, and one nodal (see Theorem 2.4 below and the ensuing discussion for a short account on our method).

Recently, elliptic boundary value problems with asymmetric reactions were studied in [15, 28, 29] (semilinear Dirichlet problems with zero potential), [18] (semilinear Neumann problem with indefinite potential), [3, 14, 23, 24] (semilinear Robin problems with indefinite potential). For nonlinear elliptic equations we mention [13, 16] (Dirichlet problems driven by the p-Laplacian), [20, 26] (Dirichlet problems driven by the $(p, 2)$-Laplacian).

Compared with the existing literature, our result is novel in a twofold sense: unlike most of the aforementioned works, our result proves existence of four nontrivial solutions with precise sign information; and it holds for a very general problem, incorporating Robin and Neumann boundary conditions and several nonlinear leading differential operators as special cases (the only exception is represented by [16], which provides four solutions but only for Dirichlet conditions and the p-Laplace operator).

For the sake of completeness, we mention some more results on nonlinear Robin problems (with symmetric reactions) contained in [19, 21, 25].

The paper has the following structure: in Section 2 we introduce our hypotheses and main result, and we also establish some preliminary results and notations; in Section 3 we deal with constant sign solutions; and in Section 4 we investigate extremal constant sign solutions and nodal solutions.

2 Hypotheses and main result

We start this section by introducing and commenting the precise hypotheses on all features of problem (1.1). We begin with the mapping a:

$\mathbf{H}_a$ $a : \mathbb{R}^N \to \mathbb{R}^N$ is defined by $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0 : \mathbb{R}_+ \to \mathbb{R}_+$, and we set for all $t \geqslant 0$

$$H_0(t) = \int_0^t a_0(\tau)\tau \, d\tau$$

and for all $y \in \mathbb{R}^N$ $H(y) = H_0(|y|)$. Moreover:

(i) $a_0 \in C^1(0,+\infty)$, $a_0(t) > 0$ for all $t > 0$, $t \mapsto a_0(t)t$ is strictly increasing on $(0,+\infty)$, and

$$\lim_{t\to 0^+} a_0(t)t = 0, \qquad \lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1;$$

(ii) there exists $\theta \in C^1(0,+\infty)$ s.t. for all $t > 0$

$$c_0 \leqslant \frac{\theta'(t)t}{\theta(t)} \leqslant c_1,\ c_2 t^{p-1} \leqslant \theta(t) \leqslant c_3(t^{\sigma-1} + t^{p-1}) \qquad (c_0, c_1, c_2, c_3 > 0,\ 1 \leqslant \sigma < p),$$

and for all $y \in \mathbb{R}^N \setminus \{0\}$

$$|\nabla a(y)| \leqslant c_4 \frac{\theta(|y|)}{|y|} \qquad (c_4 > 0);$$

(iii) for all $y, z \in \mathbb{R}^N$, $y \neq 0$

$$\langle \nabla a(y)z, z\rangle \geqslant \frac{\theta(|y|)|z|^2}{|y|};$$

(iv) there exists $r \in (1, p]$ s.t. $t \mapsto H_0(t^{\frac{1}{r}})$ is convex,

$$\limsup_{t\to 0^+} \frac{rH_0(t)}{t^r} \leqslant c_5 \qquad (c_5 > 0),$$

and for all $t \geqslant 0$

$$pH_0(t) - a_0(t)t^2 \geqslant -c_6 \qquad (c_6 > 0).$$

Hypotheses $\mathbf{H}_a$ (i)–(iii) are dictated by the nonlinear regularity theory of [12] and the nonlinear maximum principle of [27]. Hypothesis $\mathbf{H}_a$ (iv) serves the needs of our problem but is general enough to include several cases of interest (see Example 2.2 below). As a whole, $\mathbf{H}_a$ implies that H_0 is strictly convex and increasing on $\mathbb{R}_+$, and H is convex with $H(0) = 0$, $\nabla H(0) = 0$, and $\nabla H(y) = a(y)$ for all $y \in \mathbb{R}^N \setminus \{0\}$, i.e., H is the primitive of a. This, along with convexity, clearly implies for all $y \in \mathbb{R}^N$

$$H(y) \leqslant \langle a(y), y\rangle. \tag{2.1}$$

Hypotheses $\mathbf{H}_a$ (i)–(iii) and (2.1) lead to the following properties of a and H.

Lemma 2.1. *If $\mathbf{H}_a$ (i)–(iii) hold, then*

(i) *$a : \mathbb{R}^N \to \mathbb{R}^N$ is continuous and monotone (hence maximal monotone);*

(ii) *$|a(y)| \leqslant c_7(1 + |y|^{p-1})$ for all $y \in \mathbb{R}^N$ ($c_7 > 0$);*

(iii) *$\langle a(y), y\rangle \geqslant \frac{c_2|y|^p}{p-1}$ for all $y \in \mathbb{R}^N$;*

(iv) *$\frac{c_2|y|^p}{p(p-1)} \leqslant H(y) \leqslant c_8(1 + |y|^p)$ for all $y \in \mathbb{R}^N$ ($c_8 > 0$).*

In what follows we shall denote $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ the nonlinear differential operator defined for all $u, v \in W^{1,p}(\Omega)$ by

$$\langle A(u), v\rangle = \int_\Omega \langle a(\nabla u), \nabla v\rangle\, dx,$$

which is well defined by virtue of $\mathbf{H}_a$ *(ii)*. Such operator enjoys the $(S)_+$-property, i.e., whenever (u_n) is a sequence in $W^{1,p}(\Omega)$ s.t. $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and

$$\limsup_n \langle A(u_n), u_n - u \rangle \leqslant 0,$$

then $u_n \to u$ in $W^{1,p}(\Omega)$ (see [17, p. 405]). Here follow some examples.

Example 2.2. The following maps $a : \mathbb{R}^N \to \mathbb{R}^N$ satisfy $\mathbf{H}_a$:

(a) $a(y) = |y|^{p-2}y$, corresponding to the p-Laplace operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u);$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ $(1 < q < p < +\infty)$, corresponding to the (p,q)-Laplace operator

$$\Delta_p u + \Delta_q u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\big).$$

Such operators arise in problems of mathematical physics, see [2] (reaction-diffusion equations), [4] (elementary particles), [30] (plasma physics). Further:

(c) $a(y) = (1+|y|^2)^{\frac{p-2}{2}}y$, corresponding to the generalized p-mean curvature operator

$$\operatorname{div}\big((1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\big);$$

(d) $a(y) = \big(2\ln(1+|y|^p) + (1+|y|^p)^{-1}\big)y$, corresponding to the operator

$$\operatorname{div}\left(2\ln(1+|\nabla u|^p)\nabla u + \frac{\nabla u}{1+|\nabla u|^p}\right);$$

(e) $a(y) = |y|^{p-2}y + |y|^{p-2}(1+|y|^p)^{-1}y$, corresponding to the operator

$$\Delta_p u + \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{1+|\nabla u|^p}\right).$$

Such operators arise in problems of nonlinear elasticity [7] and plasticity.

The other ingredients of (1.1) are subject to the following hypotheses:

$\mathbf{H}_\xi$ $\xi \in L^\infty(\Omega)$;

$\mathbf{H}_\beta$ $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$, $\beta(x) \geqslant 0$ for all $x \in \partial\Omega$.

We note that the potential ξ may change sign, and that for $\beta = 0$ we recover the Neumann problem. Finally we introduce our hypotheses on the reactions, starting with g:

$\mathbf{H}_g$ $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, for all $(x,t) \in \Omega \times \mathbb{R}$ we set

$$G(x,t) = \int_0^t g(x,\tau)\,d\tau.$$

Moreover:

(i) for all $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ s.t. for a.a. $x \in \Omega$, all $|t| \leqslant \rho$

$$|g(x,t)| \leqslant a_\rho(x);$$

(ii) $\lim_{|t|\to+\infty} \frac{g(x,t)}{|t|^{p-2}t} = 0$ uniformly for a.a. $x \in \Omega$;

(iii) there exists $q \in (1, r)$ s.t. for a.a. $x \in \Omega$, all $t \in \mathbb{R}$

$$g(x,t)t \geqslant c_9|t|^q \qquad (c_9 > 0);$$

(iv) $\limsup_{t\to 0} \frac{g(x,t)}{|t|^{q-2}t} \leqslant c_{10}$ uniformly for a.a. $x \in \Omega$ ($c_{10} > 0$);

(v) there exists $\delta_0 > 0$ s.t. for a.a. $x \in \Omega$, all $|t| \leqslant \delta_0$

$$g(x,t)t \leqslant qG(x,t).$$

We set $\xi_0 = (p-1)\xi/c_2$, $\beta_0 = (p-1)\beta/c_2$ ($c_2 > 0$ as in $\mathbf{H}_a$ (ii)) and we denote by $\hat{\lambda}_1 = \hat{\lambda}_1(p, \xi_0, \beta_0) > 0$ the first eigenvalue of the auxiliary problem

$$\begin{cases} -\Delta_p u + \xi_0(x)|u|^{p-2}u = \lambda|u|^{p-2}u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} + \beta_0(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

where

$$\frac{\partial u}{\partial n_p} = \langle |\nabla u|^{p-2}\nabla u, n\rangle$$

(n being as usual the outward unit normal to $\partial\Omega$). Now we consider the asymmetric term f:

$\mathbf{H}_f$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, for all $(x,t) \in \Omega \times \mathbb{R}$ we set

$$F(x,t) = \int_0^t f(x,\tau)\, d\tau.$$

Moreover:

(i) for all $\rho > 0$ there exists $b_\rho \in L^\infty(\Omega)_+$ s.t. for a.a. $x \in \Omega$, all $|t| \leqslant \rho$

$$|f(x,t)| \leqslant b_\rho(x);$$

(ii) $\lim_{t\to+\infty} \frac{f(x,t)}{t^{p^*-1}} = 0$ uniformly for a.a. $x \in \Omega$;

(iii) $\lim_{t\to+\infty} \frac{f(x,t)}{t^{p-1}} = +\infty$ uniformly for a.a. $x \in \Omega$;

(iv) $f(x,t) \leqslant c_{11}(t^{p^*-1} + t^{r-1}) - c_{12}t^{p-1}$ for a.a. $x \in \Omega$, all $t \geqslant 0$ ($c_{11}, c_{12} > 0$);

(v) uniformly for a.a. $x \in \Omega$

$$-c_{13} \leqslant \liminf_{t\to-\infty} \frac{f(x,t)}{|t|^{p-2}t} \leqslant \limsup_{t\to-\infty} \frac{f(x,t)}{|t|^{p-2}t} \leqslant \frac{c_2\hat{\lambda}_1}{p-1} \qquad (c_{13} > 0);$$

(vi) $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$ uniformly for a.a. $x \in \Omega$;

(vii) there exists $\delta_1 > 0$ s.t. for a.a. $x \in \Omega$, all $|t| \leqslant \delta_1$

$$f(x,t)t \geqslant 0.$$

Finally, we set for all $\lambda > 0$ and all $(x, t) \in \Omega \times \mathbb{R}$

$$e_\lambda(x,t) = \lambda g(x,t)t + f(x,t)t - p\big[\lambda G(x,t) + F(x,t)\big]$$

and we assume the following condition:

$\mathbf{H}_e$ for all $\lambda > 0$

(i) there exists $\eta_\lambda \in L^1(\Omega)_+$ s.t. for a.a. $x \in \Omega$, all $0 \leqslant t \leqslant t'$

$$e_\lambda(x,t) \leqslant e_\lambda(x,t') + \eta_\lambda(x);$$

(ii) $\lim\limits_{t\to-\infty} e_\lambda(x,t) = +\infty$ uniformly for a.a. $x \in \Omega$.

We will write $\mathbf{H}$ to mean all hypotheses $\mathbf{H}_a$, $\mathbf{H}_\xi$, $\mathbf{H}_\beta$, $\mathbf{H}_g$, $\mathbf{H}_f$, and $\mathbf{H}_e$.

