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New conditions for the exponential stability of fractionally perturbed ODEs

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Abstract. The aim of this paper is to present some results on the exponential stability of the zero solution for a class of fractionally perturbed ordinary differential equations, whose right-hand sides involve the Riemann–Liouville substantial fractional integrals of different orders and we assume that they are polynomially bounded. In their proofs we apply a method recently developed by Rigoberto Medina. We also prove an existence result for this type of equations.

Keywords: fractional differential equation, Riemann–Liouville integral, exponential stability.

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1 Introduction

It is well known that the system of linear fractional differential equations

$$D^{\alpha}x(t) = Ax(t), \qquad x(t) \in \mathbb{R}^N,\ \alpha \in (0,1), \tag{1.1}$$

where $D^{\alpha}x(t)$ is the Riemann–Liouville or the Caputo derivative of $x(t)$ of the order $\alpha \in (0,1)$ and A is a constant matrix, do not have exponentially stable solutions, but asymptotically stable only. The equilibrium $x = 0$ of this equation is asymptotically stable if and only if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ for all eigenvalues λ of the matrix A. In this case all components of $x(t)$ decay towards 0 like $t^{-\alpha}$ (see e.g. [8]).

In the paper [3] a sufficient condition for the exponential stability of the zero solution of nonlinear fractional systems of equations of the following class

$$\dot{x}(t) = Ax(t) + g\big(t, x(t), {}^{RL}I^{\alpha_1}x(t), \dots, {}^{RL}I^{\alpha_m}x(t)\big), \qquad x(t) \in \mathbb{R}^N, \tag{1.2}$$

is proved. Here A is a constant matrix and

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$$^{RL}I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds \tag{1.3}$$

is the Riemann–Liouville fractional integral of order α of the function $x(t)$. The aim of this paper is to prove a result of that type for the following class of fractional system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + F\Big(t, x(t)\Big) + f\Big(t, I^{(\alpha_1,\beta_1)}x(t), \dots, I^{(\alpha_m,\beta_m)}x(t)\Big), \\ &t \geq 0,\ x(t) \in R^N,\ x(t_0) = x_0, \end{aligned} \tag{1.4}$$

where

$$I^{(\alpha,\beta)}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\beta(t-s)}x(s)ds \tag{1.5}$$

is the so-called fractional substantial integral of the function $x(t)$ of order $\alpha > 0$ with a parameter $\beta > 0$ (see e.g. [5]). This integral is more general than integrals defining the following fractional derivations:

$$^{RL}D^{\alpha}x(t) := \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t (t-s)^{\alpha-1}x(s)ds \quad \text{(Riemann–Liouville)}, \tag{1.6}$$

$$^{C}D^{\gamma}x(t) := \frac{1}{\Gamma(1-\gamma)}\int_0^t (t-s)^{-\gamma}\dot{x}(s)ds, \quad \gamma = 1-\alpha \quad \text{(Caputo)}, \tag{1.7}$$

$$^{CF}D^{\beta}x(t) := \frac{1}{1-\beta}\int_0^t e^{-\frac{\beta}{1-\beta}(t-s)}\dot{x}(s)ds \quad \text{(Caputo–Fabrizio)}. \tag{1.8}$$

We remark that the substantial fractional derivative, corresponding to the substantial fractional integral is defined as

$$D^{(\alpha,\beta)}x(t) = \frac{1}{\Gamma(\alpha)}\left[\frac{\partial}{\partial t} + \beta\right]\int_0^t (t-s)^{\alpha-1}e^{-\beta(t-s)}x(s)ds, \qquad 0 < \alpha < 1,\ \beta > 0. \tag{1.9}$$

Definition 1.1. We say that $x(t)$ is a solution of the initial value problem (2.1), defined on the interval $[t_0, T)$ it it is C^1-differentiable, the fractional integrals in this equation exists, $x(t)$ fulfils the equality (2.1) for all $t \in (0, T)$ with $x(0) = x_0$. It is called maximal, if there no its proper continuation, i.e. there is no $\epsilon > 0$, such that there exists a solution $y(t)$ of this problem, defined on the interval $[t_0, T+\epsilon)$ with $y(t) = x(t)$ for all $t \in [t_0, T)$. If $T = \infty$, the this solution is called global.

In the paper [4] the problem of exponential stability of fractional differential equations of the type (1.2), where instead of the Riemann–Liouville fractional integrals there are Caputo–Fabrizio fractional integrals, is studied.

The aim of this paper is to prove a result on the exponential stability of the zero solution of equations of the form (1.2), where instead of the constant matrix A there is a time-dependent matrix $A(t)$ and instead the Riemann–Liouville fractional integrals there are the Riemann–Liouville substantial fractional integrals. These integrals have some better properties, convenient for the study asymptotic properties of solutions, than the Riemann–Liouville fractional integrals.

