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Classification and evolution of bifurcation curves for a one-dimensional Neumann–Robin problem and its applications

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Abstract. We study the classification and evolution of bifurcation curves of positive solutions for the one-dimensional Neumann–Robin boundary value problem

$$\begin{cases} u''(x) + \lambda f(u(x)) = 0, & 0 < x < 1, \\ u'(0) = 0 \text{ and } u'(1) + \alpha u(1) = 0, \end{cases}$$

where $\lambda > 0$ is a bifurcation parameter, $\alpha > 0$ is an evolution parameter, and nonlinearity f satisfies $f(0) \geq 0$ and $f(u) > 0$ for $u > 0$. We obtain the multiplicity of positive solutions for $\alpha > 0$ and $\lambda > 0$. Applications are given.

Keywords: bifurcation, multiplicity, positive solution, Neumann–Robin problem, S-shaped bifurcation curve, $\subset$-shaped bifurcation curve, time map.

2010 Mathematics Subject Classification: 34B18, 74G35.

1 Introduction

We study the classification and evolution of bifurcation curves of positive solutions for the one-dimensional Neumann–Robin boundary value problem

$$\begin{cases} u''(x) + \lambda f(u(x)) = 0, & 0 < x < 1, \\ u'(0) = 0 \text{ and } u'(1) + \alpha u(1) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a bifurcation parameter and $\alpha > 0$ is an evolution parameter. We basically assume that nonlinearity f satisfies the following hypothesis:

(H) $f(0) \geq 0$, $f(u) > 0$ for $0 < u < \eta$, and $f \in C[0,\eta) \cap C^2(0,\eta)$, where $\eta \in (0,\infty]$. In addition, $f(\eta) = 0$ if $0 < \eta < \infty$. (Note that we allow η to be either a finite positive number or infinite.)

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If $\alpha = 0$, (1.1) becomes a Neumann boundary value problem. In this case, for all $\lambda > 0$, it is easy to show that $u(x) \equiv \eta < \infty$ is the unique positive solution. If $\alpha = \infty$, (1.1) becomes the Neumann–Dirichlet boundary value problem

$$\begin{cases} u''(x) + \lambda f(u(x)) = 0, & 0 < x < 1, \\ u'(0) = 0 \text{ and } u(1) = 0, \end{cases}$$

which is equivalent to the Dirichlet boundary value problem

$$\begin{cases} u''(x) + \lambda f(u(x)) = 0, & -1 < x < 1, \\ u(-1) = 0 \text{ and } u(1) = 0, \end{cases} \tag{1.2}$$

due to symmetry of positive solutions u on $(-1, 1)$.

We first observe that u is a positive solution of (1.1) with fixed $\alpha > 0$. Since $u''(x) = -\lambda f(u(x)) < 0$ on $(0, 1)$ and $u'(0) = 0$, hence $u(x)$ is strictly concave and strictly decreasing on $(0, 1)$. This implies that

$$u(0) > u(1). \tag{1.3}$$

We study the classification and evolution of bifurcation curves of positive solutions of (1.1) defined by

$$S_\alpha = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)}\} \tag{1.4}$$

on the $(\lambda, \|u\|_\infty)$-plane when the evolution parameter α varies from 0^+ to infinity.

It should be noticed that shapes of bifurcation curves of positive solutions for one-dimensional Dirichlet boundary value problem (1.2) defined by

$$\bar{S} = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.2)}\} \tag{1.5}$$

on the $(\lambda, \|u\|_\infty)$-plane have been studied extensively, see e.g. [5,6,8–13] and references therein. While shapes of bifurcation curves of positive solutions for one-dimensional Neumann–Robin problem (1.1) are much less considered; see [1–4, 14]. Anuradha, Maya and Shivaji [4] first studied the existence of positive solutions for (1.1) where the parameters α and λ satisfy $\alpha, \lambda > 0$, and $f \in C^2[0, \infty)$ is strictly convex, non-decreasing, and superlinear with the positone $(f(0) > 0)$ case as well as the semipositone $(f(0) < 0)$ case. They also showed that, for each $\alpha > 0$, S_α lies on the left hand side of $\bar{S}$ on the $(\lambda, \|u\|_\infty)$-plane, see [4, Figs. 2.3 and 3.4]. Afrouzi and Khaleghy Moghaddam [1–3] studied the existence and multiplicity of positive solutions for (1.1) where the parameters α and λ satisfy $\alpha < 0 < \lambda$, and $f \in C^2[0, \infty)$ is strictly convex and increasing with the semipositone $(f(0) < 0)$ case. Yang and Yang [14] extended some results in Anuradha, Maya and Shivaji [4] by replacing u'' in (1.1) by the one-dimensional p-Laplacian operator with $p > 1$.

Before going into further discussions on problems (1.1) and (1.2), we first introduce following terminologies, which also hold for $\bar{S}$ if S_α is replaced by $\bar{S}$.

Monotone increasing and strictly increasing: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is *monotone increasing* if S_α is a continuous curve and for each pair of points $(\lambda_1, \|u_{\lambda_1}\|_\infty)$ and $(\lambda_2, \|u_{\lambda_2}\|_\infty)$ of S_α, $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ implies $\lambda_1 \le \lambda_2$, and it is *strictly increasing* if $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ implies $\lambda_1 < \lambda_2$.

Monotone decreasing and strictly decreasing: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is *monotone decreasing* if S_α is a continuous curve and for each pair of points $(\lambda_1, \|u_{\lambda_1}\|_\infty)$ and $(\lambda_2, \|u_{\lambda_2}\|_\infty)$ of S_α, $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ implies $\lambda_1 \ge \lambda_2$, and it is *strictly decreasing* if $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ implies $\lambda_1 > \lambda_2$.

S-shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is *S-shaped* if S_α has *at least two* turning points, say $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, satisfying $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty$, and

- **(i)** at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve S_α turns to the *left*,
- **(ii)** at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the bifurcation curve S_α turns to the *right*,
- **(iii)** S_α initially continues to the *right* and eventually continues to the *right*.

Exactly S-shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is *exactly S-shaped* if S_α is S-shaped and it has *exactly two* turning points; see Fig. 1.1 (I).

Reversed S-shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is *reversed S-shaped* if S_α has *at least two* turning points, say $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, satisfying $\lambda_* < \lambda^*$ and $\|u_{\lambda^*}\|_\infty > \|u_{\lambda_*}\|_\infty$, and

- **(i)** at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve S_α turns to the *left*,
- **(ii)** at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$ the bifurcation curve S_α turns to the *right*,
- **(iii)** S_α initially continues to the *left* and eventually continues to the *left*.

Exactly reversed S-shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is *exactly reversed S-shaped* if S_α is reversed S-shaped and it has *exactly two* turning points.

$\subset$-shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is *$\subset$-shaped* if S_α has *at least one* turning point, say $(\lambda^*, \|u_{\lambda^*}\|_\infty)$, satisfying

- **(i)** at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve S_α turns to the *right*,
- **(ii)** S_α initially continues to the *left* and eventually continues to the *right*.

Exactly $\subset$-shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is *exactly $\subset$-shaped* if S_α is $\subset$-shaped and it has *exactly one* turning point; see Fig. 1.1(II).

Reversed $\subset$-shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is *$\subset$-shaped* if S_α has *at least one* turning point, say $(\lambda^*, \|u_{\lambda^*}\|_\infty)$, satisfying

- **(i)** at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ the bifurcation curve S_α turns to the *left*,
- **(ii)** S_α initially continues to the *right* and eventually continues to the *left*.

Exactly reversed $\subset$-shaped: We say that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is *exactly reversed $\subset$-shaped* if S_α is reversed $\subset$-shaped and it has *exactly one* turning point.