By $\mathbf{H}_g$ (ii) $g(x,\cdot)$ is strictly $(p-1)$-sublinear at $\pm\infty$, so it gives a 'concave' contribution to the reaction of (1.1). Hypotheses $\mathbf{H}_f$ (iii), (v) imply that $f(x,\cdot)$ has an asymmetric behavior at $\pm\infty$. More precisely, $\mathbf{H}_f$ (iii) means that $f(x,\cdot)$ is strictly $(p-1)$-superlinear at $+\infty$, so on $\mathbb{R}_+$ it represents a 'convex' contribution to the reaction, leading to a competition phenomenon (concave-convex nonlinearities). We point out that the $(p-1)$-superlinearity of $f(x,\cdot)$ is not coupled with the usual Ambrosetti–Rabinowitz (AR) condition. Instead we use the less restrictive quasimonotonicity condition $\mathbf{H}_e$ (i), which includes in our framework $(p-1)$-superlinear reactions with a slower growth at $+\infty$, that fail to satisfy (AR). Note that $\mathbf{H}_e$ (i) holds whenever we can find $\rho > 0$ s.t. the mapping

$$t \mapsto \frac{\lambda g(x,t) + f(x,t)}{t^{p-1}}$$

is nondecreasing in $[\rho, +\infty)$ for a.a. $x \in \Omega$ [11]. On the negative semiaxis $\mathbb{R}_-$, by $\mathbf{H}_f$ (v) the mapping $f(x,\cdot)$ is asymptotically $(p-1)$-linear at $-\infty$, and in the special case of the p-Laplacian (Example 2.2 (a) with $c_2 = p-1$) resonance with the principal eigenvalue is allowed. Resonance occurs from the left, so by $\mathbf{H}_e$ (ii) problem (1.1) is coercive on the negative direction, which permits the use of the direct method of the calculus of variations. Finally we remark that by $\mathbf{H}_f$ (ii) $f(x,\cdot)$ does not satisfy the usual subcritical growth. Instead we have 'almost-critical' growth, namely for all $\varepsilon > 0$ we can find $c_\varepsilon > 0$ s.t. for a.a. $x \in \Omega$, all $t \in \mathbb{R}$

$$|f(x,t)| \leqslant \varepsilon|t|^{p^*-1} + c_\varepsilon.$$

This kind of growth is a source of technical difficulties, since $W^{1,p}(\Omega)$ is not compactly embedded into $L^{p^*}(\Omega)$. We shall overcome such difficulties by using Vitali's theorem.

Example 2.3. The following functions (of the type $\lambda g + f$, $\lambda > 0$) satisfy hypotheses $\mathbf{H}_g$, $\mathbf{H}_f$, and $\mathbf{H}_e$:

$$(a)\ \ t \mapsto \lambda|t|^{q-2}t + \begin{cases} \hat{\lambda}_1|t|^{p-2}t & \text{if } t \leqslant 0 \\ t^{s-1} + t^{r-1} & \text{if } t > 0 \end{cases} \qquad (q < r < p < s < p^*);$$

$$(b)\ \ t \mapsto \lambda|t|^{q-2}t + \begin{cases} \hat{\lambda}_1|t|^{p-2}t & \text{if } t \leqslant 0 \\ t^{p-1}\ln(1+t) - t^{r-1} & \text{if } t > 0 \end{cases} \qquad (q < r < p);$$

$$(c)\ \ t \mapsto \lambda|t|^{q-2}t + \begin{cases} \hat{\lambda}_1|t|^{p-2}t & \text{if } t \leqslant 0 \\ \dfrac{t^{p^*-1}}{\ln(1+t^p)} - \dfrac{pt^{p^*+p-1}}{p^*(1+t^p)\ln(1+t^p)^2} - t^{r-1} & \text{if } t > 0 \end{cases} \quad (q < r < p);$$

Note that (a) satisfies (AR) while (b) does not, and that (c) has an almost critical growth at $+\infty$.

Our main result is the following.

Theorem 2.4. *If* **H** *hold, then there exists $\lambda^* > 0$ s.t. for all $\lambda \in (0, \lambda^*)$ problem (1.1) admits at least four nontrivial solutions $u_+, v_+, u_-, \tilde{u} \in C^1(\overline{\Omega})$ with u_+, v_+ positive in $\overline{\Omega}$, u_- negative in $\overline{\Omega}$, and $\tilde{u}$ nodal.*

Our approach is entirely variational, based on critical point theory. For all $\lambda > 0$, we define an energy functional φ_λ for problem (1.1), which always admits 0 as a critical point. Then we introduce two truncated/perturbed functionals $\hat{\varphi}_\lambda^\pm$, by replacing the indefinite potential ξ with a positive one and truncating the reactions at 0: thus, nontrivial critical points of $\hat{\varphi}_\lambda^+$ (resp. $\hat{\varphi}_\lambda^-$) are positive (resp. negative) solutions of (1.1). Then we study separately the critical sets of such functionals: for $\lambda > 0$ small enough, $\hat{\varphi}_\lambda^+$ turns out to admit at least two nontrivial critical points, namely, a local minimizer v_+ and another critical point u_+ produced by the mountain pass theorem; while $\hat{\varphi}_\lambda^-$ contributes a negative global minimizer u_-.

Then we go further, proving existence of a smallest positive solution $\underline{u}_+$ and a biggest negative solution $\overline{u}_-$ of (1.1), and we truncate again the reactions. The resulting functional $\tilde{\varphi}_\lambda$ selects solutions lying in the interval $[\overline{u}_-, \underline{u}_+]$, and admits a critical point $\tilde{u}$ of mountain pass type. By computing the critical groups at 0 and at $\tilde{u}$, we see that $\tilde{u} \neq 0$, hence it must be nodal.

2.1 Notation

We establish some notation: we set $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0]$; $c_0, c_1, \ldots$ denote positive constants; for all $t \in \mathbb{R}$ we set

$$t^\pm = \max\{0, \pm t\}.$$

In any Banach space X, $\rightharpoonup$ denotes weak convergence and $\to$ strong convergence; if X is a function space on the domain D, then we denote the positive order cone by

$$X_+ = \{u \in X : \ u(x) \geqslant 0 \text{ for a.a. } x \in D\}.$$

We will say that a functional $\varphi \in C^1(X)$ satisfies the Cerami condition (C), if any sequence (u_n) s.t. $(\varphi(u_n))$ is bounded in $\mathbb{R}$ and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^*, admits a (strongly) convergent subsequence. We will denote the set of critical points of φ by

$$K(\varphi) = \{u \in X : \ \varphi'(u) = 0\}.$$

We also recall the basic notion from Morse theory: let $\varphi \in C^1(X)$ and $u \in K(\varphi)$ be an isolated critical point, namely there exists a neighborhood $U \subset X$ of u s.t. $K(\varphi) \cap U = \{u\}$, and $\varphi(u) = c$. Then, for all $k \in \mathbb{N}$, the k-th critical group of φ at u is defined by

$$C_k(\varphi, u) = H_k\big(\{v \in U : \ \varphi(v) \leqslant c\}, \{v \in U : \ \varphi(v) \leqslant c, \, v \neq u\}\big),$$

where $H_k(\cdot, \cdot)$ denotes the k-th singular homology group of a topological pair.

We shall use the function spaces $(W^{1,p}(\Omega), \|\cdot\|)$ and $(C^1(\overline{\Omega}), \|\cdot\|_{C^1(\overline{\Omega})})$, endowed with the usual norms. Brackets $\langle\cdot,\cdot\rangle$ denote both the inner product of $\mathbb{R}^N$ and the duality between $W^{1,p}(\Omega)^*$ and $W^{1,p}(\Omega)$, with no possible confusing arising. We shall also use the Lebesgue spaces $(L^\nu(\Omega), \|\cdot\|_\nu)$ for all $\nu \in [1, +\infty]$, and the trace space $(L^p(\partial\Omega), \|\cdot\|_{L^p(\partial\Omega)})$ (any $u \in W^{1,p}(\Omega)$ will be identified with its trace on $\partial\Omega$). We set

$$D_+ = \{u \in C^1(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \overline{\Omega}\},$$

noting that $D_+ \subseteq \operatorname{int}(C^1(\overline{\Omega})_+)$.

3 Constant sign solutions

For all $\lambda > 0$, $u \in W^{1,p}(\Omega)$ we set

$$\varphi_\lambda(u) = \int_\Omega H(\nabla u)\,dx + \frac{1}{p}\int_\Omega \xi(x)|u|^p\,dx + \frac{1}{p}\int_{\partial\Omega}\beta(x)|u|^p\,d\sigma - \int_\Omega [\lambda G(x,u) + F(x,u)]\,dx$$

(the integral on $\partial\Omega$ is computed with respect to the $(N-1)$-dimensional Hausdorff measure). By Lemma 2.1 *(iv)*, $\mathbf{H}_\xi$, $\mathbf{H}_\beta$, $\mathbf{H}_g$ *(i)* *(ii)*, $\mathbf{H}_f$ *(i)* *(ii)* *(v)*, we have $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$. Moreover, φ_λ is the energy functional for problem (1.1). Indeed, for all $u \in K(\varphi_\lambda)$, $v \in W^{1,p}(\Omega)$ we have

$$\langle A(u), v\rangle + \int_\Omega \xi(x)|u|^{p-2}uv\,dx + \int_{\partial\Omega}\beta(x)|u|^{p-2}uv\,d\sigma = \int_\Omega [\lambda g(x,u) + f(x,u)]v\,dx, \tag{3.1}$$

i.e., u is a (weak) solution of (1.1). Besides, let

$$\mu > \max\left\{1, \frac{p-1}{c_2}\right\}\|\xi\|_\infty \tag{3.2}$$

and for all $(x,t) \in \Omega \times \mathbb{R}$ set

$$k_\lambda(x,t) = \lambda g(x,t) + f(x,t) + \mu|t|^{p-2}t,$$

$$k_\lambda^\pm(x,t) = k_\lambda(x, \pm t^\pm),$$

and the primitives

$$K_\lambda^{(\pm)}(x,t) = \int_0^t k_\lambda^{(\pm)}(x,\tau)\,d\tau.$$

Now we define two truncated/perturbed functionals by setting for all $u \in W^{1,p}(\Omega)$

$$\hat{\varphi}_\lambda^\pm(u) = \int_\Omega H(\nabla u)\,dx + \frac{1}{p}\int_\Omega (\xi(x)+\mu)|u|^p\,dx + \frac{1}{p}\int_{\partial\Omega}\beta(x)|u|^p\,d\sigma - \int_\Omega K_\lambda^\pm(x,u)\,dx$$

(note that $\xi + \mu$ is positive by (3.2)). We shall study separately the properties of $\hat{\varphi}_\lambda^+$ and $\hat{\varphi}_\lambda^-$, which are different by the asymmetry of f.

Lemma 3.1. *If* $\mathbf{H}$ *hold, then for all* $\lambda > 0$ $\hat{\varphi}_\lambda^+ \in C^1(W^{1,p}(\Omega))$ *satisfies* (C).

Proof. Clearly $k_\lambda^+ : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, with a growth defined by $\mathbf{H}_g$ *(ii)*, $\mathbf{H}_f$ *(ii)* *(v)*, so $\hat{\varphi}_\lambda^+ \in C^1(W^{1,p}(\Omega))$.

Let (u_n) be a sequence in $W^{1,p}(\Omega)$ s.t. $|\hat{\varphi}_\lambda^+(u_n)| \leqslant c_{14}$ for all $n \in \mathbb{N}$ $(c_{14} > 0)$ and $(1 + \|u_n\|)(\hat{\varphi}_\lambda^+)'(u_n) \to 0$ in $W^{1,p}(\Omega)^*$. We can find a sequence (ε_n) in $\mathbb{R}$ s.t. $\varepsilon_n \to 0^+$ and for all $n \in \mathbb{N}$, $v \in W^{1,p}(\Omega)$

$$\left| \langle A(u_n), v\rangle + \int_\Omega (\xi(x) + \mu)|u_n|^{p-2}u_n v\,dx + \int_{\partial\Omega} \beta(x)|u_n|^{p-2}u_n v\,d\sigma - \int_\Omega k_\lambda^+(x,u_n)v\,dx \right| \leqslant \frac{\varepsilon_n \|v\|}{1+\|u_n\|}. \tag{3.3}$$

Choosing $v = -u_n^-$ in (3.3) and using Lemma 2.1 *(iii)* we get for all $n \in \mathbb{N}$

$$\frac{c_2}{p-1}\|\nabla u_n^-\|_p^p + \int_\Omega (\xi(x)+\mu)(u_n^-)^p\,dx \leqslant \varepsilon_n.$$

Passing to the limit we see that $u_n^- \to 0$ in $W^{1,p}(\Omega)$. Now we deal with u_n^+. By definition of K_λ^+ we have for all $n \in \mathbb{N}$

$$\begin{aligned} pc_{14} &\geqslant p\int_\Omega H(\nabla u_n)\,dx + \int_\Omega (\xi(x)+\mu)|u_n|^p\,dx + \int_{\partial\Omega}\beta(x)|u_n|^p\,d\sigma - p\int_\Omega K_\lambda^+(x,u_n)\,dx \\ &\geqslant p\int_\Omega H(\nabla u_n^+)\,dx + \int_\Omega \xi(x)(u_n^+)^p\,dx + \int_{\partial\Omega}\beta(x)(u_n^+)^p\,d\sigma - p\int_\Omega [\lambda G(x,u_n^+) + F(x,u_n^+)]\,dx, \end{aligned}$$

while (3.3) with $v = -u_n^+$ yields

$$-\langle A(u_n^+), u_n^+\rangle - \int_\Omega \xi(x)(u_n^+)^p\,dx - \int_{\partial\Omega}\beta(x)(u_n^+)^p\,d\sigma + \int_\Omega [\lambda g(x,u_n^+) + f(x,u_n^+)]u_n^+\,dx \leqslant \varepsilon_n.$$