In the papers [7] a sufficient condition for the asymptotic stability of the zero solution of the equation

$$^{RL}D^{\alpha}x(t) = f(t, x(t)), \qquad \alpha \in (0,1),\ x \in \mathbb{R}, \tag{1.10}$$

where

$$|f(t,x)| \leq t^{\mu}\Phi(t)e^{-\sigma t}|x|^m, \qquad \mu \geq 0,\ m > 1,\ \sigma > 0, \tag{1.11}$$

f, Φ are continuous functions, are proved. In this case solutions decay toward 0 as $t \to \infty$ like $t^{-\alpha}$. It is proven in the paper [13] that solutions of the equation

$$u''(t) + a^C D^{\alpha} u(t) + bu(t) = 0, \qquad \alpha \in (0,1),\ a > 0,\ b > 0 \tag{1.12}$$

have the same asymptotic properties. This equation can be written in the form of the system (1.4) and this means that there is a chance to obtain some conditions for the exponential stability of the zero solution of a fractional perturbation of the equation (1.12), or the corresponding system, only if we consider time dependent coefficients a, b. We consider this type of equations in [3,4] with the Riemann–Liouville and Caputo–Fabrizio fractional integrals and in this paper we study equations of this type with the Riemann–Liouville substantial fractional integrals.

2 Existence result

In this section, we prove a local existence and uniqueness result concerning the initial value problem

$$\dot{x}(t) = A(t)x(t) + \mathcal{F}\Big(t, x(t), I^{(\alpha_1,\beta_1)}x(t), \dots, I^{(\alpha_m,\beta_m)}x(t)\Big), \qquad x(t_0) = x_0, \tag{2.1}$$

where $A(t)$ is a continuous matrix function and $\mathcal{F}(t, x, v_1, v_2, \dots, v_m)$ is a continuous mapping in the variables $(t, x, v_1, v_2, \dots, v_m)$ in all variables $t \geq 0, v_1, v_2, \dots, v_m \in \mathbb{R}^N$.

Theorem 2.1. *Let $G \subset \mathbb{R} \times \mathbb{R}^N$ be a region, $H_m \subset \mathbb{R}^m$ is a region with $0 \in H_m$ and $\mathcal{F} \in C(G \times H_m, \mathbb{R}^N)$ be a continuous locally Lipschitz mapping. Then for any $(t_0, x_0) \in G, t_0 \geq 0$, there exists a $\delta > 0$ such that the initial value problem (2.1) has a unique solution $x(t)$ on the interval $I_\delta = [t_0, t_0 + \delta)$.*

Proof. Let

$$\begin{aligned} G_0 = \big\{ &(t, x, u_1, \dots, u_m) \in G \times H_m : t_0 \leq t \leq t_0 + a, t_0 \geq 0, \\ &\|x - x_0\| \leq b,\ |u_i| \leq \|x_0\| + b,\ i = 1, 2, \dots, m \big\}, \end{aligned} \tag{2.2}$$

for some $a > 0, b > 0$. Let

$$\begin{aligned} M_1 &= \max_{\|x-x_0\| \leq b, t_0 \leq t \leq t_0+a} \|A(t)x\|\}, \\ M_2 &= \max_{(t,x,u_1,\dots,u_m) \in G_0} \|\mathcal{F}(t, x, u_1, \dots, u_m)\| \\ M_3 &= \max_{t_0 \leq t \leq t_0+a} \|A(t)\| \end{aligned} \tag{2.3}$$

and the mapping $\mathcal{F}$ satisfies the condition

$$\|\mathcal{F}(t, x, u_1, u_2, \dots, u_m) - \mathcal{F}(t, y, v_1, v_2, \dots, v_m)\| \leq L_0\|x - y\| + \sum_{i=1}^{m} L_i\|u_i - v_i\| \tag{2.4}$$

for all $(t, x, u_1, u_2, \dots, u_m), (t, y, v_1, v_2, \dots, v_m) \in G_0$. Let

$$0 < \delta = \min\left\{a, \frac{b}{M_1 + M_2},\ c,\ \frac{1}{M_3 + L_0 + \sum_{i=1}^{m} L_i}\right\}, \tag{2.5}$$

where $c = \min_{1\le i\le m}\left[\Gamma(\alpha_i)\alpha_i\right]^{\frac{1}{\alpha_i}}$. Let $C_\delta := C(I_\delta, \mathbb{R}^N), I_\delta = [t_0, t_0+\delta]$, be the Banach space of continuous mappings from I_δ into $\mathbb{R}^N$ endowed with the metrics $d(h,g) := \|h-g\| := \max_{t\in I_\delta}\|h(t)-g(t)\|$. Let us define the successive approximations $\{x_n\}_{n=0}^\infty$, $x_n \in C_\delta := C(I_\delta, \mathbb{R}^N)$, by

$$\begin{aligned} x_0(t) &\equiv x_0,\\ x_{n+1}(t) &= x_0 + \int_{t_0}^t A(s)x_n(s)ds\\ &\quad + \int_{t_0}^t \mathcal{F}\Big(s, x_n(s), \frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}e^{-\beta_1(s-\tau)}x_n(\tau)d\tau,\\ &\qquad \dots, \frac{1}{\Gamma(\alpha_m)}\int_0^s (s-\tau)^{\alpha_m-1}e^{-\beta_m(s-\tau)}x_n(\tau)d\tau\Big)ds,\\ &\quad t\in I_\delta,\ n = 1,2,\dots \end{aligned} \tag{2.6}$$