In Section 2 below, we study the classification and evolution of bifurcation curves S_α of positive solutions for (1.1) with general nonlinearity f satisfying hypothesis (H). In addition, as applications, we study the classification and evolution of bifurcation curves S_α of positive solution for (1.1) with two particular nonlinearities

$$f(u) = \exp\left(\frac{u}{1+\varepsilon u}\right), \qquad \varepsilon > 0 \tag{1.6}$$

and

$$f(u) = u\,(1 - \sin u) + u^p, \qquad p \geq 1 \tag{1.7}$$

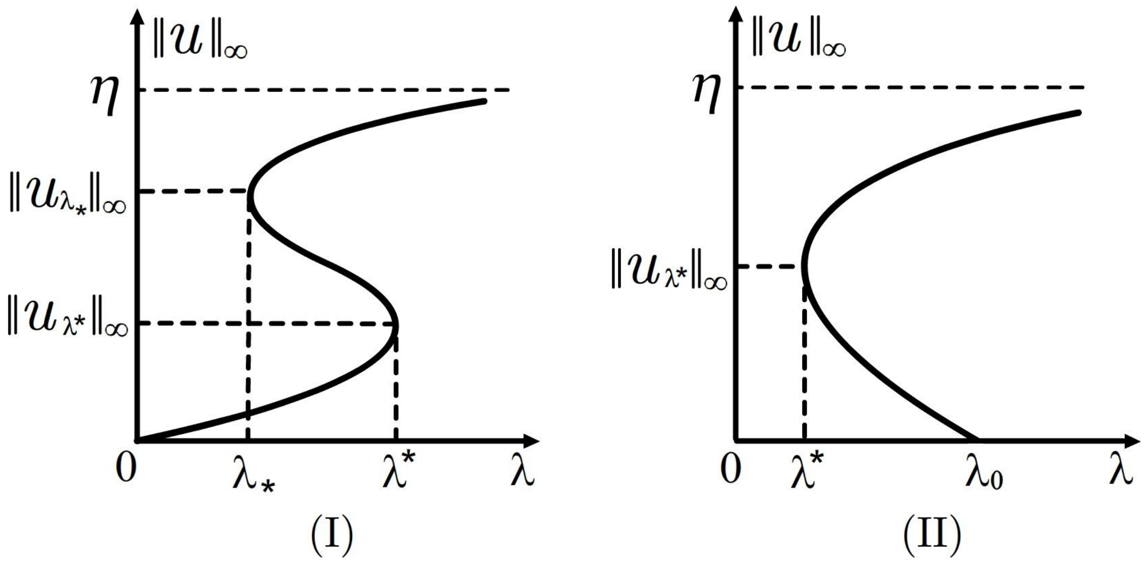

Figure 1.1: (I) Exactly S-shaped bifurcation curve S_α with exactly two turning points, which starts at $(0, 0)$ and goes to infinity along the horizontal line $\|u\|_\infty = \eta$. (II) Exactly $\subset$-shaped bifurcation curve S_α with exactly one turning point, which starts at $(\lambda_0, 0)$ and goes to infinity along the horizontal line $\|u\|_\infty = \eta$.

which satisfy hypothesis (H) with $\eta = \infty$.

Huang and Wang [6] studied the evolution and qualitative behaviors of bifurcation curves $\bar{S}$ of positive solutions of one-dimensional perturbed Gelfand problem (1.2), (1.6).

Theorem 1.1 ([6, Theorem 2.1 and Fig. 1])**.** *Consider* (1.2), (1.6). *Then there exists a positive* $\tilde{\varepsilon} \approx 0.2457$ *such that, on the* $(\lambda, \|u\|_\infty)$*-plane, such that the following assertions (i)–(iii) holds:*

(i) For $0 < \varepsilon < \tilde{\varepsilon}$*, the bifurcation curve* $\bar{S}$ *is exactly S-shaped.*

(ii) For $\varepsilon = \tilde{\varepsilon}$*, the bifurcation curve* $\bar{S}$ *is strictly increasing and* (1.2), (1.6) *has exactly one degenerate positive solution.*

(iii) For $\varepsilon > \tilde{\varepsilon}$*, the bifurcation curve* $\bar{S}$ *is strictly increasing and all positive solutions of* (1.2), (1.6) *are nondegenerate.*

Wang [13] studied the evolution and qualitative behaviors of bifurcation curves $\bar{S}$ of positive solution of Dirichlet problem (1.2), (1.7).

Theorem 1.2 ([13, Theorem 2.1 and Figs. 1–2])**.** *Consider* (1.2), (1.7). *Then, on the* $(\lambda, \|u\|_\infty)$*-plane, the bifurcation curve* $\bar{S}$ *satisfies the following assertions (i)–(iv).*

(i) For $p = 1$*, the bifurcation curve* $\bar{S}$ *starts at* $\left(\frac{\pi^2}{8}, 0\right)$ *and goes to infinity oscillationally along the vertical line* $\lambda = \frac{\pi^2}{8}$*, and it has infinitely many turning points.*

(ii) For $1 < p < 2$*, the bifurcation curve* $\bar{S}$ *starts at* $\left(\frac{\pi^2}{4}, 0\right)$ *and goes to infinity along the vertical line* $\lambda = 0$*, and it is reversed S-shaped.*

(iii) *For $p = 2$, the bifurcation curve $\bar{S}$ starts at $\left(\frac{\pi^2}{4}, 0\right)$ and goes to infinity along the vertical line $\lambda = 0$, and it is strictly decreasing.*

(iv) *For $p > 2$, the bifurcation curve $\bar{S}$ starts at $\left(\frac{\pi^2}{4}, 0\right)$ and goes to infinity along the vertical line $\lambda = 0$, and it is exactly reversed $\subset$-shaped.*

The paper is organized as follows. Section 2 contains statements of main results. Section 3 contains lemmas needed to prove the main results. Finally, Section 4 contains the proofs of the main results.

2 Main results

The main results in this paper are next Theorems 2.1–2.6 and 2.8. In Theorems 2.1 and 2.3 for (1.1), under (H) for nonlinearity f, for all $\alpha > 0$, we present some basic properties of bifurcation curves S_α on the $(\lambda, \|u\|_\infty)$-plane. In particular, in Theorem 2.1(iii), we show that, on the $(\lambda, \|u\|_\infty)$-plane, S_α moves to right strictly as α increases and S_α tends to the $\|u\|_\infty$-axis as α approaches 0^+ and tends to $\bar{S}$ as α approaches infinity. In Theorem 2.1 (iv), we prove an interesting comparison result, cf. Remark 2.2 stated behind. In Theorems 2.4 and 2.5, under (H) and some suitable hypotheses on f, for $\alpha > 0$, we give a classification of bifurcation curves S_α on the $(\lambda, \|u\|_\infty)$-plane. In Theorems 2.6 and 2.8, as applications of Theorems 2.1–2.5, we study the classification and evolution of bifurcation curves S_α for problem (1.2), (1.6) and problem (1.2), (1.7), respectively with α varying from 0^+ to infinity.

We first define the number $\lambda^* = \lambda^*(\alpha) \in \left(0, \frac{\pi^2}{4}\right)$ satisfying

$$\alpha = \sqrt{\lambda^*} \tan \sqrt{\lambda^*}. \tag{2.1}$$

Notice that λ^* is a strictly increasing function of $\alpha > 0$.

Theorem 2.1. *Consider (1.1) with fixed $\alpha > 0$. Assume that f satisfies (H). Then the bifurcation curve S_α is a continuous curve on the $(\lambda, \|u\|_\infty)$-plane. Moreover, on the $(\lambda, \|u\|_\infty)$-plane, S_α satisfies the following assertions (i)–(iv).*

(i) *If there exist $s_0 \geq 0$ and $0 < L_0 < \infty$ such that*

$$\lim_{u \to 0^+} \frac{f(u)}{u^{s_0}} = L_0,$$

then S_α starts from infinity along the horizontal line $\|u\|_\infty = 0$ if $s_0 > 1$, from the point $\left(\frac{\lambda^*}{2L_0}, 0\right)$ if $s_0 = 1$, and from the origin $(0,0)$ if $0 \leq s_0 < 1$.

(ii) *(a) If $\eta = \infty$ and there exist $s_\infty \geq 0$ and $0 < L_\infty < \infty$ such that*

$$\lim_{u \to \infty} \frac{f(u)}{u^{s_\infty}} = L_\infty, \tag{2.2}$$

then S_α goes to infinity along the vertical line $\lambda = 0$ if $s_\infty > 1$, to infinity along the vertical line $\lambda = \frac{\lambda^*}{2L_\infty}$ if $s_\infty = 1$, and to infinity as $\lambda \to \infty$ if $0 \leq s_\infty < 1$.

(b) If $0 < \eta < \infty$, $f(\eta) = 0$ and there exist $s_\eta > 0$ and $L_\eta > 0$ such that

$$\lim_{u \to \eta^-} \frac{f(u)}{(\eta - u)^{s_\eta}} = L_\eta,$$

then S_α goes to infinity along the horizontal line $\|u\|_\infty = \eta$ if $s_\eta \geq 1$, and ends at some point (λ_η, η) with $\lambda_\eta > 0$ if $0 < s_\eta < 1$.

(iii) *For any two positive numbers* $\alpha_1 < \alpha_2$, S_{α_2} *lies strictly on the right hand side of* S_{α_1} *on the* $(\lambda, \|u\|_\infty)$*-plane (So* $S_{\alpha_1} \cap S_{\alpha_2} = \varnothing$*). In addition,*

(a) As α *approaches* 0^+, S_α *tends to the positive* $\|u\|_\infty$*-axis if* $\eta = \infty$ *and* S_α *tends to the segment (on the positive* $\|u\|_\infty$*-axis) connecting the origin* $(0,0)$ *and the point* $(0,\eta)$ *if* $0 < \eta < \infty$.

(b) As α *approaches infinity,* S_α *tends to* $\bar{S}$.