Adding up we get

$$\int_\Omega [pH(\nabla u_n^+) - \langle a(\nabla u_n^+), \nabla u_n^+\rangle]\,dx + \int_\Omega e_\lambda(x,u_n^+)\,dx \leqslant c_{15}\ (c_{15} > 0),$$

which by $\mathbf{H}_a$ *(iv)* implies

$$\int_\Omega e_\lambda(x,u_n^+)\,dx \leqslant c_{16}\ (c_{16} > 0). \tag{3.4}$$

We claim that (u_n^+) is bounded in $W^{1,p}(\Omega)$. Arguing by contradiction, we may assume that (passing if necessary to a subsequence) $\|u_n^+\| \to +\infty$. Then we set for all $n \in \mathbb{N}$ $w_n = u_n^+\|u_n^+\|^{-1}$, so $w_n \in W^{1,p}(\Omega)$ with $\|w_n\| = 1$. Passing again to a subsequence we have $w_n \rightharpoonup w$ in $W^{1,p}(\Omega)$ and $w_n \to w$ both in $L^p(\Omega)$ and in $L^p(\partial\Omega)$ (due to the compact embeddings $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, $L^p(\partial\Omega)$). Clearly $w \in W^{1,p}(\Omega)_+$. Two cases may occur:

(a) First we assume $w \neq 0$. Let

$$\Omega_+ = \{x \in \Omega : w(x) > 0\},$$

then $|\Omega_+| > 0$ and for a.a. $x \in \Omega_+$ we have $u_n^+(x) \to +\infty$. By $\mathbf{H}_f$ *(iii)* we have for a.a. $x \in \Omega_+$

$$\frac{F(x,u_n^+(x))}{\|u_n^+\|^p} = \frac{F(x,u_n^+(x))}{u_n^+(x)^p}w_n(x)^p \to +\infty$$

By Fatou's lemma we have

$$\lim_n \int_{\Omega_+} \frac{F(x,u_n^+)}{\|u_n^+\|^p}\,dx = +\infty.$$

By $\mathbf{H}_f$ *(i)* *(iii)* we have for a.a. $x \in \Omega$, all $t \geqslant 0$

$$F(x,t) \geqslant t^p - c_{17} \qquad (c_{17} > 0),$$

so we have

$$\int_{\Omega\setminus\Omega_+} \frac{F(x,u_n^+)}{\|u_n^+\|^p}\,dx \geqslant \int_{\Omega\setminus\Omega_+} w_n^p\,dx - \frac{c_{17}|\Omega|}{\|u_n^+\|^p},$$

and the latter is bounded from below. Summarizing,

$$\lim_n \int_\Omega \frac{F(x,u_n^+)}{\|u_n^+\|^p}\,dx = +\infty. \tag{3.5}$$

Besides, $\mathbf{H}_g$ *(ii)* implies, as above,

$$\lim_n \int_{\Omega_+} \frac{G(x,u_n^+)}{\|u_n^+\|^p}\,dx = 0.$$

By $\mathbf{H}_g$ *(i)* *(ii)*, for any $\varepsilon > 0$ we can find $c_{18} = c_{18}(\varepsilon) > 0$ s.t. for a.a. $x \in \Omega$, all $t \geqslant 0$

$$G(x,t) \leqslant \frac{\varepsilon}{p} t^p + c_{18}.$$

So we have

$$\limsup_n \int_{\Omega\setminus\Omega_+} \frac{G(x,u_n^+)}{\|u_n^+\|^p}\,dx \leqslant \limsup_n \int_{\Omega\setminus\Omega_+} \left(\frac{\varepsilon}{p} w_n^p + \frac{c_{18}}{\|u_n^+\|^p}\right) dx \leqslant \frac{\varepsilon}{p}\|w\|_p^p.$$

since $\varepsilon > 0$ is arbitrary, adding the two integrals we get

$$\lim_n \int_\Omega \frac{G(x,u_n^+)}{\|u_n^+\|^p}\,dx = 0. \tag{3.6}$$

Now (3.5), (3.6) imply

$$\lim_n \int_\Omega \frac{\lambda G(x,u_n^+) + F(x,u_n^+)}{\|u_n^+\|^p}\,dx = +\infty.$$

But again from boundedness of $(\hat{\varphi}_\lambda^+(u_n))$, and recalling that $u_n^- \to 0$ in $W^{1,p}(\Omega)$, we have for all $n \in \mathbb{N}$

$$\begin{aligned}\int_\Omega H(\nabla u_n^+)\,dx + \frac{1}{p}\int_\Omega \xi(x)(u_n^+)^p\,dx + \frac{1}{p}\int_{\partial\Omega} \beta(x)(u_n^+)^p\,d\sigma - \int_\Omega \left[\lambda G(x,u_n^+) + F(x,u_n^+)\right] dx \\ \geqslant -c_{19} \qquad (c_{19} > 0),\end{aligned}$$

which, along with Lemma 2.1 *(iv)*, implies

$$\begin{aligned}&\int_\Omega \frac{\lambda G(x,u_n^+) + F(x,u_n^+)}{\|u_n^+\|^p}\,dx \\ &\quad\leqslant \frac{c_{19}}{\|u_n^+\|^p} + \int_\Omega \frac{c_8(1 + |\nabla u_n^+|^p)}{\|u_n^+\|^p}\,dx + \frac{1}{p}\int_\Omega \xi(x) w_n^p\,dx + \frac{1}{p}\int_{\partial\Omega} \beta(x) w_n^p\,d\sigma \\ &\quad\leqslant c_{20}(1 + \|w_n\|^p) \qquad (c_{20} > 0),\end{aligned}$$

and the latter is bounded from above. Thus we reach a contradiction.

(b) Now we assume $w = 0$. Fix $M > 0$ and set $\hat{w}_n = (Mp)^{\frac{1}{p}} w_n$ for all $n \in \mathbb{N}$, so $\hat{w}_n \rightharpoonup 0$ in $W^{1,p}(\Omega)$ and $\hat{w}_n \to 0$ in $L^p(\Omega)$ and $L^p(\partial\Omega)$. By $\mathbf{H}_g$ *(i) (ii)* we have for a.a. $x \in \Omega$, all $t \in \mathbb{R}$

$$|G(x,t)| \leqslant c_{21}(1 + |t|^p) \qquad (c_{21} > 0).$$

So we get

$$\lim_n \int_\Omega G(x, \hat{w}_n)\, dx = 0. \tag{3.7}$$

Clearly $(\hat{w}_n)$ is bounded in $L^{p^*}(\Omega)$, so set

$$K_0 = \sup_{n \in \mathbb{N}} \|\hat{w}_n\|_{p^*}^{p^*}.$$

By $\mathbf{H}_f$ *(i) (ii)*, for any $\varepsilon > 0$ we can find $c_{21} = c_{22}(\varepsilon) > 0$ s.t. for a.a. $x \in \Omega$, all $t \in \mathbb{R}$

$$|F(x,t)| \leqslant \frac{\varepsilon}{2K_0}|t|^{p^*} + c_{22}.$$

So, the sequence $(F(\cdot, \hat{w}_n))$ is bounded in $L^1(\Omega)$. Furthermore, for any measurable set $B \subset \Omega$ with $|B| \leqslant \varepsilon(2c_{22})^{-1}$ we have for all $n \in \mathbb{N}$

$$\int_B |F(x, \hat{w}_n)|\, dx \leqslant \frac{\varepsilon}{2K_0}\|\hat{w}_n\|_{p^*}^{p^*} + c_{22}|B| \leqslant \varepsilon.$$

So the sequence $(F(\cdot, \hat{w}_n))$ is uniformly integrable in Ω (see [8, Problem 1.6]). Passing to a subsequence, we have $F(x, \hat{w}_n(x)) \to 0$ as $n \to \infty$, for a.a. $x \in \Omega$. By Vitali's theorem [8, p. 5] we have

$$\lim_n \int_\Omega F(x, \hat{w}_n)\, dx = 0. \tag{3.8}$$

Since $\|u_n^+\| \to +\infty$, for $n \in \mathbb{N}$ big enough we have

$$0 < \frac{(Mp)^{\frac{1}{p}}}{\|u_n^+\|} \leqslant 1. \tag{3.9}$$

Let $\hat{\psi}_\lambda^+ \in C^1(W^{1,p}(\Omega))$ be defined for all $u \in W^{1,p}(\Omega)$ by

$$\hat{\psi}_\lambda^+(u) = \frac{c_2}{p(p-1)}\|\nabla u\|_p^p + \frac{1}{p}\int_\Omega (\xi(x) + \mu)|u|^p\, dx + \frac{1}{p}\int_{\partial\Omega} \beta(x)|u|^p\, d\sigma - \int_\Omega K_\lambda^+(x,u)\, dx.$$

For all $n \in \mathbb{N}$ there exists $t_n \in [0,1]$ s.t.

$$\hat{\psi}_\lambda^+(t_n u_n^+) = \max_{t \in [0,1]} \hat{\psi}_\lambda^+(t u_n^+).$$

In particular, by (3.9) we have for $n \in \mathbb{N}$ big enough

$$\begin{aligned}
\hat{\psi}_\lambda^+(t_n u_n^+) &\geqslant \hat{\psi}_\lambda^+(\hat{w}_n) \\
&\geqslant \frac{c_2 M}{p-1}\|\nabla w_n\|_p^p + M\int_\Omega (\xi(x) + \mu) w_n^p\, dx - \int_\Omega \left[\lambda G(x, \hat{w}_n) + F(x, \hat{w}_n) + M\mu w_n^p\right] dx \\
&\geqslant M(c_{23} - \mu\|w_n\|_p^p) - \int_\Omega \left[\lambda G(x, \hat{w}_n) + F(x, \hat{w}_n)\right] dx \qquad (c_{23} > 0)
\end{aligned}$$

(recall that $\mu > \|\xi\|_\infty$ and $\|w_n\| = 1$). Now by (3.7), (3.8) we have for $n \in \mathbb{N}$ even bigger

$$\hat{\psi}_\lambda^+(t_n u_n^+) \geqslant M c_{24} \qquad (c_{24} > 0)$$

which by arbitrarity of $M > 0$ implies $\hat{\psi}_\lambda^+(t_n u_n^+) \to +\infty$ as $n \to \infty$. By Lemma 2.1 *(iv)* we have $\hat{\varphi}_\lambda^+(u) \geqslant \hat{\psi}_\lambda^+(u)$ for all $u \in W^{1,p}(\Omega)$, hence the sequence $(\hat{\psi}_\lambda^+(u_n^+))$ is bounded from above. Besides, clearly $\hat{\psi}_\lambda^+(0) = 0$. So, for all $n \in \mathbb{N}$ big enough we must have $t_n \in (0,1)$. By definition of t_n, then,

$$\frac{d}{dt}\,\hat{\psi}_\lambda^+(tu_n^+)\big|_{t=t_n} = \langle(\hat{\psi}_\lambda^+)'(t_n u_n^+), u_n^+\rangle = 0.$$

Multiplying by t_n we get

$$\begin{aligned}\frac{c_2}{p-1}\|\nabla(t_n u_n^+)\|_p^p + \int_\Omega \xi(x)(t_n u_n^+)^p\,dx + \int_{\partial\Omega}\beta(x)(t_n u_n^+)^p\,d\sigma \\ = \int_\Omega [\lambda g(x, t_n u_n^+) + f(x, t_n u_n^+)]t_n u_n^+\,dx.\end{aligned}$$

By $\mathbf{H}_e$ *(i)*, $t_n < 1$, and (3.4) we have

$$\int_\Omega e_\lambda(x, t_n u_n^+)\,dx \leqslant \int_\Omega e_\lambda(x, u_n^+)\,dx + \|\eta_\lambda\|_1 < c_{25} \qquad (c_{25} > 0),$$

which implies

$$\int_\Omega [\lambda g(x, t_n u_n^+) + f(x, t_n u_n^+)]t_n u_n^+\,dx \leqslant p\int_\Omega \big[\lambda G(x, t_n u_n^+) + F(x, t_n u_n^+)\big]\,dx + c_{25}.$$

Thus, for all $n \in \mathbb{N}$ big enough we have

$$\begin{aligned}p\hat{\psi}_\lambda^+(t_n u_n^+) &= \int_\Omega [\lambda g(x, t_n u_n^+) + f(x, t_n u_n^+)]t_n u_n^+\,dx - p\int_\Omega \big[\lambda G(x, t_n u_n^+) + F(x, t_n u_n^+)\big]\,dx \\ &\leqslant c_{25},\end{aligned}$$

a contradiction.