First, let us show that $\|x_n(t)-x_0\| \le b$ for all $n\ge 1$, $t\in I_\delta$. From the definition of the number c it follows that

$$\begin{aligned} \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}e^{-\beta_i(t-s)}ds &\le \frac{1}{\Gamma(\alpha_i)}\frac{\delta^{\alpha_i}}{\alpha_i} \le \frac{1}{\Gamma(\alpha_i)}\frac{c^{\alpha_i}}{\delta^{\alpha_i}}\\ &\le \frac{1}{\Gamma(\alpha_i)}\frac{\Gamma(\alpha_i)\alpha_i}{\alpha_i} = 1, \qquad i = 1,2,\dots,m \end{aligned} \tag{2.7}$$

and so, we have

$$\begin{aligned} \left\|\frac{1}{\Gamma(\alpha_i)}\int_{t_0}^t (t-\tau)^{\alpha_i-1}e^{-\beta_i(t-s)}x_0 d\tau\right\| &\le \frac{1}{\Gamma(\alpha_i)}\frac{\delta^{\alpha_i}}{\alpha_i}[\|x_0\|+b] \le \|x_0\|+b,\\ &\quad i = 1,2,\dots,m,\ \ t\in I_\delta. \end{aligned} \tag{2.8}$$

Hence, the first approximation $x_1(t)$ is well defined and

$$\|x_1(t)-x_0\| \le M_1\delta + M_2\delta = (M_1+M_2)\delta \le (M_1+M_2)\frac{b}{M_1+M_2} = b, \qquad t\in I_\delta. \tag{2.9}$$

This yields the inequality

$$\|x_1(t)\| \le \|x_0\| + b \quad \text{for all } t\in I_\delta \tag{2.10}$$

and thus

$$\begin{aligned} &\Big(t, x_1(t), \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\tau)^{\alpha_1-1}e^{-\beta_1(t-s)}x_1(\tau)d\tau,\\ &\quad \dots, \frac{1}{\Gamma(\alpha_m)}\int_0^t (t-\tau)^{\alpha_m-1}e^{-\beta_m(t-s)}x_1(\tau)d\tau\Big) \in G_0 \end{aligned} \tag{2.11}$$

for all $t\in I_\delta$. Now, similarly as in the proof of the existence theorem in [3] we find using the Lipschitz condition (2.4) and the inequality (2.7) that

$$\|x_2 - x_1\| \le k\delta\|x_1-x_0\|, \tag{2.12}$$

where $k = M_3 + L_0 + \sum_{i=1}^m L_i$ and one can show by induction that

$$\|x_{n+1}-x_n\| \le (k\delta)^n\|x_1-x_0\|, \qquad n = 1,2\dots \tag{2.13}$$

Since

$$x_n(t) = x_0(t) + \sum_{i=1}^{n}[x_i(t) - x_{i-1}(t)] \quad \text{with } x_0(t) \equiv x_0, \tag{2.14}$$

we obtain

$$\begin{aligned} \|x_0(t) + \sum_{i=1}^{n}[x_i(t) - x_{i-1}(t)]\| &\le \|x_0\| + \sum_{i=1}^{n} \|x_i(t) - x_{i-1}(t)\| \\ &\le \Big(\|x_0\| + \sum_{i=1}^{n}(k\delta)^i\Big)\|x_1 - x_0\|, \qquad \forall\, t \in I_\delta. \end{aligned} \tag{2.15}$$

From the definition of δ it follows that $k\delta < 1$, and so the series $\|x_0\| + \sum_{i=1}^{\infty}(k\delta)^i$ is convergent. This yields the uniform convergence of the sequence $\{x_n(t)\}_{i=0}^{\infty}$ on the interval I_δ to a continuous mapping $x \in C_\delta$, which is a unique solution of the equation (2.1). □

Corollary 2.2. *For any $x_0 \in \mathbb{R}^N$ and any $t_0 \ge 0$ there exists a maximal solution of the initial value problem* (2.1).

This corollary is a consequence of Theorem 2.1.

3 Exponential stability of fractionally perturbed ODEs with linearly bounded right-hand sides

The results described in this section, is based upon a method developed by Rigoberto Medina in the paper [9] for systems of the form (1.4) without the fractional part. We extend his results to the fractional system (1.4). We will work with the logarithmic norm $\mu(B)$, of a square $N \times N$ matrix $B = (b_{ij})$ defined by

$$\mu(B) = \lim_{\epsilon \to 0^+} \frac{\|I + \epsilon B\| - 1}{\epsilon}, \tag{3.1}$$

where I is the unit matrix and $\|\cdot\|$ is a norm on $\mathbb{R}^N$. For example,

$$\mu(B) = \mu_1(B) = \max\left\{ b_{jj} + \sum_{i \ne j}^{n} |b_{ij}| \right\}, \tag{3.2}$$

with respect to the 1-norm $\|x\| := \|x\|_1 = \sum_{i=1}^{N}|x_i|, x = (x_1, x_2, \dots, x_N)$ (see [9, Lemma 5]). We will apply the following Coppel's inequality:

$$\|e^{Bt}\| \le e^{\mu(B)t}, \qquad \forall t \ge 0. \tag{3.3}$$

To established the main results we make the following assumptions:

(H1) There are positive numbers Θ, q such that

$$\|A(t) - A(s)\| \le q|t - s|^{\Theta}, \qquad \forall t, s \ge 0, \tag{3.4}$$

where $\|\cdot\|$ denotes a norm in $\mathbb{R}^N$.