(iv) *(See Fig. 2.1.) Consider (1.1) with* $f = f_1$ *and* $f = f_2$ *both satisfying (H), and denote their bifurcation curves by* $S_{\alpha,1}$ *and* $S_{\alpha,2}$ *respectively. If* $f_2(u) \geq f_1(u)$ *for* $0 < u < \eta$*, then* $S_{\alpha,2}$ *lies on the left hand side of (possibly coincides with)* $S_{\alpha,1}$ *on the* $(\lambda, \|u\|_\infty)$*-plane. In particular, if there exist two positive* $a < b < \eta$ *such that*

$$\begin{cases} f_2(u) = f_1(u) \quad \text{on} \begin{cases} [0,a] \cup [b,\eta] & \text{if } \eta < \infty, \\ [0,a] \cup [b,\eta) & \text{if } \eta = \infty, \end{cases} \\ f_2(u) > f_1(u) \quad \text{on } (a,b), \end{cases} \tag{2.3}$$

then, on the $(\lambda, \|u\|_\infty)$*-plane,* $S_{\alpha,2}$ *coincide with* $S_{\alpha,1}$ *in the region*

$$\{(\lambda, \|u\|_\infty) : \lambda > 0 \text{ and } \|u\|_\infty \in (0,a] \cup [(\alpha+1)\,b, \eta)\}$$

if $(\alpha+1)\,b < \eta$ *and in the region*

$$\{(\lambda, \|u\|_\infty) : \lambda > 0 \text{ and } \|u\|_\infty \in (0,a]\}$$

if $(\alpha+1)\,b \geq \eta$*, and* $S_{\alpha,2}$ *lies strictly on the left hand side of* $S_{\alpha,1}$ *in the striped region* $\{(\lambda, \|u\|_\infty) : \lambda > 0 \text{ and } \|u\|_\infty \in (a,b)\}$.

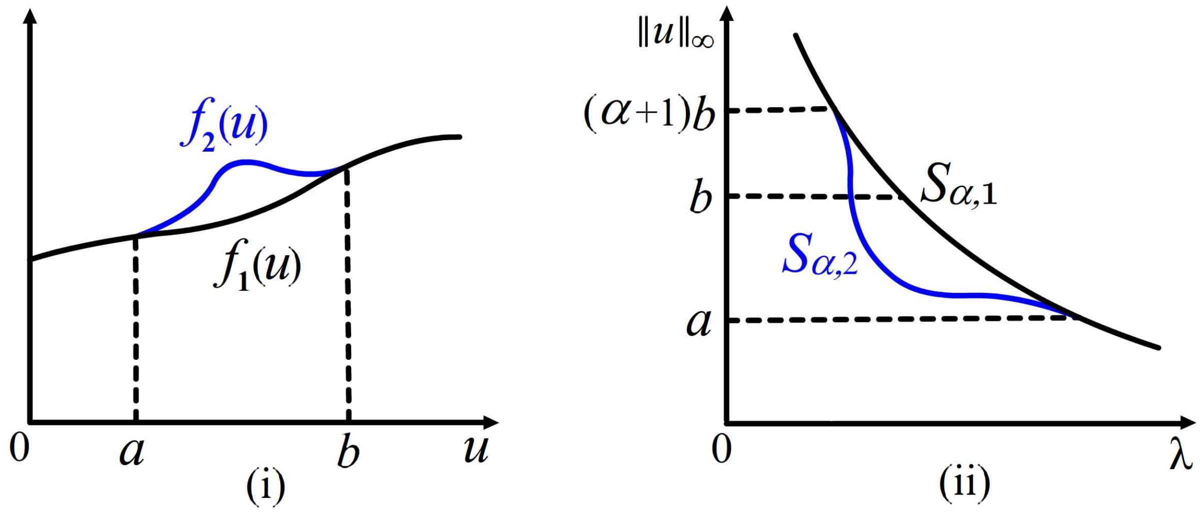

Figure 2.1: (I) Possible graphs of f_1 and f_2 satisfying $f_1(u) = f_2(u) > 0$ on $[0,a] \cup [b,\infty)$ and $f_2(u) > f_1(u) > 0$ on (a,b) with some positive $a < b < \eta = \infty$. (II) Possible corresponding bifurcation curves $S_{\alpha,1}$ and $S_{\alpha,2}$ on the $(\lambda, \|u\|_\infty)$-plane.

Remark 2.2 (Cf. Theorem 2.1 (iv) for Neumann–Robin problem (1.1))**.** Consider Dirichlet problem (1.2) with $f = f_1$ and $f = f_2$ both satisfying (H) and (2.3), and denote their bifurcation curves by $\bar{S}_1$ and $\bar{S}_2$ respectively. Then it is well-known that, on the $(\lambda, \|u\|_\infty)$-plane, $\bar{S}_2$ coincide with $\bar{S}_1$ in the striped region $\{(\lambda, \|u\|_\infty) : \lambda > 0 \text{ and } \|u\|_\infty \in (0, a]\}$ and $\bar{S}_2$ lies strictly on the left hand side of $\bar{S}_1$ in the striped region $\{(\lambda, \|u\|_\infty) : \lambda > 0 \text{ and } \|u\|_\infty \in (a, \eta)\}$.

Theorem 2.3. *Consider* (1.1). *Assume that f satisfies (H). If $f(u) - uf'(u)$ changes sign exactly k times in an interval $(a, b) \subset (0, \eta)$ with some positive $a < b \leq \eta$. Then there exist k positive numbers*

$$(0 <)\ \alpha_k < \cdots < \alpha_2 < \alpha_1$$

such that, if $0 < \alpha \leq \alpha_i$ for $i \in \{1, 2, \ldots, k\}$, then the bifurcation curve S_α has at least i turning points on the $(\lambda, \|u\|_\infty)$-plane.

For the sake of convenience, we assume the following conditions.

(C1$^+$) $f(u) - uf'(u) \geq (\not\equiv)\ 0$ on $(0, \beta_1)$ with some $\beta_1 \in (0, \eta)$.

(C1$^-$) $f(u) - uf'(u) \leq (\not\equiv)\ 0$ on $(0, \beta_1)$ with some $\beta_1 \in (0, \eta)$.

(C2$^+$) If $\eta = \infty$, $f(u) - uf'(u) \geq (\not\equiv)\ 0$ on (β_2, η) with some $\beta_2 \in (0, \eta)$. If $\eta < \infty$, the number s_η defined in Theorem 2.1(ii)(b) is equal to or larger than 1.

(C2$^-$) If $\eta = \infty$, $f(u) - uf'(u) \leq (\not\equiv)\ 0$ on (β_2, η) with some $\beta_2 \in (0, \eta)$.

(D1) (See Fig. 2.2 (I).) There exist two positive $p^* < p < \eta$ such that

$$f(u) - uf'(u) \begin{cases} > 0, & \text{if } 0 < u < p^*, \\ = 0, & \text{if } u = p^* \text{ and } u = p, \\ < 0, & \text{if } p^* < u < p. \end{cases}$$

(D2) (See Fig. 2.2 (II).) There exist two positive $p^* < p < \eta$ such that

$$f(u) - uf'(u) \begin{cases} < 0, & \text{if } 0 < u < p^*, \\ = 0, & \text{if } u = p^* \text{ and } u = p, \\ > 0, & \text{if } p^* < u < p. \end{cases}$$

Theorem 2.4. *Consider* (1.1). *Assume that f satisfies (H). Then, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α satisfies the following assertions (i)–(iv).*

(i) If $f(u) - uf'(u) > 0$ (resp. $f(u) - uf'(u) < 0$) almost everywhere for $u > 0$, then S_α is strictly increasing (resp. strictly decreasing) for all $\alpha > 0$.

(ii) If $f(u)$ satisfies (C1$^+$), (C2$^+$) and (D1) (resp. (C1$^-$), (C2$^-$) and (D2)) with positive $p^* < p < \eta$ and $\int_0^p s\,[f(s) - sf'(s)]\,ds < 0$ (resp. $\int_0^p s\,[f(s) - sf'(s)]\,ds > 0$). Then S_α is S-shaped (resp. reversed S-shaped) for all $\alpha > 0$.

(iii) (See Fig. 2.3.) If $f(u)$ satisfies (C1$^+$), (C2$^+$) and (D1) (resp. (C1$^-$), (C2$^-$) and (D2)) with positive $p^* < p < \eta$ and the following conditions (3a)–(3b) hold.