By the claim above and $u_n^- \to 0$, we see that (u_n) is bounded in $W^{1,p}(\Omega)$. Passing to a subsequence, we may assume $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$ and $L^p(\partial\Omega)$. By $\mathbf{H}_g$ *(i)* *(ii)* we have for a.a. $x \in \Omega$, all $t \in \mathbb{R}$

$$|g(x,t)| \leqslant c_{26}(1 + |t|^{p-1}) \qquad (c_{26} > 0),$$

hence by Hölder's inequality

$$\lim_n \int_\Omega g(x, u_n^+)(u_n - u)\,dx = 0. \tag{3.10}$$

Besides, (u_n) is bounded in $L^{p^*}(\Omega)$, so we set

$$K_1 = \sup_{n\in\mathbb{N}} \|u_n\|_{p^*} + \|u\|_{p^*}.$$

By $\mathbf{H}_f$ *(i)* *(ii)*, for any $\varepsilon > 0$ we can find $c_{27} = c_{27}(\varepsilon) > 0$ s.t. for a.a. $x \in \Omega$, all $t \geqslant 0$

$$|f(x,t)| \leqslant \frac{\varepsilon}{3K_1^{p^*}}t^{p^*-1} + c_{27}.$$

Passing if necessary to a subsequence, we have $f(x, u_n^+(x))(u_n(x) - u(x)) \to 0$ as $n \to \infty$, for a.a. $x \in \Omega$. Moreover, for any measurable $B \subset \Omega$ with

$$|B| \leqslant \left(\frac{\varepsilon}{6K_1 c_{27}}\right)^{(p^*)'}$$

we have by Hölder's inequality

$$\begin{aligned}\left|\int_B f(x,u_n^+)(u_n-u)\,dx\right| &\leqslant \frac{\varepsilon}{3K_1^{p^*}}\int_B (u_n^+)^{p^*-1}|u_n-u|\,dx + c_{27}\|u_n-u\|_1 \\ &\leqslant \frac{\varepsilon}{3K_1^{p^*}}\|u_n^+\|_{p^*}^{p^*-1}\|u_n-u\|_{p^*} + c_{27}|B|^{\frac{1}{(p^*)'}}\|u_n-u\|_{p^*} \\ &\leqslant \frac{2\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

So, the sequence $(f(\cdot,u_n^+)(u_n-u))$ is uniformly integrable in Ω. By Vitali's theorem we get

$$\lim_n \int_\Omega f(x,u_n^+)(u_n-u)\,dx = 0. \tag{3.11}$$

If we choose $v = u_n - u$ in (3.3), pass to the limit as $n \to \infty$, and use (3.10) and (3.11), we now get

$$\limsup_n \langle A(u_n), u_n - u\rangle = 0.$$

By the $(S)_+$-property of A we have $u_n \to u$ in $W^{1,p}(\Omega)$, which concludes the proof. $\square$

The following lemmas show that for $\lambda > 0$ small enough $\hat{\varphi}_\lambda^+$ exhibits the 'mountain pass' geometry.

Lemma 3.2. *If* **H** *hold, then there exists $\lambda^* > 0$ s.t. for all $\lambda \in (0,\lambda^*)$ there exists $\rho_\lambda > 0$ s.t.*

$$\inf_{\|u\|=\rho_\lambda} \hat{\varphi}_\lambda^+(u) = \hat{m}_\lambda^+ > 0.$$

Proof. By $\mathbf{H}_g$ *(ii) (iv)*, for all $\varepsilon > 0$ we can find $c_{28} = c_{28}(\varepsilon) > 0$ s.t. for a.a. $x \in \Omega$, all $t \geqslant 0$

$$G(x,t) \leqslant \frac{\varepsilon}{p}t^p + c_{28}t^q$$

(recall that $q < p$). By $\mathbf{H}_f$ *(iv)* we have as well for a.a. $x \in \Omega$, all $t \geqslant 0$

$$F(x,t) \leqslant \frac{c_{11}}{p^*}t^{p^*} + \frac{c_{11}}{r}t^r - \frac{c_{12}}{p}t^p.$$

Recalling that $q < r < p < p^*$ and choosing $\varepsilon < c_{12}/\lambda$ we get for a.a. $x \in \Omega$, all $t \geqslant 0$

$$\lambda G(x,t) + F(x,t) \leqslant \lambda c_{28}t^q + c_{29}t^{p^*} - \frac{c_{30}}{p}t^p,$$

where, taking $c_{28}, c_{29} > 0$ big enough, we may assume $c_{30} > \|\xi\|_\infty$. So, recalling $\mathbf{H}_\beta$ and $\mu > \|\xi\|_\infty$, for all $u \in W^{1,p}(\Omega)$ we have

$$\begin{aligned}\hat{\varphi}_\lambda^+(u) \geqslant\ & \frac{c_2}{p(p-1)}\|\nabla u^-\|_p^p + \frac{1}{p}\int_\Omega (\xi(x)+\mu)(u^-)^p\,dx \\ &+ \frac{c_2}{p(p-1)}\|\nabla u^+\|_p^p + \frac{1}{p}\int_\Omega \xi(x)(u^+)^p\,dx - \int_\Omega \left[\lambda G(x,u^+) + F(x,u^+)\right]dx \\ \geqslant\ & c_{31}\|u^-\|^p + \frac{c_2}{p(p-1)}\|\nabla u^+\|_p^p + \frac{1}{p}\int_\Omega (\xi(x)+c_{30})(u^+)^p\,dx - \lambda c_{28}\|u^+\|_q^q - c_{29}\|u^+\|_{p^*}^{p^*} \\ \geqslant\ & c_{31}\|u^-\|^p + c_{32}\|u^+\|^p - \lambda c_{33}\|u\|^q - c_{34}\|u\|^{p^*} \\ \geqslant\ & c_{35}\|u\|^p - \lambda c_{33}\|u\|^q - c_{34}\|u\|^{p^*} \qquad (c_{31},\dots,c_{35} > 0).\end{aligned}$$

Summarizing, we have

$$\hat{\varphi}^+_\lambda(u) \geqslant (c_{35} - j_\lambda(\|u\|))\|u\|^p, \tag{3.12}$$

where we have set for all $t\rho > 0$

$$j_\lambda(\rho) = \lambda c_{33}\rho^{q-p} + c_{34}\rho^{p^*-p}.$$

Since $q < p < p^*$, we have for all $\lambda > 0$

$$\lim_{\rho\to 0^+} j_\lambda(\rho) = \lim_{\rho\to+\infty} j_\lambda(\rho) = +\infty,$$

so there exists $\rho_\lambda > 0$ s.t.

$$j_\lambda(\rho_\lambda) = \inf_{\rho>0} j_\lambda(\rho).$$

In particular we have

$$0 = j'_\lambda(\rho_\lambda) = \lambda c_{33}(q-p)\rho_\lambda^{q-p-1} + c_{34}(p^*-p)\rho_\lambda^{p^*-p-1},$$

which yields

$$\rho_\lambda = \left(\frac{\lambda c_{33}(p-q)}{c_{34}(p^*-p)}\right)^{\frac{1}{p^*-q}}.$$

We are interested in the mapping $\lambda \mapsto j_\lambda(\rho_\lambda)$, which amounts to

$$j_\lambda(\rho_\lambda) = c_{36}\lambda^{\frac{p^*-p}{p^*-q}} \qquad (c_{36} > 0 \text{ independent of } \lambda),$$

and the latter tends to 0 as $\lambda \to 0^+$. So there exists $\lambda^* > 0$ s.t. for all $\lambda \in (0, \lambda^*)$ we have $j_\lambda(\rho_\lambda) < c_{35}$. Thus, by (3.12) we have for all $u \in W^{1,p}(\Omega)$ with $\|u\| = \rho_\lambda$

$$\hat{\varphi}^+_\lambda(u) \geqslant (c_{35} - j_\lambda(\rho_\lambda))\rho_\lambda^p =: \hat{m}^+_\lambda > 0,$$

which concludes the proof. □

Let $\hat{u}_1 = \hat{u}_1(p, \xi_0, \beta_0) \in D_+$ be the positive, $L^p(\Omega)$-normalized first eigenfunction of the eigenvalue problem (2.2).

Lemma 3.3. *If* **H** *hold, then for all* $\lambda > 0$

$$\lim_{t\to+\infty} \hat{\varphi}^+_\lambda(t\hat{u}_1) = -\infty.$$

Proof. Fix $\lambda > 0$. By $\mathbf{H}_g$ *(ii)*, $\mathbf{H}_f$ *(iii)*, for all $\eta > 0$ we can find $c_{37} = c_{37}(\eta) > 0$ s.t. for a.a. $x \in \Omega$, all $t \geqslant c_{37}$

$$\lambda G(x,t) + F(x,t) \geqslant \frac{\eta - \mu}{p} t^p.$$

Since $\hat{u}_1 \in D_+$, for all $t > 0$ big enough we have $t\hat{u}_1(x) \geqslant c_{37}$ for all $x \in \overline{\Omega}$. Then by Lemma 2.1 *(iv)* we have

$$\begin{aligned}
&\hat{\varphi}^+_\lambda(t\hat{u}_1) \\
&\quad\leqslant c_8 \int_\Omega (1 + t^p|\nabla\hat{u}_1|^p)\,dx + \frac{1}{p}\int_\Omega (\xi(x)+\mu)(t\hat{u}_1)^p\,dx + \frac{1}{p}\int_{\partial\Omega} \beta(x)(t\hat{u}_1)^p\,d\sigma - \int_\Omega \frac{\eta}{p}(t\hat{u}_1)^p\,dx \\
&\quad\leqslant c_{38} + \left(c_{39}\|\hat{u}_1\|^p - \frac{\eta}{p}\|\hat{u}_1\|_p^p\right)t^p \qquad (c_{38}, c_{39} > 0).
\end{aligned}$$

Choosing $\eta > 0$ big enough, the latter tends to $-\infty$ as soon as $t \to +\infty$, concluding the proof. □

The above lemmas lead, through the use of the mountain pass theorem and constrained minimization, to the existence of two positive solutions.

Proposition 3.4. *If* **H** *hold, then there exists* $\lambda^* > 0$ *s.t. for all* $\lambda \in (0, \lambda^*)$ *problem* (1.1) *admits at least two positive solutions* $u_+, v_+ \in D_+$.

Proof. Fix $\lambda \in (0, \lambda^*)$. By Lemmas 3.2, 3.3 we can find $t > 0$ s.t. $\hat{\varphi}^+_\lambda(t\hat{u}_1) < \hat{m}^+_\lambda$. Recalling also Lemma 3.1, we can apply the mountain pass theorem and find $u_+ \in K(\hat{\varphi}^+_\lambda)$ s.t. $\hat{\varphi}^+_\lambda(u_+) \geqslant \hat{m}^+_\lambda > 0$. In particular we have $u_+ \neq 0$. For all $v \in W^{1,p}(\Omega)$ we have

$$\langle A(u_+), v\rangle + \int_\Omega (\xi(x) + \mu)|u_+|^{p-2}u_+ v \, dx + \int_{\partial\Omega} \beta(x)|u_+|^{p-2}u_+ v \, d\sigma = \int_\Omega k^+_\lambda(x, u_+)v \, dx. \quad (3.13)$$

Choosing $v = -u_+^-$ in (3.13) and applying Lemma 2.1 *(iii)* yields

$$\frac{c_2}{p-1}\|\nabla u_+^-\|_p^p + \int_\Omega (\xi(x) + \mu)(u_+^-)^p \, dx \leqslant 0,$$

hence $u_+ \in W^{1,p}(\Omega)_+ \setminus \{0\}$. So (3.13) becomes for all $v \in W^{1,p}(\Omega)$

$$\langle A(u_+), v\rangle + \int_\Omega \xi(x)u_+^{p-1} v \, dx + \int_{\partial\Omega} \beta(x)u_+^{p-1} v \, d\sigma = \int_\Omega [\lambda g(x, u_+) + f(x, u_+)]v \, dx,$$

i.e., u_+ is a solution of (1.1). Reasoning as in [19,21] we have $u_+ \in L^\infty(\Omega)$. By the nonlinear regularity theory of [12] we have $u_+ \in C^1(\overline{\Omega})_+ \setminus \{0\}$. For a.a. $x \in \Omega$ we have by $\mathbf{H}_g$ *(iii)*

$$g(x, u_+(x)) \geqslant 0,$$

while by $\mathbf{H}_f$ *(ii)* *(vi)* we have

$$f(x, u_+(x)) \geqslant -c_{40}u_+(x)^{p^*-1} \qquad (c_{40} > 0).$$

So we have in Ω (in a weak sense)

$$\operatorname{div} a(\nabla u_+) \leqslant \|\xi\|_\infty u_+^{p-1} + c_{40}u_+^{p^*-1} \leqslant c_{41}u_+^{p-1} \qquad (c_{41} > 0),$$

since u_+ is bounded. By the nonlinear maximum principle [27, pp. 111, 120] we have $u_+ \in D_+$.