(H2) For any logarithmic norm μ, the matrix $A(t)$ satisfies

$$\rho = -\sup_{t \ge 0} \mu(A(t)) > 0. \tag{3.5}$$

(H3) For a positive constant $r \le \infty$, there is a constant $\gamma = \gamma(r)$ such that $\rho > \gamma$ and

$$\|F(t,u)\| \le \gamma\|u\|, \qquad \forall t \ge 0,\ \forall u \in \Omega(r), \tag{3.6}$$

where $\Omega(r) = \{h \in \mathbb{R}^N : \|h\| < r\}$.

(H4) There are positive constants $\eta_i, \mu_i > \rho, i = 1,2,\dots,m$ such that

$$\|f(t,v_1,v_2,\dots,v_m)\| \le \sum_{i=1}^{m} \eta_i e^{-\mu_i t}\|v_i\|, \qquad \forall t \ge 0,\ \forall v_i \in \Omega(r),\ i = 1,2,\dots,m. \tag{3.7}$$

Theorem 3.1. *Suppose that the conditions (H1)–(H4) are satisfied. In addition, let*

$$G(A(.),F,f) := q\frac{\Gamma(\Theta+1)}{\rho^{\Theta+1}} + \frac{\gamma}{\rho} + \frac{1}{\rho}\sum_{i=1}^{m}\frac{\eta_i}{\beta_i^{\alpha_i}(\mu_i-\rho)} < 1, \tag{3.8}$$

where

$$\Gamma(z) = \int_0^\infty \tau^{z-1}e^{-\tau}d\tau \tag{3.9}$$

is the Euler's Gamma function. Then the zero solution of the equation (1.4) is exponentially stable with respect to the ball $\Omega(\lambda)$, with $\lambda = r(1-G(A(\cdot),F,f))$, provided that

$$\|x(0)\| < \lambda. \tag{3.10}$$

Proof. Let $x(t)$ be a solution of the equation (1.4) on the interval $[0,T), 0 < T < \infty$ with the initial value $x(0) \in \Omega(\lambda)$. Rewrite this system in the form

$$\dot{x}(t) = A(\tau)x(t) + [A(t)-A(\tau)]x(t) + F\Big(t,x(t)\Big) + f\Big(t, I^{(\alpha_1,\beta_1)}x(t),\dots,I^{(\alpha_m,\beta_m)}x(t)\Big), \tag{3.11}$$

regarding an arbitrary $\tau \ge 0$ as fixed. Then

$$\begin{aligned} x(t) &= e^{A(\tau)t}x(0) \\ &\quad + \int_0^t e^{A(\tau)(t-s)}[A(s)-A(\tau)]x(s)ds \\ &\quad + \int_0^t e^{A(\tau)(t-s)}F(s,x(s))ds \\ &\quad + \int_0^t e^{A(\tau)(t-s)}f\Big(s, I^{(\alpha_1,\beta_1)}x(s),\dots,I^{(\alpha_m,\beta_m)}x(s)\Big)ds. \end{aligned} \tag{3.12}$$

There are two cases to consider: $r = \infty$ and $r < \infty$. First, assume that $r = \infty$. Then we obtain the relation

$$\begin{aligned} \|x(t)\| \le e^{\mu(A(\tau))t}\|x(0)\| &+ \int_0^t e^{\mu(A(\tau))(t-s)}q|s-\tau|^\Theta\|x(s)\|ds \\ &+ \int_0^t e^{\mu(A(\tau))(t-s)}\gamma\|x(s)\|ds \\ &+ \int_0^t e^{\mu(A(\tau))(t-s)}\Big(\sum_{i=1}^{m}\eta_i\big\|I^{(\alpha_i,\beta_i)}x(s)\big\|ds\Big), \end{aligned} \tag{3.13}$$

where

$$\big\|I^{(\alpha_i,\beta_i)}x(s)\big\| \le \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}e^{-\beta_i(t-s)}\|x(s)\|ds. \tag{3.14}$$

Hence, we have

$$\begin{aligned}\|x(t)\| \le{}& e^{\mu(A(\tau))t}\|x(0)\| \\ &+ \int_0^t e^{\mu(A(\tau))(t-s)} q|s-\tau|^{\Theta}\|x(s)\|ds \\ &+ \int_0^t e^{\mu(A(\tau))(t-s)}\gamma\|x(s)\|ds \\ &+ \int_0^t e^{\mu(A(\tau))(t-s)}\left(\sum_{i=1}^m \eta_i e^{-\mu_i s}\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\sigma)^{\alpha_i-1}e^{-\beta_i(s-\sigma)}\|x(\sigma)\|d\sigma\right)ds.\end{aligned} \tag{3.15}$$

Denote by $\Psi(t)$ the right-hand side of this inequality. Then

$$\begin{aligned}\|x(t)\| \le \Psi(t) \le{}& e^{\mu(A(\tau))t}\|x(0)\| \\ &+ \int_0^t e^{\mu(A(\tau))(t-s)} q|s-\tau|^{\Theta}\Psi(s)ds \\ &+ \gamma\int_0^t e^{\mu(A(\tau))(t-s)}\Psi(s)ds \\ &+ \int_0^t e^{\mu(A(\tau))(t-s)}\left(\sum_{i=1}^m \eta_i e^{-\mu_i s}\frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\sigma)^{\alpha_i-1}e^{-\beta_i(s-\sigma)}\Psi(\sigma)d\sigma\right)ds.\end{aligned} \tag{3.16}$$