(3a) The bifurcation curve $\bar{S}$ of (1.2) *is S-shaped (resp. reversed S-shaped),*

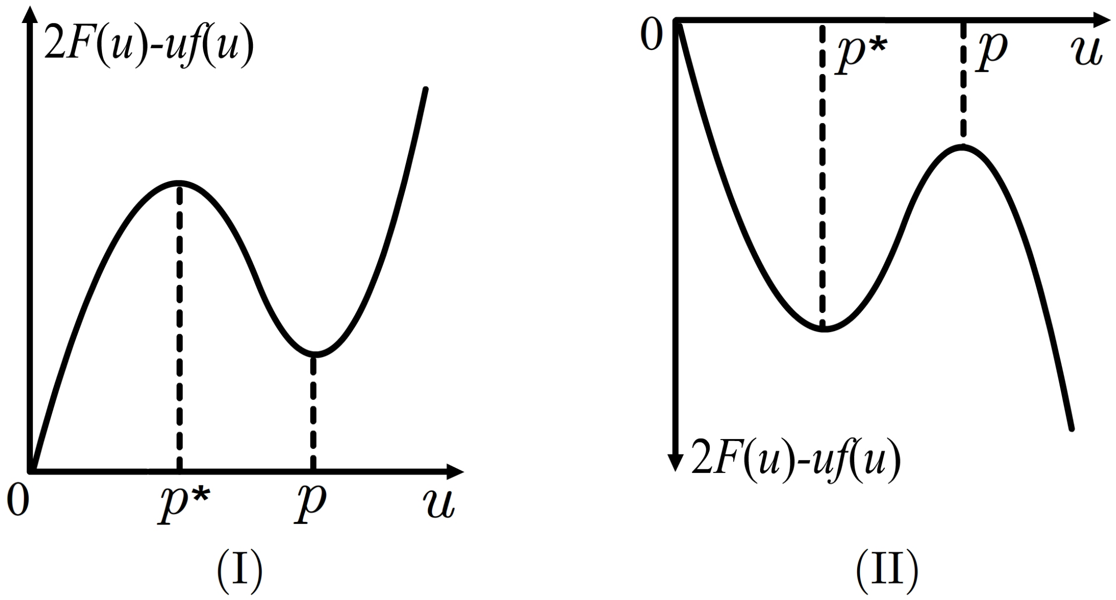

Figure 2.2: (I) Graph of $2F(u) - uf(u)$ satisfying condition (D1). (II) Graph of $2F(u) - uf(u)$ satisfying condition (D2).

(3b) $p^* < \|u_{\lambda_1}\|_\infty < p < \|u_{\lambda_2}\|_\infty$ *where* $(\lambda_1, \|u_{\lambda_1}\|_\infty)$ *and* $(\lambda_2, \|u_{\lambda_2}\|_\infty)$ *are turning points of* $\bar{S}$ *such that* $\bar{S}$ *turns to the left (resp. to the right) at* $(\lambda_1, \|u_{\lambda_1}\|_\infty)$ *and* $\bar{S}$ *turns to the right (resp. to the left) at* $(\lambda_2, \|u_{\lambda_2}\|_\infty)$.

Then S_α *is S-shaped (resp. reversed S-shaped) for all* $\alpha > 0$. *Furthermore, there exist two points* $(\lambda_3, \|u_{\lambda_3}\|_\infty)$ *and* $(\lambda_4, \|u_{\lambda_4}\|_\infty)$ *on* S_α *such that* $\|u_{\lambda_3}\|_\infty \le \|u_{\lambda_1}\|_\infty < p \le \|u_{\lambda_4}\|_\infty$ *and the portion of* S_α *connecting* $(\lambda_3, \|u_{\lambda_3}\|_\infty)$ *and* $(\lambda_4, \|u_{\lambda_4}\|_\infty)$ *is monotone decreasing (resp. monotone increasing), where* $(\lambda_3, \|u_{\lambda_3}\|_\infty)$ *is a turning point to the left (resp. to the right) of* S_α *and* $(\lambda_4, \|u_{\lambda_4}\|_\infty)$ *is a turning point to the right (resp. to the left) of* S_α.

(iv) If $f(u)$ *satisfies (C1*$^-$*) and (C2*$^+$*) (resp. (C1*$^+$*) and (C2*$^-$*)), then* S_α *is* $\subset$*-shaped (resp. reversed* $\subset$*-shaped) for all* $\alpha > 0$.

Theorem 2.5. *Consider (1.1). Assume that* f *satisfies (H). Then, on the* $(\lambda, \|u\|_\infty)$*-plane, the bifurcation curve* S_α *satisfies the following assertions (i)–(iii).*

(i) If $f(u)$ *satisfies (C1*$^+$*) and (C2*$^+$*) and* $f(\tilde{p}) - \tilde{p}f'(\tilde{p}) < 0$ *with some* $\tilde{p} \in (0, \eta)$*, then* S_α *is S-shaped for* $\alpha > 0$ *small enough.*

(ii) If $f(u)$ *satisfies (C1*$^-$*) and (C2*$^-$*) and* $f(\tilde{p}) - \tilde{p}f'(\tilde{p}) > 0$ *with some* $\tilde{p} \in (0, \eta)$*, then* S_α *is reversed S-shaped for* $\alpha > 0$ *small enough.*

(iii) Assume that $f(u)$ *satisfies (C1*$^+$*) (resp. (C1*$^-$*)) with some* $\beta_1 \in (0, \eta)$*, the bifurcation curve* $\bar{S}$ *of (1.2) is strictly increasing (resp. strictly decreasing) on the* $(\lambda, \|u\|_\infty)$*-plane, and (1.2) has no degenerate solution. Then, on the* $(\lambda, \|u\|_\infty)$*-plane, for every* $\rho_0 \in (\beta_1, \eta)$*, there exists* $\alpha^* = \alpha^*(\rho_0) > 0$ *such that, for* $\alpha \ge \alpha^*$*,* S_α *is strictly increasing (resp. strictly decreasing) in the striped region* $\{(\lambda, \|u\|_\infty) : \lambda > 0 \text{ and } \|u\|_\infty \in (0, \rho_0]\}$*. In addition, assume that there exists a constant* $\bar{\rho}_0 \in (\beta_1, \eta)$ *such that*

$$0 < \theta(u) < \theta(\bar{\rho}_0) < \theta(\bar{u}_1) < \theta(\bar{u}_2) \text{ for } 0 < u < \bar{\rho}_0 < \bar{u}_1 < \bar{u}_2 < \eta, \tag{2.4}$$

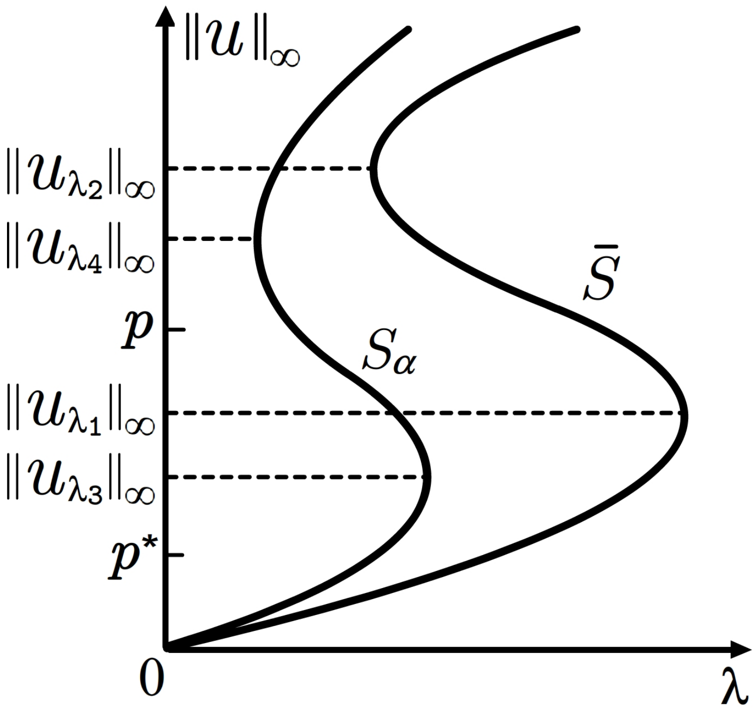

Figure 2.3: Illustration of S-shaped bifurcation curves S_α and $\bar{S}$ in Theorem 2.4 (iii).