Now let $\delta_0 > 0$ be as in $\mathbf{H}_g$ *(v)*. By $\mathbf{H}_a$ *(iv)* we can find $\delta_2 \in (0, \delta_0)$, $c_{42} > c_5$ s.t. for all $y \in \mathbb{R}^N$, $|y| \leqslant \delta_2$

$$H(y) \leqslant c_{42}|y|^r.$$

Fix any $u \in D_+$. For $t > 0$ small enough we have $\|tu\|_{C^1(\overline{\Omega})} \leqslant \delta_2$, in particular $|\nabla(tu)(x)| \leqslant \delta_2$ for all $x \in \overline{\Omega}$. Thus, recalling $\mathbf{H}_g$ *(iii)* *(v)*, $\mathbf{H}_f$ *(vi)*, we have

$$\begin{aligned}\hat{\varphi}^+_\lambda(tu) &= \int_\Omega H(\nabla(tu)) \, dx + \frac{t^p}{p}\int_\Omega \xi(x)u^p \, dx + \frac{t^p}{p}\int_{\partial\Omega} \beta(x)u^p \, d\sigma - \int_\Omega \left[\lambda G(x, tu) + F(x, tu)\right] dx \\ &\leqslant c_{42}t^r\|\nabla u\|_r^r + c_{43}\frac{t^p}{p}\|u\|^p - \lambda c_{44}\frac{t^q}{q}\|u\|^q \qquad (c_{43}, c_{44} > 0).\end{aligned}$$

Recalling that $q < r \leqslant p$, we deduce that $\hat{\varphi}^+_\lambda(tu) < 0$ for all $t > 0$ small enough. Let $\rho_\lambda > 0$ be as in Lemma 3.2, then we have

$$\inf_{\|u\| \leqslant \rho_\lambda} \hat{\varphi}^+_\lambda(u) < 0.$$

By $\mathbf{H}_a$ *(iv)* and the compact embeddings $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega), L^p(\partial\Omega)$, $\hat{\varphi}^+_\lambda$ is sequentially weakly l.s.c. in $W^{1,p}(\Omega)$. So we can find $v_+ \in W^{1,p}(\Omega)$ s.t. $\|v_+\| \leqslant \rho_\lambda$ and

$$\hat{\varphi}^+_\lambda(v_+) = \inf_{\|u\|\leqslant\rho_\lambda} \hat{\varphi}^+_\lambda(u) < 0 < \hat{m}^+_\lambda.$$

In particular $v_+ \neq 0, u_+$ and $\|v_+\| < \rho_\lambda$. Therefore $v_+ \in K(\hat{\varphi}^+_\lambda)$. As above we deduce that $v_+ \in D_+$ and is a solution of (1.1). □

The case of $\hat{\varphi}^-_\lambda$ is simpler.

Lemma 3.5. *If* $\mathbf{H}$ *hold, then for all* $\lambda > 0$ $\hat{\varphi}^-_\lambda \in C^1(W^{1,p}(\Omega))$ *is coercive and satisfies* (C).

Proof. Preliminarily we prove that uniformly for a.a. $x \in \Omega$

$$\lim_{t\to-\infty}\Big(\frac{c_2}{p(p-1)}\hat{\lambda}_1|t|^p - K^-_\lambda(x,t)\Big) = +\infty. \tag{3.14}$$

Indeed, by $\mathbf{H}_e$ *(i)*, for all $\eta > 0$ we can find $c_{45} = c_{45}(\eta) > 0$ s.t. for a.a. $x \in \Omega$, all $t \leqslant -c_{45}$

$$\lambda g(x,t)t + f(x,t)t - p\big(\lambda G(x,t) + F(x,t)\big) \geqslant \eta.$$

Recalling the definition of K^-_λ, for a.a. $x \in \Omega$, all $t < 0$ we have

$$\begin{aligned}\frac{d}{dt}\frac{K^-_\lambda(x,t)}{|t|^p} &= \frac{(\lambda g(x,t) + f(x,t) + \mu|t|^{p-2}t)|t|^p - (\lambda G(x,t) + F(x,t) + \mu/p|t|^p)p|t|^{p-2}t}{|t|^{2p}}\\ &= \frac{\lambda g(x,t)t + f(x,t)t - p\big(\lambda G(x,t) + F(x,t)\big)}{|t|^p t} \leqslant \frac{\eta}{|t|^p t}\end{aligned}$$

(recall that $t < 0$). So, for a.a. $x \in \Omega$ and all $t' < t \leqslant -c_{45}$ we have

$$\frac{K^-_\lambda(x,t)}{|t|^p} - \frac{K^-_\lambda(x,t')}{|t'|^p} \leqslant \int_{t'}^t \frac{\eta}{|\tau|^p\tau}\,d\tau \leqslant \frac{\eta}{p}\Big(\frac{1}{|t'|^p} - \frac{1}{|t|^p}\Big). \tag{3.15}$$

Besides, by $\mathbf{H}_g$ *(ii)*, $\mathbf{H}_f$ *(v)* we have uniformly for a.a. $x \in \Omega$

$$-c_{46} \leqslant \liminf_{t\to-\infty}\frac{K^-_\lambda(x,t)}{|t|^p} \leqslant \limsup_{t\to-\infty}\frac{K^-_\lambda(x,t)}{|t|^p} \leqslant \frac{c_2\hat{\lambda}_1}{p(p-1)} + \frac{\mu}{p}\ (c_{46} > 0). \tag{3.16}$$

Passing to the limit in (3.15) as $t' \to -\infty$ and applying (3.16), we see that for a.a. $x \in \Omega$, all $t \leqslant -c_{45}$

$$\frac{c_2\hat{\lambda}_1}{p(p-1)}|t|^p - K^-_\lambda(x,t) \leqslant \frac{\mu}{p}|t|^p - \frac{\eta}{p},$$

and the latter tends to $+\infty$ as $t \to -\infty$, yielding (3.14).

Now we prove coercivity, arguing by contradiction. Let (u_n) be a sequence in $W^{1,p}(\Omega)$ s.t. $\|u_n\| \to +\infty$ and $|\hat{\varphi}^-_\lambda(u_n)| \leqslant c_{47}$ $(c_{47} > 0)$. For all $n \in \mathbb{N}$ set $w_n = \|u_n\|^{-1}u_n$, so $\|w_n\| = 1$. Passing if necessary to a subsequence, we have $w_n \rightharpoonup w$ in $W^{1,p}(\Omega)$ and $w_n \to w$ in $L^p(\Omega)$ and $L^p(\partial\Omega)$. By Lemma 2.1 *(iv)* we have for all $n \in \mathbb{N}$

$$\begin{aligned}&\frac{c_2}{p(p-1)}\|\nabla w_n\|_p^p + \frac{1}{p}\int_\Omega(\xi(x)+\mu)|w_n|^p\,dx + \frac{1}{p}\int_{\partial\Omega}\beta(x)|w_n|^p\,d\sigma - \int_\Omega\frac{K^-_\lambda(x,u_n)}{\|u_n\|^p}\,dx\\ &\quad\leqslant \frac{c_{47}}{\|u_n\|^p}.\end{aligned} \tag{3.17}$$

By $\mathbf{H}_g$ (i) (ii), $\mathbf{H}_f$ (i) (v) we have for a.a. $x \in \Omega$, all $t \in \mathbb{R}$

$$|K_\lambda^-(x,t)| \leqslant c_{48}(1+|t|^p) \qquad (c_{48} > 0).$$

Thus, the sequence $(K_\lambda^-(\cdot,u_n)\|u_n\|^{-p})$ is uniformly integrable in Ω. By the Dunford–Pettis theorem, then, it admits a weakly convergent subsequence in $L^1(\Omega)$. More precisely, using (3.16) and arguing as in [1, Proposition 30], we can find $\theta_\lambda^- \in L^\infty(\Omega)$ s.t. $\theta_\lambda^-(x) \leqslant \hat{\lambda}_1$ for a.a. $x \in \Omega$ and

$$\frac{K_\lambda^-(\cdot,u_n)}{\|u_n\|^p} \rightharpoonup \left(\frac{c_2}{p(p-1)}\theta_\lambda^- + \frac{\mu}{p}\right)(w^-)^p \quad \text{in } L^1(\Omega).$$

Passing to the limit in (3.17) as $n \to \infty$, and recalling that by convexity

$$\|\nabla w\|_p^p \leqslant \liminf_n \|\nabla w_n\|_p^p,$$

we get

$$\frac{c_2}{p(p-1)}\|\nabla w\|_p^p + \frac{1}{p}\int_\Omega (\xi(x)+\mu)|w|^p\,dx + \frac{1}{p}\int_{\partial\Omega}\beta(x)|w|^p\,d\sigma$$
$$\leqslant \int_\Omega \left(\frac{c_2}{p(p-1)}\theta_\lambda^-(x) + \frac{\mu}{p}\right)(w^-)^p\,dx.$$

Operating on both sides and recalling the definitions of ξ_0, β_0, we have

$$\|\nabla w^-\|_p^p + \int_\Omega \xi_0(x)(w^-)^p\,dx + \int_{\partial\Omega}\beta_0(x)(w^-)^p\,d\sigma \leqslant \int_\Omega \theta_\lambda^-(x)(w^-)^p\,dx. \tag{3.18}$$

Now we distinguish two cases:

(a) If $\theta_\lambda^- \not\equiv \hat{\lambda}_1$ (non-resonance), then we can find $c_{49} > 0$ s.t. for all $v \in W^{1,p}(\Omega)$

$$\|\nabla v\|_p^p + \int_\Omega \xi_0(x)|v|^p\,dx + \int_{\partial\Omega}\beta_0(x)|v|^p\,d\sigma - \int_\Omega \theta_\lambda^-(x)|v|^p\,dx \geqslant c_{49}\|v\|^p.$$

So, by (3.18) we have $w^- = 0$, i.e., $w \in W^{1,p}(\Omega)_+$. Passing to the limit in (3.17), we get

$$\frac{c_2}{p(p-1)}\|\nabla w\|_p^p + \frac{1}{p}\int_\Omega (\xi(x)+\mu)w^p\,dx + \frac{1}{p}\int_{\partial\Omega}\beta(x)w^p\,d\sigma \leqslant 0,$$

hence $w = 0$. Thus, using again (3.17) we see that $\|\nabla w_n\|_p \to 0$, which along with $w_n \to 0$ in $L^p(\Omega)$ yields $w_n \to 0$ in $W^{1,p}(\Omega)$, against $\|w_n\| = 1$.

(b) If $\theta_\lambda^- \equiv \hat{\lambda}_1$ (resonance), then (3.18) becomes

$$\|\nabla w^-\|_p^p + \int_\Omega \xi_0(x)(w^-)^p\,dx + \int_{\partial\Omega}\beta_0(x)(w^-)^p\,d\sigma \leqslant \hat{\lambda}_1\|w^-\|_p^p.$$

By the Lagrange multiplier rule and the definition of $\hat{\lambda}_1$, either $w^- = 0$, or $w^- \in W^{1,p}(\Omega)_+$ is a principal eigenfunction of (2.2). If $w^- = 0$, arguing as in case (a) we reach a contradiction. So, let $w^- \in W^{1,p}(\Omega)_+ \setminus \{0\}$ be a principal eigenfunction of (2.2). Then $w^- \in D_+$, which implies $u_n(x) \to -\infty$ for a.a. $x \in \Omega$. Using (3.14) and Fatou's lemma, we have

$$\lim_n \int_\Omega \left(\frac{c_2}{p(p-1)}\hat{\lambda}_1|u_n|^p - K_\lambda^-(x,u_n)\right)dx = +\infty.$$

So, for all $n \in \mathbb{N}$ big enough

$$\hat{\varphi}_\lambda^-(u_n) \geqslant \frac{c_2}{p(p-1)}\left(\|\nabla u_n\|_p^p + \int_\Omega \xi_0(x)|u_n|^p\,dx + \int_{\partial\Omega} \beta_0(x)|u_n|^p\,d\sigma - \hat{\lambda}_1\|u_n\|_p^p\right) + \int_\Omega \left(\frac{c_2}{p(p-1)}\hat{\lambda}_1|u_n|^p - K_\lambda^-(x,u_n)\right)dx,$$

and the latter tends to $+\infty$ as $n \to \infty$, against $|\hat{\varphi}_\lambda^-(u_n)| \leqslant c_{47}$.