Since the function $\Psi(t)$ is nondecreasing and

$$\begin{aligned}\int_0^t e^{\mu(A(\tau))(t-s)}|s-\tau|^{\Theta}ds &\le \int_0^t e^{-\rho(t-s)}|T-s|^{\Theta}ds \\ &\le \int_0^\infty e^{-\rho\zeta}\zeta^{\Theta}d\zeta \\ &= \frac{1}{\rho^{\Theta+1}}\int_0^\infty z^{\Theta}e^{-z}dz = \frac{\Gamma(\Theta+1)}{\rho^{\Theta+1}},\end{aligned} \tag{3.17}$$

$$\int_0^t e^{\mu(A(\tau))(t-s)}ds \le \int_0^\infty e^{-\rho\zeta}d\zeta = \frac{1}{\rho}, \tag{3.18}$$

$$\int_0^s (s-\sigma)^{\alpha_i-1}e^{-\beta_i(s-\sigma)}d\sigma \le \int_0^\infty \zeta^{\alpha_i-1}e^{-\beta_i\zeta}d\zeta = \frac{\Gamma(\alpha_i)}{\beta_i^{\alpha_i}}, \tag{3.19}$$

we obtain the inequality

$$\Psi(t) \le e^{-\rho t}\|x(0)\| + \left(q\frac{\Gamma(\Theta+1)}{\rho^{\Theta+1}}\right) + \frac{\gamma}{\rho} + \sum_{i=1}^m \frac{\eta_i}{\beta_i^{\alpha_i}(\mu_i-\rho)}\Bigg)\Psi(t). \tag{3.20}$$

Hence, we have the inequality

$$\Psi(t)\Big(1-G(A(\cdot),F,f)\Big) \le e^{-\rho t}\|x(0)\|, \tag{3.21}$$

i.e.

$$\|x(t)\| \le \Psi(t) \le e^{-\rho t}\Big(1-G(A(\cdot),F,f)\Big)^{-1}\|x(0)\| \qquad \forall t\in[0,T), \tag{3.22}$$

where $G(A(\cdot),F,f)$ is given by (3.8). Since the right-hand side of (3.8) is independent of T this inequality holds for all $t\in[0,\infty)$.

Hence the condition $\|x(0)\| < \lambda = r[1 - G(A(\cdot), F, f)]$, ensure the exponential stability of the solution $x(t)$ with respect to the ball $\Omega(\lambda)$.

If $r < \infty$, then using the Uryson's lemma [2, Lemma 10.2], we get the exponential stability in this case. □

4 Example 1

Let us illustrate Theorem 3.1 by the following example, which is a fractional perturbation of the [9, Example 9, p. 4]:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) \\ &\quad + F\Big(x(t)\Big) + f\Big(t, I^{(\alpha_1,\beta_1)}x(t), I^{(\alpha_2,\beta_2)}x(t)\Big), \qquad t \geq 0,\ x(t) \in R^N,\ x(t_0) = x_0,\end{aligned} \tag{4.1}$$

where

$$A(t) = \begin{bmatrix} -[a_1 + d_1(t)] & d_2(t) \\ d_1(t) & -[a_2 + d_2(t)] \end{bmatrix}, \tag{4.2}$$

where $a_1, a_2, \gamma_1, \gamma_2$ are positive constants, $d_1(t), d_2(t)$ are continuous nonnegative and bounded functions.

$$x(t) = \big(x_1(t), x_2(t)\big)^T, F(x(t)) = \big(F_1(x(t), F_2(x(t))\big) = \big(\gamma_1 x_1(t)e^{-\delta_1 x_1(t)}, \gamma_2 x_2(t)e^{-\delta_2 x_2(t)}\big), \tag{4.3}$$

where $\gamma_i, \delta_i (i = 1, 2)$ are positive constants,

$$f(t, v_1, v_2) = e^{-\mu_1 t}Bv_1 + e^{-\mu_2 t}Cv_2, \qquad v_i \in \mathbb{R}^2,\ i = 1, 2, \tag{4.4}$$

where B, C are constant 2×2 matrices and $\mu_1 > \rho, \mu_2 > \rho$ are constants.

Theorem 4.1. *Suppose that the following conditions are satisfied:*

(C1) There are positive numbers Θ, q_1, q_2 such that

$$|d_i(t) - d_i(s)| \leq q_i|t - s|^{\Theta}, \qquad \forall t, s \geq 0; \tag{4.5}$$

(C2)

$$\rho > \gamma, \tag{4.6}$$

where $\rho = \min\{a_1, a_2\}, \gamma = \max\{\gamma_1, \gamma_2\}$;

(C3) For a positive $r \leq \infty$, there is a constant $\gamma = \gamma(r)$ such that

$$\|F(u)\| \leq \gamma\|u\|, \qquad \forall t \geq 0,\ \forall u \in \Omega(r); \tag{4.7}$$

(C4)

$$S_0 = q\frac{\Gamma(\Theta+1)}{\rho^{\Theta+1}} + \frac{\gamma}{\rho} + \frac{1}{\rho}\left(\frac{\|B\|}{\beta_1^{\alpha_1}(\mu_1 - \rho)} + \frac{\|C\|}{\beta_2^{\alpha_2}(\mu_2 - \rho)}\right) < 1, \tag{4.8}$$

where $q = \max\{2q_1, 2q_2\}$.

Then the zero solution of the equation (4.1) is exponentially stable with respect to the ball $\Omega(\lambda_0)$ with $\lambda_0 = r(1 - S_0)$.