(resp.

$$0 > \theta(u) > \theta(\bar{\rho}_0) > \theta(\bar{u}_1) > \theta(\bar{u}_2) \text{ for } 0 < u < \bar{\rho}_0 < \bar{u}_1 < \bar{u}_2 < \eta), \tag{2.5}$$

where $\theta(x) \equiv 2\int_0^x f(s)ds - xf(x)$*. Then, for* $\alpha \geq \alpha^*$*,* S_α *is strictly increasing (resp. strictly decreasing) on the* $(\lambda, \|u\|_\infty)$*-plane.*

Theorem 2.6 (See Fig. 2.4.)**.** *Consider* (1.1), (1.6)*. Suppose that*

$$\varepsilon_* \equiv \frac{1}{4.25}\ (\approx 0.235) < \tilde{\varepsilon} < \varepsilon^* \equiv \frac{1}{4} = 0.25,$$

where the number $\tilde{\varepsilon} \approx 0.2457$ *exists in Theorem 1.1. Then, on the* $(\lambda, \|u\|_\infty)$*-plane, the bifurcation curve* S_α *satisfies the following assertions (i)–(iii).*

(i) For $0 < \varepsilon < \tilde{\varepsilon}$*,* S_α *is S-shaped for all* $\alpha > 0$*.*

(ii) (a) For $\tilde{\varepsilon} \leq \varepsilon < \varepsilon^*$*,* S_α *is S-shaped for* $0 < \alpha \leq \alpha_*(\varepsilon) \equiv \frac{p_2}{p_1} - 1$ *where*

$$p_1 = \frac{1 - 2\varepsilon - \sqrt{1 - 4\varepsilon}}{2\varepsilon^2} < p_2 = \frac{1 - 2\varepsilon + \sqrt{1 - 4\varepsilon}}{2\varepsilon^2},$$

are two positive zeros of quadratic polynomial $\varepsilon^2 u^2 + (2\varepsilon - 1)u + 1$*. The term* $\alpha_*(\varepsilon) = \frac{p_2}{p_1} - 1 = \frac{2\sqrt{1-4\varepsilon}}{1-2\varepsilon-\sqrt{1-4\varepsilon}}$ *is a strictly decreasing function of* $\varepsilon \in (\tilde{\varepsilon}, \varepsilon^*)$ *and satisfies*

$$\lim_{\varepsilon \to (\tilde{\varepsilon})^+} \alpha_*(\varepsilon) \approx 1.690 \quad \text{and} \quad \lim_{\varepsilon \to (\varepsilon^*)^-} \alpha_*(\varepsilon) = 0. \tag{2.6}$$

(b) For $\tilde{\varepsilon} < \varepsilon < \varepsilon^*$*,* S_α *is strictly increasing for* $\alpha \geq \alpha^*(\varepsilon)$ *with some* $\alpha^*(\varepsilon) > 0$*.*

(iii) For $\varepsilon \geq \varepsilon^*$*,* S_α *is strictly increasing for all* $\alpha > 0$*.*

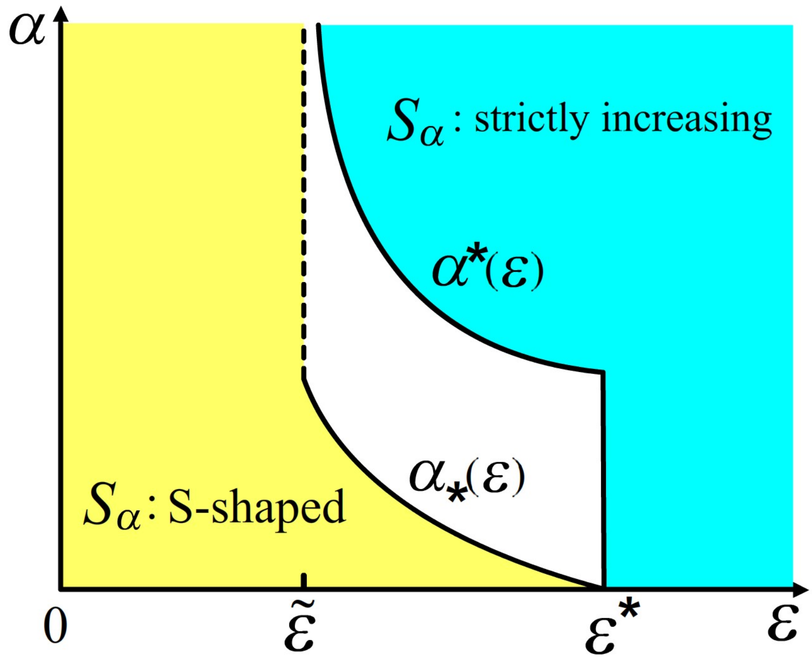

Figure 2.4: Classification of bifurcation curves S_α on the first quadrant of (ε, α)-plane for Theorem 2.6.

Conjecture 2.7. *We conjecture that, in Theorem 2.6 (ii) for (1.1), (1.6), when $\varepsilon = \tilde{\varepsilon} \approx 0.2457$, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is S-shaped for* all $\alpha > 0$. *Further investigation is needed.*

Theorem 2.8 (Cf. Theorem 1.2 for (1.2), (1.7))**.** *Consider (1.1), (1.7). Then, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curves S_α satisfies the following assertions (i)–(iv).*

(i) For $p = 1$, S_α has infinitely many turning points for all $\alpha > 0$.

(ii) For $1 < p < 2$, for any positive integer k, there exist k positive numbers

$$(0 <)\ \alpha_k < \cdots < \alpha_2 < \alpha_1$$

such that S_α has at least i turning points for $0 < \alpha \leq \alpha_i$ for $i \in \{1, 2, \ldots, k\}$.

(iii) For $p = 2$, S_α is strictly decreasing for all $\alpha > 0$.

(iv) For $p > 2$, S_α is $\subset$-shaped for all $\alpha > 0$.

Remark 2.9. For $1.9 \leq p \leq 2$, numerical simulations show that the bifurcation curve $\bar{S}$ for Dirichlet problem (1.2), (1.7) is *strictly decreasing* on the $(\lambda, \|u\|_\infty)$-plane, cf. [13, Fig. 2]. However, by Theorem 2.8 (ii), on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α for Neumann–Robin problem (1.1), (1.7) can have *arbitrarily many turning points* as desired for $\alpha > 0$ small enough.

3 Lemmas

To prove our main results for one-dimensional Neumann–Robin problem (1.1), we develop some new time-map techniques which time-map technique was used in Anuradha, Maya, and Shivaji [4]. We first define the following functions

$$F(u)=\int_0^u f(s)ds,\quad G(m,\rho)=\int_{\frac{m}{\alpha}}^{\rho}\frac{ds}{\sqrt{F(\rho)-F(s)}},\quad H(m,\rho)=\frac{m}{\sqrt{F(\rho)-F(\frac{m}{\alpha})}},\tag{3.1}$$

where $\rho\equiv u(0)=\|u\|_\infty>0$ and $m\equiv -u'(1)=\alpha u(1)\in(0,\alpha u(0))=(0,\alpha\rho)$, see [4, pp. 95–97]. We have the following properties for functions G and H in the next lemma.

Lemma 3.1 ([4, Section 2 and Theorem 2.1])**.** *Consider (1.1). Assume that f satisfies (H). Then the following assertions (i)–(iii) hold.*

(i) For fixed $\rho>0$, $G(m,\rho)$ is a strictly decreasing function of m on $(0,\alpha\rho)$. Furthermore

$$\lim_{m\to 0^+}G(m,\rho)=\int_0^{\rho}\frac{ds}{\sqrt{F(\rho)-F(s)}}>0\quad\text{and}\quad \lim_{m\to(\alpha\rho)^-}G(m,\rho)=0.$$

(ii) For fixed $\rho>0$, $H(m,\rho)$ is a strictly increasing function of m on $(0,\alpha\rho)$. Furthermore

$$\lim_{m\to 0^+}H(m,\rho)=0\quad\text{and}\quad \lim_{m\to(\alpha\rho)^-}H(m,\rho)=\infty.$$

(iii) For any $\rho>0$, there exits a unique $m=m_\rho=m_{\alpha,\rho}\in(0,\alpha\rho)$ such that $G(m_\rho,\rho)=H(m_\rho,\rho)$. (Later, for simplicity, we usually write m_ρ instead of $m_{\alpha,\rho}$ unless necessary.)

By Lemma 3.1 (iii), we see that, if $u(0)=\|u\|_\infty=\rho>0$, then (1.1) has a unique positive solution u. Since $u'(1)+\alpha u(1)=0$, we have that $u'(1)=-m_\rho<0$ and $u(1)=\frac{m_\rho}{\alpha}>0$; see [4, p. 96]. Furthermore, we define the *time-map function* for one-dimensional Neumann–Robin problem (1.1)

$$T(\rho)=\int_{\frac{m_\rho}{\alpha}}^{\rho}\frac{ds}{\sqrt{F(\rho)-F(s)}}=\frac{m_\rho}{\sqrt{F(\rho)-F(\frac{m_\rho}{\alpha})}}.\tag{3.2}$$

Then by [4, Theorem 2.1],

$$T(\rho)=\sqrt{2\lambda(\rho)},\tag{3.3}$$

and hence by (1.4), we have the bifurcation curve of positive solutions for (1.1)

$$S_\alpha=\left\{\left(\frac{1}{2}\left[T(\rho)\right]^2,\rho\right):\rho\in(0,\eta)\right\}.\tag{3.4}$$

We also define the *time-map function* for one-dimensional Dirichlet problem (1.2)

$$\bar{T}(\rho)=\int_0^{\rho}\frac{ds}{\sqrt{F(\rho)-F(s)}},\tag{3.5}$$

and we have similar results that

$$\bar{T}(\rho)=\sqrt{2\lambda(\rho)}$$

and by (1.5) the bifurcation curve of positive solutions for (1.2)

$$\bar{S}=\left\{\left(\frac{1}{2}\left[\bar{T}(\rho)\right]^2,\rho\right):\rho\in(0,\eta)\right\};\tag{3.6}$$

see e.g. [6, Eq. (11)].