In both cases we reach a contradiction, which proves that

$$\lim_{\|u\|\to+\infty} \hat{\varphi}_\lambda^-(u) = +\infty. \tag{3.19}$$

Now we prove that $\hat{\varphi}_\lambda^-$ satisfies (C). Let (u_n') be a sequence in $W^{1,p}(\Omega)$ s.t. $(\hat{\varphi}_\lambda^-(u_n'))$ is bounded in $\mathbb{R}$ and $(1+\|u_n'\|)(\hat{\varphi}_\lambda^-)'(u_n') \to 0$ in $W^{1,p}(\Omega)^*$. So, we can find a sequence (ε_n) s.t. $\varepsilon_n \to 0^+$ and for all $n \in \mathbb{N}$, $v \in W^{1,p}(\Omega)$

$$\left|\langle A(u_n'),v\rangle + \int_\Omega (\xi(x)+\mu)|u_n'|^{p-2}u_n'v\,dx + \int_{\partial\Omega} \beta(x)|u_n'|^{p-2}u_n'v\,d\sigma - \int_\Omega k_\lambda^-(x,u_n')v\,dx\right| \leqslant \frac{\varepsilon\|v\|}{1+\|u_n'\|}.$$

By (3.19) (u_n') is bounded, so passing to a subsequence we have $u_n' \rightharpoonup u'$ in $W^{1,p}(\Omega)$ and $u_n' \to u'$ in $L^p(\Omega)$ and $L^p(\partial\Omega)$. So, choosing $v = u_n' - u'$, we easily get

$$\limsup_n \langle A(u_n'), u_n' - u'\rangle \leqslant 0,$$

which by the $(S)_+$-property of A implies $u_n' \to u'$ in $W^{1,p}(\Omega)$, concluding the proof. □

By applying the direct method of the calculus of variations, we produce a negative solution.

Proposition 3.6. *If* **H** *hold, then for all* $\lambda > 0$ *problem* (1.1) *admits at least one negative solution* $u_- \in -D_+$.

Proof. Fix $\lambda > 0$. By Lemma 3.5 $\hat{\varphi}_\lambda^-$ is coercive. Besides, it is sequentially weakly l.s.c., hence we can find $u_- \in W^{1,p}(\Omega)$ s.t.

$$\hat{\varphi}_\lambda^-(u_-) = \inf_{u\in W^{1,p}(\Omega)} \hat{\varphi}_\lambda^-(u).$$

Reasoning as in Proposition 3.4 we see that $\hat{\varphi}_\lambda^-(u_-) < 0$, in particular $u_- \neq 0$. For all $v \in W^{1,p}(\Omega)$ we have

$$\langle A(u_-),v\rangle + \int_\Omega (\xi(x)+\mu)|u_-|^{p-2}u_-v\,dx + \int_{\partial\Omega} \beta(x)|u_-|^{p-2}u_-v\,d\sigma = \int_\Omega k_\lambda^-(x,u_-)v\,dx. \tag{3.20}$$

Choosing $v = u_-^+$ in (3.20) and applying $\mathbf{H}_\beta$ and Lemma 2.1 *(iii)*, we get

$$\frac{c_2}{p-1}\|\nabla u_-^+\|_p^p + \int_\Omega (\xi(x)+\mu)(u_-^+)^p\,dx \leqslant 0,$$

hence $u_- \in -W^{1,p}(\Omega)_+ \setminus \{0\}$. Then (3.20) becomes for all $v \in W^{1,p}(\Omega)$

$$\langle A(u_-), v\rangle + \int_\Omega \xi(x)|u_-|^{p-2}u_- v\,dx + \int_{\partial\Omega} \beta(x)|u_-|^{p-2}u_- v\,d\sigma = \int_\Omega [\lambda g(x,u_-) + f(x,u_-)]v\,dx,$$

i.e., u_- is a solution of (1.1). Reasoning as in [19,21] we deduce $u \in L^\infty(\Omega)$. Then nonlinear regularity theory [12] applies, yielding $u_- \in C^1(\overline{\Omega}) \setminus \{0\}$. By $\mathbf{H}_g$ *(iii)* we have for a.a. $x \in \Omega$, all $t \leqslant 0$

$$g(x,t) \leqslant 0,$$

while by $\mathbf{H}_f$ *(v)* *(vi)* we can find $c_{50} > 0$ s.t. for a.a. $x \in \Omega$, all $t \leqslant 0$

$$f(x,t)t \geqslant -c_{50}|t|^p.$$

So, for a.a. $x \in \Omega$ we have

$$\operatorname{div} a(\nabla(-u_-)) \leqslant (\|\xi\|_\infty + c_{50})(-u_-)^{p-1}.$$

By the nonlinear maximum principle [27] we get $-u_- \in D_+$, i.e., $u_- \in -D_+$. □

4 Extremal constant sign solutions and nodal solution

In this section our purpose is twofold: first we improve the results of Propositions 3.4 and 3.6 by proving that problem (1.1) admits *extremal* constant sign solutions, i.e., a smallest positive solution and a biggest negative solution. Then we use truncations and a Morse-theoretic argument to prove existence of a *nodal* solution, thus completing the proof of Theorem 2.4.

Preliminarily, we note that by $\mathbf{H}_g$ *(iii)*, $\mathbf{H}_f$ *(ii)* *(v)* we can find $c_{51} > \|\xi\|_\infty$ s.t. for all $\lambda > 0$, a.a. $x \in \Omega$, and all $t \in \mathbb{R}$

$$\lambda g(x,t)t + f(x,t)t \geqslant \lambda c_9|t|^q - c_{51}(|t|^{p^*} + |t|^p). \tag{4.1}$$

We introduce an auxiliary Robin problem (with critical growth):

$$\begin{cases} -\operatorname{div} a(\nabla u) + \xi(x)|u|^{p-2}u = \lambda c_9|u|^{q-2}u - c_{51}(|u|^{p^*-2}u + |u|^{p-2}u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_a} + \beta(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

We prove an existence/uniqueness result for constant sign solutions of (4.2).

Proposition 4.1. *If $\mathbf{H}_a$, $\mathbf{H}_\xi$, $\mathbf{H}_\beta$ hold, then for all $\lambda > 0$ (4.2) admits a unique positive solution $u_* \in D_+$ and a unique negative solution $v_* \in -D_+$.*

Proof. First we prove existence of a positive solution. Set for all $u \in W^{1,p}(\Omega)$

$$\begin{aligned}\gamma^+_\lambda(u) = &\int_\Omega H(\nabla u)\,dx + \frac{1}{p}\int_\Omega \xi(x)|u|^p\,dx + \frac{1}{p}\int_{\partial\Omega}\beta(x)|u|^p\,d\sigma + \frac{\mu}{p}\|u^-\|_p^p \\ &- \frac{\lambda c_9}{q}\|u^+\|_q^q + \frac{c_{51}}{p^*}\|u^+\|_{p^*}^{p^*} + \frac{c_{51}}{p}\|u^+\|_p^p,\end{aligned}$$

where $\mu > \|\xi\|_\infty$ is defined as in (3.2). By Lemma 2.1 *(iv)* and recalling that $c_{51} > \|\xi\|_\infty$, we have for all $u \in W^{1,p}(\Omega)$

$$\begin{aligned}\gamma_\lambda^+(u) &\geqslant \frac{c_2}{p(p-1)}\|\nabla u\|_p^p + \frac{1}{p}\int_\Omega (\xi(x)+\mu)(u^-)^p\,dx \\ &\quad + \frac{1}{p}\int_\Omega (\xi(x)+c_{51})(u^+)^p\,dx + \frac{c_{51}}{p^*}\|u^+\|_{p^*}^{p^*} - \frac{\lambda c_9}{q}\|u^+\|_q^q \\ &\geqslant c_{52}\|u\|^p - \lambda c_{53}\|u\|^q \ (c_{52}, c_{53} > 0),\end{aligned}$$

and the latter tends to $+\infty$ as $\|u\| \to +\infty$ (since $q < p$). Thus, γ_λ^+ is coercive in $W^{1,p}(\Omega)$. Besides, γ_λ^+ is sequentially weakly l.s.c. So we can find $u_* \in W^{1,p}(\Omega)$ s.t.

$$\gamma_\lambda^+(u_*) = \inf_{u \in W^{1,p}(\Omega)} \gamma_\lambda^+(u).$$

Arguing as in Proposition 3.4 we see that $\gamma_\lambda^+(u_*) < 0$, hence $u_* \neq 0$. For all $v \in W^{1,p}(\Omega)$ we have

$$\begin{aligned}&\langle A(u_*), v\rangle + \int_\Omega \xi(x)|u_*|^{p-2}u_* v\,dx + \int_{\partial\Omega} \beta(x)|u_*|^{p-2}u_* v\,d\sigma + \mu\int_\Omega (u_*^-)^{p-1}v\,dx \\ &= \lambda c_9 \int_\Omega (u_*^+)^{q-1}v\,dx - c_{51}\int_\Omega [(u_*^+)^{p^*-1} + (u_*^+)^{p-1}]v\,dx.\end{aligned}$$

Choosing $v = u_*^-$ yields, by Lemma 2.1 *(iii)*,

$$\frac{c_2}{p-1}\|\nabla u_*^-\|_p^p + \int_\Omega (\xi(x)+\mu)(u_*^-)^p\,dx \leqslant 0,$$

hence $u_* \in W^{1,p}(\Omega)_+ \setminus \{0\}$. As in previous cases we deduce that $u_* \in D_+$ and is a solution of (4.2).

Now we prove uniqueness. Set for all $u \in L^1(\Omega)$

$$\chi(u) = \begin{cases} \displaystyle\int_\Omega H(\nabla(u^{\frac{1}{r}}))\,dx + \frac{1}{p}\int_\Omega (\xi(x)+c_{51})u^{\frac{p}{r}}\,dx + \frac{1}{p}\int_{\partial\Omega}\beta(x)u^{\frac{p}{r}}\,d\sigma & \text{if } u \geqslant 0,\ u^{\frac{1}{r}} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise.}\end{cases}$$

We claim that $\chi : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is convex. Choose $u_1, u_2 \in \mathrm{dom}\,(\chi)$, $\tau \in [0,1]$, and set for all $x \in \Omega$

$$u(x) = \big((1-\tau)u_1(x) + \tau u_2(x)\big)^{\frac{1}{r}},$$

then by [5, Lemma 1] we have for a.a. $x \in \Omega$

$$|\nabla u(x)| \leqslant \big((1-\tau)|\nabla(u_1^{\frac{1}{r}})(x)|^r + \tau|\nabla(u_2^{\frac{1}{r}})(x)|^r\big)^{\frac{1}{r}}.$$

By $\mathbf{H}_a$ *(i)* *(iv)* we know that H_0 is increasing and $t \mapsto H_0(t^{\frac{1}{r}})$ is convex in $\mathbb{R}_+$. So we have for a.a. $x \in \Omega$

$$\begin{aligned}H\big(\nabla((1-\tau)u_1 + \tau u_2)^{\frac{1}{r}}(x)\big) &= H_0(|\nabla u(x)|) \leqslant H_0\Big(\big((1-\tau)|\nabla(u_1^{\frac{1}{r}})(x)|^r + \tau|\nabla(u_2^{\frac{1}{r}})(x)|^r\big)^{\frac{1}{r}}\Big) \\ &\leqslant (1-\tau)H_0(|\nabla(u_1^{\frac{1}{r}})(x)|) + \tau H_0(|\nabla(u_2^{\frac{1}{r}})(x)|) \\ &= (1-\tau)H(\nabla(u_1^{\frac{1}{r}})(x)) + \tau H(\nabla(u_2^{\frac{1}{r}})(x)),\end{aligned}$$

so the functional

$$u \mapsto \int_\Omega H(\nabla(u^{\frac{1}{r}}))\,dx$$

is convex. Besides, since $r \leqslant p$, $c_{51} > \|\xi\|_\infty$, and $\beta \geqslant 0$, the functional

$$u \mapsto \frac{1}{p}\int_\Omega (\xi(x)+c_{51})u^{\frac{p}{r}}\,dx + \frac{1}{p}\int_{\partial\Omega}\beta(x)u^{\frac{p}{r}}\,d\sigma$$

is convex as well. Summarizing, we get the claim. By Fatou's lemma, χ is l.s.c. in $L^1(\Omega)$ and Gâteaux differentiable at $C^1(\overline{\Omega})$-functions.