Proof. One can check that the condition (C1) yields the inequality

$$\|A(t) - A(s)\| \le q|t-s|^{\Theta} \qquad \forall t, s \ge 0, \tag{4.9}$$

i.e. the condition (H1) of Theorem 3.1 is fulfilled. By the formula [9, (44)] $\mu(A(t)) = -\rho, t \geqq 0$, where ρ is defined in (C2), the condition (H2) of Theorem 3.1 is also fulfilled. Since $F_i(x) = \gamma_i|x_i|, i = 1,2$, where $\gamma_1, \gamma_2 > 0$, $\|F(x)\| \le \gamma\|x\|$ with $\gamma = \max\{\gamma_1, \gamma_2\}$, the condition (H4) of Theorem 3.1 is fulfilled with $\eta_1 = \|B\|, \eta_2 = \|C\|$. If the condition (C4) is satisfied, then the condition formulated in Theorem 3.1 is satisfied and hence we have proved that the assertion of Theorem 4.1 is a consequence of Theorem 3.1. □

5 Exponential stability of fractionally perturbed ODEs with several power nonlinearities

In this section we consider the equation (1.4) under the following assumptions:

(G1) There are positive numbers Θ, q such that

$$\|A(t) - A(s)\| \le q|t-s|^{\Theta}, \qquad \forall t, s \ge 0; \tag{5.1}$$

(G2) For any logarithmic norm μ, the matrix $A(t)$ satisfies

$$\rho = -\sup_{t\ge 0} \mu(A(t)) > 0; \tag{5.2}$$

(G3) For a positive $r < \infty$, there are constants $\gamma = \gamma(r), \epsilon_i = \epsilon_i(r)$ such that

$$\|F(u)\| \le \gamma\|u\| + \sum_{i=1}^{m} \epsilon_i \|u\|^{\omega_i}, \qquad \forall t \ge 0,\ \forall u \in \Omega(r), \tag{5.3}$$

where $1 < \omega_1 < \omega_2 < \cdots < \omega_m$ are constants, independent of r such that

$$\omega_i \alpha_i > 1, \qquad \omega_i > \rho, \qquad i = 1, 2, \ldots, m. \tag{5.4}$$

(G4) There are positive constants $\eta_i, \xi_i, i = 1, 2, \ldots, m$ and $\mu_i, \mu_i > \rho, \nu_i, \nu_i > \rho, i = 1, 2, \ldots, m$ such that

$$\|f(t, v_1, v_2, \ldots, v_m)\| \le \sum_{i=1}^{m} \eta_i e^{-\mu_i t}\|v_i\| + \sum_{i=1}^{m} \xi_i e^{-\nu_i t}\|v_i\|^{\omega_i}, \qquad \forall t \ge 0,\ \forall v_i \in \Omega(r), \tag{5.5}$$

where $1 < \omega_1 < \omega_2 < \cdots < \omega_m$ are constants, independent of r with the additional property: $\omega_i \alpha_i > 1, \omega_i > \rho, i = 1, 2, \ldots, m$.

Theorem 5.1. *Let the conditions (G1)–(G4) be satisfied. In addition, let*

$$G(A(\cdot), F, f) := q\frac{\Gamma(\Theta+1)}{\rho^{\Theta+1}} + \frac{\gamma}{\rho} + \frac{1}{\rho}\sum_{i=1}^{m} \frac{\eta_i}{\beta_i^{\alpha_i}(\mu_i - \rho)} < 1 \tag{5.6}$$

Then the solution $x(t)$ of the initial value problem (2.1) with $t_0 = 0$ is global and

$$\|x(t)\| \le H(\|x(0)\|)e^{-\rho t} \qquad \forall t \in [0, \infty), \tag{5.7}$$

where

$$H(z) = zKD_1(z)D_2(z)\cdots D_m(z), \qquad z \in \Omega(r), \tag{5.8}$$

$$\begin{aligned} D_1(z) &= \Big(1-(\omega_1-1)(Kz)^{\omega_1-1}G_1\Big)^{-\frac{1}{\omega_1-1}} \\ D_i(z) &= \Big(1-(\omega_i-1)(D_{i-1})^{\omega_i-1}G_i\Big)^{-\frac{1}{\omega_i-1}}, \qquad i=2,3,\ldots,m, \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} G_i &= \frac{K}{\omega_i-\rho}\left(\epsilon_i + \frac{L_i}{\Gamma(\alpha_i)^{\omega_i}[\nu_i-\rho]}\right), \\ L_i &= \left(\frac{\omega_i-1}{\omega_i\beta_i}\right)^{\omega_i\beta_i-1}\left[\Gamma\left(\frac{\omega_i\alpha_i-1}{\omega_i-1}\right)\right]^{\omega_i-1}, \qquad i=1,2,3,\ldots,m, \\ K &= \Big(1-G(A(\cdot),F,f)\Big)^{-1}, \end{aligned} \tag{5.10}$$

provided $x(0) \in \Omega(r)$ *with*

$$r = \sup\{z : (\omega_i-1)D_i(z)^{\omega_i-1}G_i < 1,\ i=1,2,\ldots,m\}. \tag{5.11}$$