Lemma 3.2. *Consider (1.1). Fix ρ $(= u(0)) > 0$ and consider $\frac{m_\rho}{\alpha}$ $(= u(1))$ as a function of $\alpha > 0$. Then the following assertions (i)–(iii) hold.*

(i) $\frac{m_\rho}{\alpha}$ *is strictly decreasing on* $(0,\infty)$, $\lim_{\alpha\to 0^+} \frac{m_\rho}{\alpha} = \rho$, *and* $\lim_{\alpha\to\infty} \frac{m_\rho}{\alpha} = 0$.

(ii) For each $\alpha > 0$ *and* $\rho \in (0,\eta)$,

$$\frac{\rho}{\alpha+1} < \frac{m_\rho}{\alpha} < \rho \quad \left(\text{that is, } \frac{u(0)}{\alpha+1} < u(1) < u(0)\right). \tag{3.7}$$

(iii) For $s_0 \geq 0$ *defined in Theorem 2.1 (i) and* $s_\infty \geq 0$ *defined in Theorem 2.1 (ii) (a) with* $\eta = \infty$,

$$\lim_{\rho\to 0^+} \frac{m_\rho}{\alpha\rho} < 1 \tag{3.8}$$

and

$$\lim_{\rho\to\infty} \frac{m_\rho}{\alpha\rho} < 1. \tag{3.9}$$

Proof. **(I)** First, we rewrite functions H and G in (3.1) as functions of α, $\frac{m}{\alpha}$ $\left(= \frac{m_\rho}{\alpha} = \frac{m_{\alpha,\rho}}{\alpha}\right) = k$ and ρ, and we obtain that

$$G = G(\alpha,k,\rho) \equiv \int_k^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}},$$
$$H = H(\alpha,k,\rho) \equiv \frac{m}{\sqrt{F(\rho)-F(\frac{m}{\alpha})}} = \frac{k}{\sqrt{F(\rho)-F(k)}} \times \alpha.$$

We see that, for fixed k, $G(\alpha,k,\rho)$ is constant in α, but $H(\alpha,k,\rho)$ is linear in α. If $\alpha = \alpha_0$, we let $k_0 \equiv \frac{m_{\alpha_0,\rho}}{\alpha_0}$ and

$$G(\alpha_0,k_0,\rho) = \int_{k_0}^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} = H(\alpha_0,k_0,\rho) = \frac{k_0}{\sqrt{F(\rho)-F(k_0)}} \times \alpha_0.$$

Taking $\alpha > \alpha_0$, we have that

$$G(\alpha,k_0,\rho) = \int_{k_0}^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} = G(\alpha_0,k_0,\rho) < H(\alpha,k_0,\rho) = \frac{k_0}{\sqrt{F(\rho)-F(k_0)}} \times \alpha.$$

Then by Lemma 3.1 (i)–(ii), for $\alpha > \alpha_0$, we obtain that $\frac{m_{\alpha,\rho}}{\alpha} = k_\rho < k_0 = \frac{m_{\alpha_0,\rho}}{\alpha_0}$. So $\frac{m_\rho}{\alpha}$ is strictly decreasing on $(0,\infty)$.

Secondly, given any $\varepsilon > 0$, we have that

$$G(\alpha,\rho-\varepsilon,\rho) = \int_{\rho-\varepsilon}^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} > 0 \quad \text{and} \quad H(\alpha,\rho-\varepsilon,\rho) = \frac{\rho-\varepsilon}{\sqrt{F(\rho)-F(\rho-\varepsilon)}} \times \alpha > 0.$$

For

$$0 < \alpha < \delta \equiv \frac{\int_{\rho-\varepsilon}^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}}{\frac{\rho-\varepsilon}{\sqrt{F(\rho)-F(\rho-\varepsilon)}}},$$

we have that

$$H(\alpha,\rho-\varepsilon,\rho) = \frac{\rho-\varepsilon}{\sqrt{F(\rho)-F(\rho-\varepsilon)}} \times \alpha < \int_{\rho-\varepsilon}^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} = G(\alpha,\rho-\varepsilon,\rho).$$

By Lemma 3.1 (i)–(ii), for $0 < \alpha < \delta$, we see that $\rho - \varepsilon < \frac{m_\rho}{\alpha} = k < \rho$, which proves that $\lim_{\alpha \to 0^+} \frac{m_\rho}{\alpha} = \rho$. Similarly, we can prove that $\lim_{\alpha\to\infty} \frac{m_\rho}{\alpha} = 0$. The proof of part (i) is complete.

(II) By Lemma 3.1 (iii), we have that $(0 <) \frac{m_\rho}{\alpha} < \rho$, which also follows from (1.3). Thus we are left to prove that $\frac{\rho}{\alpha+1} < \frac{m_\rho}{\alpha}$. By (3.2), we have that

$$m_\rho = \int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{\sqrt{F(\rho) - F(\frac{m_\rho}{\alpha})}}{\sqrt{F(\rho) - F(s)}} ds > \int_{\frac{m_\rho}{\alpha}}^{\rho} ds = \rho - \frac{m_\rho}{\alpha},$$

which is equivalent to $\frac{\rho}{\alpha+1} < \frac{m_\rho}{\alpha}$. So part (ii) holds.

(III) The proof of part (iii) is easy but tedious and hence we put it in Appendix A.

The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Consider* (1.1) *and assume that f satisfies (H). Then, for fixed $\alpha > 0$, m_ρ is a C^3 function for $\rho > 0$ and*

$$m_\rho' \equiv \frac{\partial}{\partial \rho} m_\rho = \frac{\left(2\frac{m_\rho}{\alpha\rho} + P\right) D(\frac{m_\rho}{\alpha}) + m_\rho f(\rho)}{2\left(\frac{1}{\alpha} + 1\right) D(\frac{m_\rho}{\alpha}) + \frac{m_\rho}{\alpha} f(\frac{m_\rho}{\alpha})}, \tag{3.10}$$

where

$$\theta(x) \equiv 2F(x) - xf(x), \;\; D(x) \equiv F(\rho) - F(x), \;\; P \equiv \frac{1}{\rho} \int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{\sqrt{D(\frac{m_\rho}{\alpha})}}{D^{\frac{3}{2}}(s)} [\theta(\rho) - \theta(s)] ds. \tag{3.11}$$

In addition,

$$T'(\rho) \equiv \frac{\partial}{\partial \rho} T(\rho) = \frac{\Delta(\rho)}{\Phi(\rho)}, \tag{3.12}$$

where

$$\Delta(\rho) = P\left[D\left(\frac{m_\rho}{\alpha}\right) + \frac{m_\rho}{2\alpha} f\left(\frac{m_\rho}{\alpha}\right)\right] + \frac{m_\rho}{\alpha\rho}\left[\theta(\rho) - \theta\left(\frac{m_\rho}{\alpha}\right)\right], \tag{3.13}$$

$$\Phi(\rho) = \sqrt{D\left(\frac{m_\rho}{\alpha}\right)}\left[2\left(\frac{1}{\alpha} + 1\right) D\left(\frac{m_\rho}{\alpha}\right) + \frac{m_\rho}{\alpha} f\left(\frac{m_\rho}{\alpha}\right)\right] > 0. \tag{3.14}$$

The proof of Lemma 3.3 is easy but tedious and hence we put it in Appendix B.

By (3.12)–(3.14) and (3.2), we see that, if $\theta(\rho) - \theta(s)$ does not change sign for $s \in \left(\frac{m_\rho}{\alpha}, \rho\right)$, then we can determine the sign of $T'(\rho)$. Furthermore, by Lemma 3.2 (ii), it suffices to consider the interval $\left(\frac{\rho}{\alpha+1}, \rho\right)$ since $\frac{\rho}{\alpha+1} < \frac{m_\rho}{\alpha}$. We state this result in the following lemma.