Now we assume that $u \in W^{1,p}(\Omega)_+ \setminus \{0\}$ is a solution of (4.2). As usual we get $u \in D_+$, in particular $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. So, for all $v \in C^1(\overline{\Omega})$ and $t > 0$ small enough, we have $u^r + tv \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Taking $t > 0$ even smaller if necessary, we have as well $u_*^r + tv \in \operatorname{int}(C^1(\overline{\Omega})_+)$. By what observed above, we have

$$\langle \chi'(u^r), v\rangle = \frac{1}{r}\int_\Omega \frac{-\operatorname{div} a(\nabla u) + (\xi(x)+c_{51})u^{p-1}}{u^{r-1}}\,v\,dx + \frac{1}{r}\int_\Omega \frac{\beta(x)u^{p-1}}{u^{r-1}}\,v\,d\sigma,$$

and a similar relation holds for u_*. Since χ' is a monotone operator, and recalling that u, u_* solve (4.2), we have

$$\begin{aligned}
0 &\leqslant \langle \chi'(u^r) - \chi'(u_*^r), u^r - u_*^r\rangle \\
&\leqslant \frac{1}{r}\int_\Omega \frac{\lambda c_9 u^{q-1} - c_{51}u^{p^*-1}}{u^{r-1}}(u^r - u_*^r)\,dx - \frac{1}{r}\int_\Omega \frac{\lambda c_9 u_*^{q-1} - c_{51}u_*^{p^*-1}}{u_*^{r-1}}(u^r - u_*^r)\,dx \\
&= \frac{1}{r}\int_\Omega \big[\lambda c_9(u^{q-r} - u_*^{q-r}) - c_{51}(u^{p^*-r} - u_*^{p^*-r})\big](u^r - u_*^r)\,dx \leqslant 0
\end{aligned}$$

(recall that $q < r < p^*$). So we have $u = u_*$, i.e., u_* is the only positive solution of (4.2).

Since problem (4.2) is odd, it clearly has a unique negative solution $v_* = -u_* \in -D_+$, which concludes the proof. □

From now on, for all $u, v \in W^{1,p}(\Omega)$ we shall write $u \leqslant v$ meaning that $u(x) \leqslant v(x)$ for a.a. $x \in \Omega$. Such partial ordering makes $W^{1,p}(\Omega)$ an ordered Banach space, and for all set $S \subset W^{1,p}(\Omega)$ we will use accordingly the notions of minorant, majorant, infimum, and supremum of S. Similarly, if $u_1 \leqslant u_2$ we set

$$[u_1, u_2] = \{u \in W^{1,p}(\Omega) : u_1 \leqslant u \leqslant u_2\}.$$

Now we go back to problem (1.1). For all $\lambda > 0$ we denote by $S(\lambda)$ (resp. $S_+(\lambda)$, $S_-(\lambda)$) the set of all solutions (resp. positive, negative solutions) of (1.1). From Proposition 3.4 we know that $\emptyset \neq S_+(\lambda) \subseteq D_+$ for all $\lambda \in (0, \lambda^*)$, while Proposition 3.6 tells us that $\emptyset \neq S_-(\lambda) \subseteq -D_+$ for all $\lambda > 0$. Moreover, from [25], [6] we know that for all $\lambda > 0$ the set $S_+(\lambda)$ is downward directed, i.e., for all $u_1, u_2 \in S_+(\lambda)$ we can find $u_3 \in S_+(\lambda)$ s.t. $u_3 \leqslant u_1$ and $u_3 \leqslant u_2$. Similarly, $S_-(\lambda)$ is upward directed. Now we prove a lower bound for $S_+(\lambda)$ and an upper bound for $S_-(\lambda)$, respectively.

Lemma 4.2. *If* **H** *hold, then*

(*i*) *for all* $\lambda \in (0, \lambda^*)$ *and all* $u \in S_+(\lambda)$, $u \geqslant u_*$;

(*ii*) *for all* $\lambda > 0$ *and all* $u \in S_-(\lambda)$, $u \leqslant v_*$.

Proof. We prove (*i*). Fix $\lambda \in (0, \lambda^*)$, $u \in S_+(\lambda)$. Since $u \in D_+$, we can set for all $(x, t) \in \Omega \times \mathbb{R}$

$$h_\lambda^+(x,t) = \begin{cases} 0 & \text{if } t < 0 \\ \lambda c_9 t^{q-1} - c_{51} t^{p^*-1} - (c_{51} - \mu) t^{p-1} & \text{if } 0 \leqslant t \leqslant u(x) \\ \lambda c_9 u(x)^{q-1} - c_{51} u(x)^{p^*-1} - (c_{51} - \mu) u(x)^{p-1} & \text{if } t > u(x) \end{cases}$$

and

$$H_\lambda^+(x,t) = \int_0^t h_\lambda^+(x,\tau)\, d\tau.$$

Then we define a functional $\hat{\gamma}_\lambda^+ \in C^1(W^{1,p}(\Omega))$ by setting for all $u \in W^{1,p}(\Omega)$

$$\hat{\gamma}_\lambda^+(u) = \int_\Omega H(\nabla u)\, dx + \frac{1}{p}\int_\Omega (\xi(x) + \mu)|u|^p\, dx + \frac{1}{p}\int_{\partial\Omega} \beta(x)|u|^p\, d\sigma - \int_\Omega H_\lambda^+(x,u)\, dx.$$

Since $\mu > \|\xi\|_\infty$, and $h_\lambda^+(x,\cdot)$ is bounded in $\mathbb{R}$ for a.a. $x \in \mathbb{R}$, $\hat{\gamma}_\lambda^+$ is coercive and sequentially weakly l.s.c. So there exists $\hat{u}_* \in W^{1,p}(\Omega)$ s.t.

$$\hat{\gamma}_\lambda^+(\hat{u}_*) = \inf_{u \in W^{1,p}(\Omega)} \hat{\gamma}_\lambda^+(u) < 0,$$

in particular $\hat{u}_* \neq 0$. For all $v \in W^{1,p}(\Omega)$ we have

$$\langle A(\hat{u}_*), v\rangle + \int_\Omega (\xi(x) + \mu)|\hat{u}_*|^{p-2}\hat{u}_* v\, dx + \int_{\partial\Omega} \beta(x)|\hat{u}_*|^{p-2}\hat{u}_* v\, d\sigma = \int_\Omega h_\lambda^+(x,\hat{u}_*) v\, dx. \tag{4.3}$$

Choosing $v = -\hat{u}_*^-$ in (4.3), we have by Lemma 2.1 (*iii*)

$$\frac{c_2}{p-1}\|\nabla \hat{u}_*^-\|_p^p + \int_\Omega (\xi(x) + \mu)(\hat{u}_*^-)^p\, dx \leqslant 0,$$

hence $\hat{u}_* \in W^{1,p}(\Omega)_+ \setminus \{0\}$. Instead, choosing $v = (\hat{u}_* - u)^+$ in (4.3), applying (4.1), and recalling that $u \in S_+(\lambda)$ yields

$$\begin{aligned} &\langle A(\hat{u}_*), (\hat{u}_* - u)^+\rangle + \int_\Omega (\xi(x) + \mu)\hat{u}_*^{p-1}(\hat{u}_* - u)^+\, dx + \int_{\partial\Omega} \beta(x)\hat{u}_*^{p-1}(\hat{u}_* - u)^+\, d\sigma \\ &\quad = \int_\Omega [\lambda c_9 u^{q-1} - c_{51} u^{p^*-1} - (c_{51} - \mu)u^{p-1}](\hat{u}_* - u)^+\, dx \\ &\quad \leqslant \int_\Omega [\lambda g(x,u) + f(x,u) + \mu u^{p-1}](\hat{u}_* - u)^+\, dx \\ &\quad = \langle A(u), (\hat{u}_* - u)^+\rangle + \int_\Omega (\xi(x) + \mu)u^{p-1}(\hat{u}_* - u)^+\, dx + \int_{\partial\Omega} \beta(x)u^{p-1}(\hat{u}_* - u)^+\, d\sigma, \end{aligned}$$

hence

$$\langle A(\hat{u}_*) - A(u), (\hat{u}_* - u)^+\rangle + \int_\Omega (\xi(x) + \mu)(\hat{u}_*^{p-1} - u^{p-1})(\hat{u}_* - u)^+\, dx \leqslant 0.$$

By Lemma 2.1 (*i*), this implies $\hat{u}_* \leqslant u$. So (4.3) becomes for all $v \in W^{1,p}(\Omega)$

$$\langle A(\hat{u}_*), v\rangle + \int_\Omega \xi(x)\hat{u}_*^{p-1} v\, dx + \int_{\partial\Omega} \beta(x)\hat{u}_*^{p-1} v\, d\sigma = \int_\Omega [\lambda c_9 \hat{u}_*^{q-1} - c_{51}(\hat{u}_*^{p^*-1} + \hat{u}_*^{p-1}] v\, dx.$$

We conclude that $\hat{u}_*$ is a positive solution of (4.2), hence by Proposition 4.1 we have $\hat{u}_* = u_*$. Thus, we have $u \geqslant u_*$.

Similarly we prove (*ii*). $\square$

Using these bounds, we can detect extremal constant sign solutions of (1.1).

Proposition 4.3. *If* **H** *hold, then*

(*i*) *for all* $\lambda \in (0, \lambda^*)$ *there exists* $\underline{u}_+ \in S_+(\lambda)$ *s.t.* $\underline{u}_+ = \inf S_+(\lambda)$*;*

(*ii*) *for all* $\lambda > 0$ *there exists* $\overline{u}_- \in S_-(\lambda)$ *s.t.* $\overline{u}_- = \sup S_-(\lambda)$*.*

Proof. We prove (*i*). By [10, Lemma 3.10, p. 178] we can find a sequence (u_n) in $S_+(\lambda)$, pointwise decreasing, s.t.

$$\inf_{n\in\mathbb{N}} u_n = \inf S_+(\lambda). \tag{4.4}$$

For all $n \in \mathbb{N}$, $v \in W^{1,p}(\Omega)$ we have

$$\langle A(u_n), v\rangle + \int_\Omega \xi(x) u_n^{p-1} v\, dx + \int_{\partial\Omega} \beta(x) u_n^{p-1} v\, d\sigma = \int_\Omega [\lambda g(x, u_n) + f(x, u_n)] v\, dx. \tag{4.5}$$

Choosing $v = u_n$ in (4.5), recalling that $0 \leqslant u_n \leqslant u_1$, and using Lemma 2.1 (*iii*), we see that (u_n) is bounded in $W^{1,p}(\Omega)$. Passing to a subsequence, we have $u_n \rightharpoonup \underline{u}_+$ in $W^{1,p}(\Omega)$, $u_n \to \underline{u}_+$ in $L^p(\Omega)$ and $L^p(\partial\Omega)$. In particular $\underline{u}_+ \in W^{1,p}(\Omega)_+$. Choosing $v = u_n - \underline{u}_+$ in (4.5) and passing to the limit as $n \to \infty$ then provides

$$\lim_n \langle A(u_n), u_n - \underline{u}_+\rangle = 0,$$

which by the $(S)_+$ property of A implies $u_n \to \underline{u}_+$ in $W^{1,p}(\Omega)$. Once again we use (4.5) and for all $v \in W^{1,p}(\Omega)$ we have

$$\langle A(\underline{u}_+), v\rangle + \int_\Omega \xi(x) \underline{u}_+^{p-1} v\, dx + \int_{\partial\Omega} \beta(x) \underline{u}_+^{p-1} v\, d\sigma = \int_\Omega [\lambda g(x, \underline{u}_+) + f(x, \underline{u}_+)] v\, dx.$$

So $\underline{u}_+ \in S(\lambda)$. Lemma 4.2 (*i*) implies $\underline{u}_+ \geqslant u_* > 0$, so $\underline{u}_+ \in S_+(\lambda)$. Then by (4.4) we have $u \geqslant \underline{u}_+$ for all $u \in S_+(\lambda)$.

Similarly we prove (*ii*). □

Let us recall a basic notion from Morse theory (see [15, Definition 6.43]). Let φ be a C^1-functional defined on a Banach space X, and $u \in X$ be an isolated critical point of φ, i.e., there exists a neighborhood U of u s.t. u is the only critical point of φ in U. For all $k \in \mathbb{N}$, we define the *k-th critical group* of φ at u as

$$C_k(\varphi, u) = H_k\big(\{v \in U : \varphi(v) \leqslant \varphi(u)\}, \{v \in U : \varphi(v) \leqslant \varphi(u),\, v \neq u\}\big),$$

where $H_k(\cdot,\cdot)$ denotes the k-th singular homology group of a topological pair of sets (such definition is independent of U). Now we can perform our final step and produce a nodal solution.