Proof. Let $x(t)$ be a solution of the initial value problem (2.1) with $x(0)=x_0$. Then

$$\begin{aligned} \|x(t)\| \le{}& \|x(0)\|e^{-\rho t} + q\int_0^t e^{-\rho(t-s)}|s-\tau|^{\Theta}\|x(s)\|ds + \gamma\int_0^t e^{-\rho(t-s)}\|x(s)\|ds \\ &+ \int_0^t e^{-\rho(t-s)}\left(\sum_{i=1}^m \eta_i \frac{1}{\Gamma(\alpha_i)}\int_0^s (s-\sigma)^{\alpha_i-1}e^{-\beta_i(s-\sigma)}\|x(\sigma)\|d\sigma\right)ds \\ &+ \sum_{i=1}^m \epsilon_i \int_0^t e^{-\rho(t-s)}\|x(s)\|^{\omega_i}ds \\ &+ \int_0^t e^{-\rho(t-s)}\sum_{i=1}^m \xi_i e^{-\nu_i s}\frac{1}{\Gamma(\alpha_i)^{\omega_i}}\left(\int_0^s (s-\sigma)^{\alpha_i-1}e^{-\beta_i(s-\sigma)}\|x(\sigma)\|d\sigma\right)^{\omega_i}ds. \end{aligned} \tag{5.12}$$

The first three integrals are the same as in the linear case studied in Section 4. Therefore we can apply the same procedure as in the proof of Theorem 3.1. Denote by $\Phi(t)$ the right-hand side of the inequality (5.12). Hence, if $K = \big(1-G(A(\cdot),F,f)\big)^{-1}$, then from this inequality we have

$$\begin{aligned} \|x(t)\| &\le \Phi(t) \\ &\le e^{-\rho t}K\|x(0)\| + K\sum_{i=1}^m \epsilon_i\int_0^t e^{-\rho(t-s)}\Phi(s)^{\omega_i}ds \\ &\quad + K\int_0^t e^{-\rho(t-s)}\sum_{i=1}^m \xi_i e^{-\nu_i s}\frac{1}{\Gamma(\alpha_i)^{\omega_i}}\left(\int_0^s (s-\sigma)^{\alpha_i-1}e^{-\beta_i(s-\sigma)}\Phi(\sigma)d\sigma\right)^{\omega_i}ds. \end{aligned} \tag{5.13}$$

Now, let us apply the desingularization method suggested in the paper [10] (see also [11,12]). Using the Hölder inequality with ω_i and $\kappa_i = \frac{\omega_i}{\omega_i-1}$ we obtain the estimate:

$$\begin{aligned}&\left(\int_0^s (s-\sigma)^{\alpha_i-1}e^{-\beta_i(s-\sigma)}\Phi(\sigma)d\sigma\right)^{\omega_i}\\&\qquad\le\left(\int_0^s (s-\sigma)^{\kappa_i(\alpha_i-1)}e^{-\kappa_i\beta_i(s-\sigma)}d\sigma\right)^{\frac{\omega_i}{\kappa_i}}\int_0^s\Phi(\sigma)^{\omega_i}d\sigma.\end{aligned}\tag{5.14}$$

We have the following estimate:

$$\begin{aligned}\int_0^s (s-\sigma)^{\kappa_i(\alpha_i-1)}e^{-\kappa_i\beta_i(s-\sigma)}d\sigma&=\int_0^s\eta^{\kappa_i(\alpha_i-1)}e^{-\kappa_i\beta_i\eta}d\eta\\&=\frac{1}{(\kappa_i\beta_i)^{\kappa_i(\alpha_i-1)+1}}\int_0^{\kappa_i\beta_i s}z^{\kappa_i(\alpha_i-1)}e^{-z}dz\\&\le\frac{1}{(\kappa_i\beta_i)^{\kappa_i(\alpha_i-1)+1}}\Gamma(\kappa_i(\alpha_i-1)+1).\end{aligned}\tag{5.15}$$

Since $\kappa_i=\frac{\omega_i}{\omega_i-1},\frac{\omega_i}{\kappa_i}=\omega_i-1,\kappa_i(\alpha_i-1)+1=\frac{\omega_i\alpha_i-1}{\omega_i-1},\kappa_i\beta_i=\frac{\omega_i\beta_i}{\omega_i-1}$, we obtain the inequality

$$\left(\int_0^s (s-\sigma)^{\alpha_i-1}e^{-\beta_i(s-\sigma)}\Phi(\sigma)d\sigma\right)^{\omega_i}\le L_i\int_0^s\Phi(\sigma)^{\omega_i}d\sigma,\tag{5.16}$$

where

$$L_i=\left(\frac{\omega_i-1}{\omega_i\beta_i}\right)^{\omega_i\alpha_i-1}\Gamma\left(\frac{\omega_i\alpha_i-1}{\omega_i-1}\right)^{\omega_i-1}.\tag{5.17}$$

Using this inequality we obtain from the inequality (5.13):

$$\begin{aligned}\|x(t)\|\le\Phi(t)\le e^{-\rho t}K\|x(0)\|+Ke^{-\rho t}\sum_{i=1}^m\epsilon_i\int_0^t e^{\rho s}\Phi(s)^{\omega_i}ds\\+\left(Ke^{-\rho t}\int_0^t e^{-(\nu_i-\rho)s}\sum_{i=1}^m\xi_i\frac{L_i}{\Gamma(\alpha_i)^{\omega_i}}ds\right)\int_0^t\Phi(\sigma)^{\omega_i}d\sigma.\end{aligned}\tag{5.18}$$