Lemma 3.4. *Consider* (1.1) *and assume that f satisfies (H). Then, for fixed $\alpha > 0$ and for $u \in \left(\frac{\rho}{\alpha+1}, \rho\right)$ and $\rho \in (0, \eta)$,*

$$T'(\rho) \begin{cases} > 0, & \text{if } \theta(\rho) - \theta(u) \geq (\not\equiv)\, 0, \\ < 0, & \text{if } \theta(\rho) - \theta(u) \leq (\not\equiv)\, 0. \end{cases}$$

Remark 3.5. Letting $\alpha \to \infty$ in (3.12), by Lemma 3.2 (i), the time-map $\bar{T}(\rho)$ for (1.2) satisfies

$$\bar{T}'(\rho) \equiv \frac{\partial}{\partial \rho} \bar{T}(\rho) = \frac{1}{2\rho} \int_0^{\rho} \frac{\theta(\rho) - \theta(s)}{[D(s)]^{3/2}} ds.$$

And similarly, for $u \in (0, \rho)$,

$$\bar{T}'(\rho) \begin{cases} > 0, & \text{if } \theta(\rho) - \theta(u) > 0, \\ < 0, & \text{if } \theta(\rho) - \theta(u) < 0. \end{cases}$$

Lemma 3.6. *Consider* (1.1) *with fixed* $\alpha > 0$. *For* $f(u) = u^b$ *with* $b \geq 0$, $T(\rho)$ *satisfies the following assertions (i)–(iii).*

(i) If $b > 1$, *then* $\lim_{\rho\to 0^+} T(\rho) = \infty$, $\lim_{\rho\to\infty} T(\rho) = 0$, *and* $T'(\rho) < 0$ *for* $\rho \in (0, \infty)$.

(ii) If $b = 1$, *then* $T(\rho) \equiv \sqrt{\lambda^*}$ *on* $(0, \infty)$, *where* $\lambda^* \in (0, \frac{\pi^2}{4})$ *is defined in* (2.1).

(iii) If $0 \leq b < 1$, *then* $\lim_{\rho\to 0^+} T(\rho) = 0$, $\lim_{\rho\to\infty} T(\rho) = \infty$, *and* $T'(\rho) > 0$ *for* $\rho \in (0, \infty)$.

Proof. **(I)** First, for $f(u) = u^b$ with $b \geq 0$, we calculate that

$$F(u) = \frac{1}{b+1} u^{b+1},$$

$$D(u) = F(\rho) - F(u) = \frac{1}{b+1}\rho^{b+1} - \frac{1}{b+1} u^{b+1},$$

$$\theta(u) = 2F(u) - uf(u) = \frac{1-b}{b+1} u^{b+1},$$

$$\theta(\rho) - \theta(u) = \frac{1-b}{b+1}\rho^{b+1} - \frac{1-b}{b+1} u^{b+1} = (1-b)\, D(u).$$

We observe that

$$D(u) = \frac{1}{b+1}(\rho^{b+1} - u^{b+1}) \in \left[\frac{u^b}{b+1}(\rho - u), \frac{\rho^{b+1}}{b+1}\right) \quad \text{for } \rho > u, \tag{3.15}$$

$$\begin{aligned} P &= \frac{1}{\rho}\int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{\sqrt{D(\frac{m_\rho}{\alpha})}}{D^{\frac{3}{2}}(s)} \left[\theta(\rho) - \theta(s)\right] ds \\ &= \frac{1}{\rho}\int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{\sqrt{D(\frac{m_\rho}{\alpha})}}{D^{\frac{3}{2}}(s)} (1-b)\, D(s) ds = \frac{1}{\rho}(1-b)\sqrt{D(\frac{m_\rho}{\alpha})} T(\rho). \end{aligned}$$

For $b \geq 0$, by (3.15), we have that

$$\begin{aligned} T(\rho) &= \int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{ds}{\sqrt{D(s)}} \geq \frac{\sqrt{b+1}}{\rho^{\frac{b+1}{2}}}\rho\left(1 - \frac{m_\rho}{\alpha\rho}\right) \\ &= \frac{\sqrt{b+1}}{\rho^{\frac{b-1}{2}}}\left(1 - \frac{m_\rho}{\alpha\rho}\right). \end{aligned} \tag{3.16}$$

On the other hand, by (3.15), we have that

$$\begin{aligned} T(\rho) &= \int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{ds}{\sqrt{D(s)}} \leq \frac{\sqrt{b+1}}{\left(\frac{m_\rho}{\alpha}\right)^{\frac{b}{2}}} \int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{ds}{\sqrt{\rho - s}} \\ &\leq \frac{2\sqrt{b+1}}{\left(\frac{m_\rho}{\alpha}\right)^{\frac{b}{2}}}\sqrt{\rho - \frac{m_\rho}{\alpha}} \leq 2\sqrt{b+1}\,(\alpha+1)^{\frac{b}{2}}\sqrt{\frac{1 - \frac{m_\rho}{\alpha\rho}}{\rho^{b-1}}}. \end{aligned} \tag{3.17}$$

Then, since $b > 1$ and by Lemma 3.2 (iii), we have that $\lim_{\rho\to 0^+} T(\rho) = \infty$ and $\lim_{\rho\to\infty} T(\rho) = 0$. In addition, by (3.12)–(3.14), the numerator of $T'(\rho)$ is

$$\begin{aligned}
\Delta(\rho) &= \frac{1}{2}\left[\frac{m_\rho}{\alpha} f\left(\frac{m_\rho}{\alpha}\right) + 2D\left(\frac{m_\rho}{\alpha}\right)\right] P + \frac{m_\rho}{\alpha\rho}\left[\theta(\rho) - \theta\left(\frac{m_\rho}{\alpha}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{m_\rho}{\alpha}\right)^{b+1} + 2D\left(\frac{m_\rho}{\alpha}\right)\right] \frac{1-b}{\rho}\sqrt{D\left(\frac{m_\rho}{\alpha}\right)} T(\rho) + \frac{m_\rho}{\alpha\rho}(1-b) D\left(\frac{m_\rho}{\alpha}\right) \\
&= (1-b)\left\{\frac{1}{2\rho}\left[\left(\frac{m_\rho}{\alpha}\right)^{b+1} + 2D\left(\frac{m_\rho}{\alpha}\right)\right]\sqrt{D\left(\frac{m_\rho}{\alpha}\right)} T(\rho) + \frac{m_\rho}{\alpha\rho} D\left(\frac{m_\rho}{\alpha}\right)\right\} \qquad (3.18) \\
&< 0
\end{aligned}$$

since $b > 1$. Thus $T'(\rho) < 0$ for $\rho \in (0, \infty)$. So part (i) holds.

(II) We prove part (ii). For $f(u) = u$, we compute $T(\rho)$ in (3.2) and we have that

$$T(\rho) = \int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{ds}{\sqrt{\frac{1}{2}\rho^2 - \frac{1}{2}s^2}} = \frac{m_\rho}{\sqrt{\frac{1}{2}\rho^2 - \frac{1}{2}\left(\frac{m_\rho}{\alpha}\right)^2}}. \qquad (3.19)$$

So we obtain that

$$\arccos\left(\frac{m_\rho}{\alpha\rho}\right) = \frac{\frac{m_\rho}{\alpha\rho}}{\sqrt{1 - \left(\frac{m_\rho}{\alpha\rho}\right)^2}} \times \alpha, \qquad (3.20)$$

which holds for every $\rho > 0$, with positive α fixed. Thus $\frac{m_\rho}{\alpha\rho}$ is constant in ρ. By (3.3), if we consider $\sqrt{\lambda} \in (0, \frac{\pi}{2})$ as an angle of a triangle and $\frac{m_\rho}{\alpha\rho}$ as its cosine value in the triangle, then

$$\tan\sqrt{\lambda} = \frac{\sqrt{1 - \left(\frac{m_\rho}{\alpha\rho}\right)^2}}{\frac{m_\rho}{\alpha\rho}}$$

and $\sqrt{\lambda}\tan\sqrt{\lambda} = \alpha$ by (3.19)–(3.20). So $T(\rho) \equiv \sqrt{\lambda^*}$, where $\lambda^* \in (0, \frac{\pi^2}{4})$ is defined in (2.1). So part (ii) holds.

(III) Part (iii) for $f(u) = u^b$ with $0 \le b < 1$ follows easily by applying (3.16)–(3.18).

The proof of Lemma 3.6 is complete. □

Lemma 3.7. *Consider (1.1) with s_0, s_∞, L_0, and L_∞ defined in Theorem 2.1 (i)–(ii). Then the following assertions (i) and (ii) hold:*

(i) There exists a function $R_0(\rho)$ such that

$$T(\rho) = \int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{[1 + R_0(\rho)]\, ds}{\sqrt{\frac{L_0}{s_0+1}\rho^{s_0+1} - \frac{L_0}{s_0+1}s^{s_0+1}}},$$

where $\lim_{\rho\to 0^+} R_0(\rho) = 0$.

(ii) There exists a function $R_\infty(\rho)$ such that

$$T(\rho) = \int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{[1 + R_\infty(\rho)]\, ds}{\sqrt{\frac{L_\infty}{s_\infty+1}\rho^{s_\infty+1} - \frac{L_\infty}{s_\infty+1}s^{s_\infty+1}}},$$

where $\lim_{\rho\to\infty} R_\infty(\rho) = 0$.