Proposition 4.4. *If* **H** *hold, then for all* $\lambda \in (0, \lambda^*)$ *problem* (1.1) *admits a nodal solution* $\tilde{u} \in C^1(\overline{\Omega}) \setminus \{0\}$ *s.t. for all* $x \in \Omega$

$$\overline{u}_-(x) \leqslant \tilde{u}(x) \leqslant \underline{u}_+(x).$$

Proof. Fix $\lambda \in (0, \lambda^*)$, and let $\overline{u}_- \in S_-(\lambda)$, $\underline{u}_+ \in S_+(\lambda)$ be given by Proposition 4.3. We set for all $(x, t) \in \Omega \times \mathbb{R}$

$$\tilde{k}_\lambda(x,t) = \begin{cases} \lambda g(x, \overline{u}_-(x)) + f(x, \overline{u}_-(x)) + \mu|\overline{u}_-(x)|^{p-2}\overline{u}_-(x) & \text{if } t < \overline{u}_-(x) \\ \lambda g(x,t) + f(x,t) + \mu|t|^{p-2}t & \text{if } \overline{u}_-(x) \leqslant t \leqslant \underline{u}_+(x) \\ \lambda g(x, \underline{u}_+(x)) + f(x, \underline{u}_+(x)) + \mu \underline{u}_+(x)^{p-1} & \text{if } t > \underline{u}_+(x) \end{cases}$$

(with $\mu > 0$ given by (3.2)), as well as

$$\tilde{K}_\lambda(x,t) = \int_0^t \tilde{k}_\lambda(x,\tau)\,d\tau.$$

Now we set for all $u \in W^{1,p}(\Omega)$

$$\tilde{\varphi}_\lambda(u) = \int_\Omega H(\nabla u)\,dx + \frac{1}{p}\int_\Omega (\xi(x)+\mu)|u|^p\,dx + \frac{1}{p}\int_{\partial\Omega} \beta(x)|u|^p\,d\sigma - \int_\Omega \tilde{K}_\lambda(x,u)\,dx.$$

Clearly $\tilde{\varphi}_\lambda \in C^1(W^{1,p}(\Omega))$. We study now the properties of its critical set:

$$K(\tilde{\varphi}_\lambda) \subseteq [\overline{u}_-, \underline{u}_+] \cap S(\lambda). \tag{4.6}$$

Indeed, let $u \in K(\tilde{\varphi}_\lambda)$. For all $v \in W^{1,p}(\Omega)$ we have

$$\langle A(u), v\rangle + \int_\Omega (\xi(x)+\mu)|u|^{p-2}uv\,dx + \int_{\partial\Omega} \beta(x)|u|^{p-2}uv\,d\sigma = \int_\Omega \tilde{k}_\lambda(x,u)v\,dx. \tag{4.7}$$

Choosing $v = (u - \underline{u}_+)^+$ in (4.7) we have

$$\begin{aligned}
&\langle A(u), (u-\underline{u}_+)^+\rangle + \int_\Omega (\xi(x)+\mu)|u|^{p-2}u(u-\underline{u}_+)^+\,dx + \int_{\partial\Omega} \beta(x)|u|^{p-2}u(u-\underline{u}_+)^+\,d\sigma \\
&\quad = \int_\Omega [\lambda g(x,\underline{u}_+) + f(x,\underline{u}_+) + \mu \underline{u}_+^{p-1}](u-\underline{u}_+)^+\,dx \\
&\quad = \langle A(\underline{u}_+), (u-\underline{u}_+)^+\rangle + \int_\Omega (\xi(x)+\mu)\underline{u}_+^{p-1}(u-\underline{u}_+)^+\,dx + \int_{\partial\Omega} \beta(x)\underline{u}_+^{p-1}(u-\underline{u}_+)^+\,d\sigma,
\end{aligned}$$

i.e.,

$$\langle A(u) - A(\underline{u}_+), (u-\underline{u}_+)^+\rangle + \int_\Omega (\xi(x)+\mu)(u^{p-1} - \underline{u}_+^{p-1})(u-\underline{u}_+)^+\,dx \leqslant 0.$$

Since $\mu > \|\xi\|_\infty$, we have $u \leqslant \underline{u}_+$. Similarly, choosing $v = (\overline{u}_- - u)^+$ in (4.7) we get $u \geqslant \overline{u}_-$. Thus, (4.7) becomes for all $v \in W^{1,p}(\Omega)$

$$\langle A(u), v\rangle + \int_\Omega \xi(x)|u|^{p-2}uv\,dx + \int_{\partial\Omega} \beta(x)|u|^{p-2}uv\,d\sigma = \int_\Omega [\lambda g(x,u) + f(x,u)]v\,dx,$$

hence $u \in S(\lambda)$. In particular, by nonlinear regularity theory [12] we have $u \in C^1(\overline{\Omega})$.

Now set for all $(x,t) \in \Omega \times \mathbb{R}$

$$\tilde{k}_\lambda^\pm(x,t) = \tilde{k}_\lambda(x, \pm t^\pm), \qquad \tilde{K}_\lambda^\pm(x,t) = \int_0^t \tilde{k}_\lambda^\pm(x,\tau)\,d\tau,$$

and the corresponding functionals $\tilde{\varphi}_\lambda^\pm \in C^1(W^{1,p}(\Omega))$ defined for all $u \in W^{1,p}(\Omega)$ by

$$\tilde{\varphi}_\lambda^\pm(u) = \int_\Omega H(\nabla u)\,dx + \frac{1}{p}\int_\Omega (\xi(x)+\mu)|u|^p\,dx + \frac{1}{p}\int_{\partial\Omega} \beta(x)|u|^p\,d\sigma - \int_\Omega \tilde{K}_\lambda^\pm(x,u)\,dx.$$

For the critical sets of such functionals we have a complete description:

$$K(\tilde{\varphi}_\lambda^+) = \{0, \underline{u}_+\}. \tag{4.8}$$

Indeed, clearly $0, \underline{u}_+ \in K(\tilde{\varphi}_\lambda^+)$. Besides, reasoning as above we see that for all $u \in K(\tilde{\varphi}_\lambda^+) \setminus \{0\}$

$$u \in [0, \underline{u}_+] \cap S_+(\lambda).$$

Then, by Proposition 4.3 *(i)* we have $u \geqslant \underline{u}_+$, hence $u = \underline{u}_+$. Similarly we get

$$K(\tilde{\varphi}_\lambda^-) = \{0, \overline{u}_-\}. \tag{4.9}$$

We prove now that $\underline{u}_+$, $\overline{u}_-$ are local minimizers of $\tilde{\varphi}_\lambda$. We only deal with $\underline{u}_+$, as the case of $\overline{u}_-$ is analogous. Since $\tilde{k}_\lambda^+(x,\cdot)$ is bounded in $\mathbb{R}$ for a.a. $x \in \Omega$, the functional $\tilde{\varphi}_\lambda^+$ is coercive, beside being sequentially weakly l.s.c. So we can find $\tilde{u}_+ \in W^{1,p}(\Omega)$ s.t.

$$\tilde{\varphi}_\lambda^+(\tilde{u}_+) = \inf_{u \in W^{1,p}(\Omega)} \tilde{\varphi}_\lambda^+(u) < 0,$$

in particular $\tilde{u}_+ \neq 0$. By (4.8), then, we have $\tilde{u}_+ = \underline{u}_+$, i.e., $\underline{u}_+$ is a global minimizer of $\tilde{\varphi}_\lambda^+$. We recall that

$$\underline{u}_+ \in D_+ \subseteq \operatorname{int}(C^1(\overline{\Omega})_+),$$

and clearly the functionals $\tilde{\varphi}_\lambda$, $\tilde{\varphi}_\lambda^+$ agree on $C^1(\overline{\Omega})_+$. So $\underline{u}_+$ is a $C^1(\overline{\Omega})$-local minimizer of $\tilde{\varphi}_\lambda$, namely there exists $\rho > 0$ s.t. for all $u \in C^1(\overline{\Omega})$ with $\|u - \underline{u}_+\|_{C^1(\overline{\Omega})} < \rho$ we have

$$\tilde{\varphi}_\lambda(u) \geqslant \tilde{\varphi}_\lambda(\underline{u}_+).$$

By the results of [21], $\underline{u}_+$ is as well a $W^{1,p}(\Omega)$-local minimizer of $\tilde{\varphi}_\lambda$, namely there exists $c_{54} = c_{54}(\rho) > 0$ s.t. for all $u \in W^{1,p}(\Omega)$ with $\|u - \underline{u}_+\| < c_{54}$ we have

$$\tilde{\varphi}_\lambda(u) \geqslant \tilde{\varphi}_\lambda(\underline{u}_+).$$

Without loss of generality we may assume $\tilde{\varphi}_\lambda(\overline{u}_-) \leqslant \tilde{\varphi}_\lambda(\underline{u}_+)$, and that the set $K(\tilde{\varphi}_\lambda)$ is finite. Clearly $\tilde{\varphi}_\lambda$ satisfies (C). By [1, Proposition 29], then, we can find $\tilde{\rho}_\lambda \in (0, \|\overline{u}_- - \underline{u}_+\|)$ s.t., summarizing,

$$\tilde{\varphi}_\lambda(\overline{u}_-) \leqslant \tilde{\varphi}_\lambda(\underline{u}_+) < \tilde{m}_\lambda = \inf_{\|u - \underline{u}_+\| = \tilde{\rho}_\lambda} \tilde{\varphi}_\lambda(u). \tag{4.10}$$

Applying a convenient version of the mountain pass theorem [9], we see that there exists $\tilde{u} \in K(\tilde{\varphi}_\lambda)$ of mountain pass type, namely, s.t. the set

$$\big\{u \in W^{1,p}(\Omega) : \|u - \tilde{u}\| \leqslant \rho,\ \tilde{\varphi}_\lambda(u) < \tilde{\varphi}_\lambda(\tilde{u})\big\}$$

is path disconnected for all $\rho > 0$ small enough, and $\tilde{\varphi}_\lambda(\tilde{u}) \geqslant \tilde{m}_\lambda$. By (4.10) we have $\tilde{u} \neq \underline{u}_+, \overline{u}_-$, so it remains to prove that $\tilde{u} \neq 0$. Indeed, since $\tilde{u}$ is of mountain pass type, by [17, Corollary 6.81] we have

$$C_1(\tilde{\varphi}_\lambda, \tilde{u}) \neq 0. \tag{4.11}$$

Let $\delta_0, \delta_1 > 0$ be as in $\mathbf{H}_g$ *(v)*, $\mathbf{H}_f$ *(vii)*, respectively, and set

$$\delta_3 = \min\Big\{\delta_0, \delta_1,\ \min_{\overline{\Omega}} \underline{u}_+,\ \min_{\overline{\Omega}}(-\overline{u}_-)\Big\} > 0$$

(recall that $\underline{u}_+, -\overline{u}_- \in D_+$). Fix $c_{55} > q$, $\delta \in (0, \delta_3)$. Then for a.a. $x \in \Omega$, all $|t| \leqslant \delta$ we have by $\mathbf{H}_g$ *(iii)* *(v)*, $\mathbf{H}_f$ *(vi)* *(vii)*

$$\begin{aligned}
&c_{55}\big[\lambda G(x,t) + F(x,t)\big] - \big[\lambda g(x,t)t + f(x,t)t\big] \\
&\quad \geqslant \lambda\big[(c_{55} - q)G(x,t) + \big(qG(x,t) - g(x,t)t\big)\big] - f(x,t)t \\
&\quad \geqslant \lambda\frac{(c_{55} - q)c_9}{q}|t|^q - c_{56}|t|^p \ (c_{56} > 0),
\end{aligned}$$

and the latter is positive for $\delta > 0$ small enough (as $q < p$). Thus we can apply [22, Proposition 6] and get for all $k \in \mathbb{N}$

$$C_k(\tilde{\varphi}_\lambda, 0) = 0. \tag{4.12}$$

Comparing (4.11) and (4.12) we see that $\tilde{u} \neq 0$. Furthermore, by (4.6) we have $\overline{u}_- \leqslant \tilde{u} \leqslant \underline{u}_+$, so as above we get $\tilde{u} \in S(\lambda)$. Proposition 4.3 now implies that $\tilde{u}$ is nodal. □

Now we can prove our main result at once.

Proof of Theorem 2.4. It follows at once from Propositions 3.4, 3.6, and 4.4. □

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