From this inequality it follows the following inequality for $v(t)=\Phi(t)e^{\rho t}$:

$$v(t)\le K\|x(0)\|+\sum_{i=1}^m\int_0^t F_i(s)v(s)^{\omega_i}ds,\tag{5.19}$$

where

$$F_i(t)=Ke^{-[\omega_i-\rho]t}\left(\epsilon_i+\frac{L_i}{\Gamma(\alpha_i)^{\omega_i}[\nu_i-\rho]}\right),\qquad i=1,2,\dots,m.\tag{5.20}$$

From Pinto's inequality [14], which is a generalization of the Bihari inequality [1], it follows an integral inequality, corresponding to several power nonlinearities, formulated and proved in [3] (see [3, Lemma 3.1]), we obtain the inequality:

$$v(t)\le H(\|x(0)\|),\tag{5.21}$$

where

$$H(z) = zKD_1(z)D_2(z)\dots D_m(z), \qquad z \in \Omega(r), \tag{5.22}$$

$$\begin{aligned} D_1(z) &= \Big(1-(\omega_1-1)(Kz)^{\omega_1-1}G_1\Big)^{-\frac{1}{\omega_1-1}} \\ D_i(z) &= \Big(1-(\omega_i-1)(D_{i-1})^{\omega_i-1}G_i\Big)^{-\frac{1}{\omega_i-1}}, \qquad i=2,3,\dots,m, \end{aligned} \tag{5.23}$$

where

$$G_i = \int_0^\infty F_i(s)ds = \frac{K}{\omega_i-\rho}\bigg(\epsilon_i + \frac{L_i}{\Gamma(\alpha_i)^{\omega_i}[\nu_i-\rho]}\bigg), \qquad i=1,2,\dots,m \tag{5.24}$$

and

$$r = \sup\{z : (\omega_i-1)D_i(z)^{\omega_i-1}G_i < 1,\ i=1,2,\dots,m\}. \tag{5.25}$$

This yields the inequality

$$\|x(t)\| \le \Phi(t) = v(t)e^{-\rho t} \le H(\|x(0)\|)e^{-\rho t} \qquad \forall t \in [0,\infty) \tag{5.26}$$

and since the function $H(z)$ is continuous on $\Omega(r)$ and $H(0)=0$, from the inequality (5.26) it follows that the maximal solution $x(t)$ is global and that if $x(0)\in\Omega(r)$, then $\lim_{t\to\infty}\|x(t)\| = 0$. □

6 Example 2

Consider the system (4.1) with $A(t)$ defined by (4.4), $F(x)$ defined by (4.3), $\epsilon_1 = 0$, i.e. F is linearly bounded),

$$f(t,w) = \eta e^{-\mu t}w + \xi e^{-\nu t}(w_1^2, w_2^2), \quad w=(w_1,w_2),\ \xi>0,\ \eta>0, \tag{6.1}$$

$$\|f(t,w)\| \le \eta e^{-\mu t}\|w\| + \xi e^{-\nu t}\|w\|^2, \quad \forall t\ge 0,\ w\in\mathbb{R}^2, \tag{6.2}$$

$$\begin{aligned} &m=1, \quad \Theta=1, \quad \alpha_1=\alpha=\frac{2}{3}, \quad \beta_1=\beta=2, \quad \omega_1=\omega=2, \\ &\rho=\min\{a_1,a_2\}<\omega=2, \quad \mu_1=\mu>\rho, \quad \nu_1=\nu>\rho. \end{aligned} \tag{6.3}$$

Assume that

$$\begin{aligned} G(A(\cdot),F,f) &= \big(\max\{2q_1,2q_2\}\big)\frac{1}{(\min\{a_1,a_2\})^2} + \frac{\gamma}{\min\{a_1,a_2\}} \\ &+ \frac{\eta}{2^{\frac{2}{3}}(\min\{a_1,a_2\})(\mu-\min\{a_1,a_2\})} < 1. \end{aligned} \tag{6.4}$$

We have

$$H(z) = zKD_i(z) = zK\Big(1-\xi(Kz)G_1\Big)^{-1}, \tag{6.5}$$

where $K = G(A(\cdot),F,f)^{-1}$,

$$G_1 = \frac{\xi KL_1}{(2-\min\{a_1,a_2\})\Gamma(\frac{2}{3})[\nu-\min\{a_1,a_2\}]}, \tag{6.6}$$

where

$$L_1 = L_1 = \left(\frac{\omega_1 - 1}{\omega_1\beta_1}\right)^{\omega_1\beta_1}\left[\Gamma\left(\frac{\omega_1\alpha_1 - 1}{\omega_1 - 1}\right)\right]^{\omega_1 - 1} = \left(\frac{1}{4}\right)^4\Gamma\left(\frac{4}{3}\right). \tag{6.7}$$

The function $H(z)$ is obviously defined for all $z \in \Omega(r)$ with

$$r = \sup\left\{z : |z| < \frac{1}{\xi K G_1}\right\} = \frac{G(A(\cdot), F, f)}{\xi G_1}. \tag{6.8}$$

If $x(t)$ is a solution of the initial value problem (4.1), then by Theorem 5.1

$$\|x(t)\| \le H(\|x(0)\|)e^{-(\min\{a_1,a_2\})t} \qquad \forall t \ge 0 \tag{6.9}$$

for any $x(0) \in \Omega(r)$.

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