Proof. We simply prove part (ii). The proof of part (i) is similar. We compute that

$$\begin{aligned} T(\rho) &= \int_{\frac{m_\rho}{\alpha}}^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} \\ &= \rho \int_{\frac{m_\rho}{\alpha\rho}}^{1} \left\{ \frac{L_\infty}{s_\infty+1}\left[\rho^{s_\infty+1}-(\rho s)^{s_\infty+1}\right] + \int_{\rho s}^{\rho} f(u) - L_\infty u^{s_\infty} du \right\}^{-1/2} ds \\ &= \rho \int_{\frac{m_\rho}{\alpha\rho}}^{1} \left\{ \frac{L_\infty}{s_\infty+1}\left[\rho^{s_\infty+1}-(\rho s)^{s_\infty+1}\right] \right\}^{-1/2} \left[1 - \frac{\sqrt{1+\phi(\rho,s)}-1}{\sqrt{1+\phi(\rho,s)}}\right] ds, \end{aligned}$$

where

$$\phi(\rho,s) = \frac{\int_{\rho s}^{\rho} f(u) - L_\infty u^{s_\infty} du}{\frac{L_\infty}{s_\infty+1}\left[\rho^{s_\infty+1}-(\rho s)^{s_\infty+1}\right]}.$$

Since $s \geq \frac{m_\rho}{\alpha\rho} > \frac{1}{\alpha+1}$, we have that

$$\begin{aligned} 0 \leq |\phi(\rho,s)| &= \left| \frac{\int_{\rho s}^{\rho} f(u) - L_\infty u^{s_\infty} du}{\frac{L_\infty}{s_\infty+1}\left[\rho^{s_\infty+1}-(\rho s)^{s_\infty+1}\right]} \right| \\ &\leq \frac{\int_{\rho s}^{\rho} u^{s_\infty} du}{\frac{L_\infty}{s_\infty+1}\left[\rho^{s_\infty+1}-(\rho s)^{s_\infty+1}\right]} \sup_{t\in\left(\frac{\rho}{\alpha+1},\rho\right)} \left| \frac{f(t)}{t^{s_\infty}} - L_\infty \right| = \frac{1}{L_\infty} \sup_{t\in\left(\frac{\rho}{\alpha+1},\rho\right)} \left| \frac{f(u)}{t^{s_\infty}} - L_\infty \right|. \end{aligned}$$

We see that $\phi(\rho,s)$ approaches zero as $\rho \to \infty$, uniformly in s. So

$$R_\infty(\rho) \equiv -\frac{\sqrt{1+\phi(\rho,s)}-1}{\sqrt{1+\phi(\rho,s)}} \to 0 \quad \text{as } \rho \to \infty,$$

and hence part (ii) holds.

The proof of Lemma 3.7 is complete. □

Lemma 3.8. *Consider* (1.1) *with s_0, s_∞, L_0, and L_∞ defined in Theorem* 2.1 *(i)–(ii) and λ^* defined in* (2.1). *Then*

$$\lim_{\rho\to 0^+} T(\rho) = \begin{cases} \infty, & \text{if } s_0 > 1, \\ \sqrt{\frac{\lambda^*}{L_0}}, & \text{if } s_0 = 1, \\ 0, & \text{if } 0 \leq s_0 < 1, \end{cases} \tag{3.21}$$

$$\lim_{\rho\to \infty} T(\rho) = \begin{cases} 0, & \text{if } s_\infty > 1, \\ \sqrt{\frac{\lambda^*}{L_\infty}}, & \text{if } s_\infty = 1, \\ \infty, & \text{if } 0 \leq s_\infty < 1. \end{cases} \tag{3.22}$$

Proof. If $s_0 \geq 0$ and $s_0 \neq 1$, Eq. (3.21) follows by Lemmas 3.6 and 3.7. If $s_0 = 1$, we rewrite (1.1) as

$$-u''(x) = \lambda f(u(x)) = [L_0\lambda]\left[\frac{f(u(x))}{L_0}\right] = \tilde{\lambda}\tilde{f}(u(x)),$$

with $\tilde{\lambda} \equiv L_0\lambda$ and $\tilde{f}(u) \equiv \frac{f(u)}{L_0}$. Then $\lim_{u\to 0^+} \frac{\tilde{f}(u)}{u} = \lim_{u\to 0^+} \frac{f(u)}{uL_0} = 1$, and we can apply Lemmas 3.6 and 3.7 to get $\lim_{\rho\to 0^+} T(\rho) = \sqrt{\frac{\lambda^*}{L_0}}$. The proof of Eq. (3.21) is complete. Eq. (3.22) can be proved similarly. We omit it here.

The proof of Lemma 3.8 is complete. □

Lemma 3.9. *Consider* (1.1) *and assume that* f *satisfies (H) with* $f(\eta) = 0$ *for some finite* $\eta > 0$. *Let* $s_\eta > 0$ *and* $L_\eta > 0$ *be defined in Theorem* 2.1 *(ii) (b). Then*

$$\lim_{\rho \to \eta^-} T(\rho) \begin{cases} = \infty & \text{if } s_\eta \geq 1, \\ \in (0, \infty) & \text{if } 0 < s_\eta < 1. \end{cases}$$

The proof of Lemma 3.9 is easy but tedious and hence we put it in Appendix C.

Lemma 3.10. *Consider* (1.1). *Then the following assertions (i) and (ii) hold:*

(i) Assume that f *satisfies (D1) with positive* $p^* < p < \eta$ *and* $\int_0^p s\,[f(s) - sf'(s)]\,ds < 0$. *Then* $T'(p) < 0$ *for all* $\alpha > 0$.

(ii) Assume that f *satisfies (D2) with positive* $p^* < p < \eta$ *and* $\int_0^p s\,[f(s) - sf'(s)]\,ds > 0$. *Then* $T'(p) > 0$ *for all* $\alpha > 0$.

The proof of Lemma 3.10 is easy but tedious and hence we put it in Appendix D.

Lemma 3.11. *Consider* (1.1). *Let* $k = \frac{m}{\alpha}$,

$$B(k) \equiv \frac{m}{\alpha} f\left(\frac{m}{\alpha}\right) + 2D\left(\frac{m}{\alpha}\right) = kf(k) + 2D(k), \tag{3.23}$$

and

$$\Delta(\rho) = \Delta(\rho, k, \alpha) = P\left[D(k) + \frac{k}{2} f(k)\right] + \frac{k}{\rho}\left[\theta(\rho) - \theta(k)\right]$$

be the function Δ *defined in Lemma* 3.3. *Then the following assertions (i) and (ii) hold.*

(i) If f *satisfies (D1) with positive* $p^* < p < \eta$, *then*

$$\frac{\partial \Delta(\rho, k, \alpha)}{\partial k} + \frac{2}{B(k)}\left[\frac{f(k)}{4D(k)} B(k) + \frac{1}{2}\theta'(k)\right] \Delta(\rho, k, \alpha) < 0 \ \text{ for } \rho = p > p^* > k > 0.$$

(ii) If f *satisfies (D2) with positive* $p^* < p < \eta$, *then*

$$\frac{\partial \Delta(\rho, k, \alpha)}{\partial k} + \frac{2}{B(k)}\left[\frac{f(k)}{4D(k)} B(k) + \frac{1}{2}\theta'(k)\right] \Delta(\rho, k, \alpha) > 0 \ \text{ for } \rho = p > p^* > k > 0.$$

The proof of Lemma 3.11 is easy but tedious and hence we put it in Appendix E.

Lemma 3.12. *Consider* (1.1). *Then, for all* $\alpha > 0$, *on the* $(\lambda, \|u\|_\infty)$*-plane, the bifurcation curve* S_α *satisfies the following assertions (i)–(iv):*

(i) If f *satisfies* $(C1^+)$, *then* S_α *continues to the right initially.*

(ii) If f *satisfies* $(C1^-)$, *then* S_α *continues to the left initially.*

(iii) If f *satisfies* $(C2^+)$, *then* S_α *continues to the right eventually.*

(iv) If f *satisfies* $(C2^-)$, *then* S_α *continues to the left eventually.*

Proof. **(I)** If f satisfies (C1$^+$) with some $\beta_1 \in (0,\eta)$, then since $\theta'(u) = f(u) - uf'(u)$ and by Lemma 3.4, $T'(\rho) > 0$ for $0 < \rho < \beta_1$. Thus, by (3.3), the bifurcation curve S_α continues to the right initially for each $\alpha > 0$. This proves part (i). Similarly, part (ii) can be proved. We omit it here.

(II) For $\eta = \infty$, if f satisfies (C2$^+$) with some $\beta_2 \in (0,\infty)$, then for each $\alpha > 0$, by Lemma 3.4, $T'(\rho) > 0$ for $\rho > (\alpha+1)\beta_2$. Thus the bifurcation curve S_α continues to the right eventually for each $\alpha > 0$. For $\eta < \infty$, by Lemma 3.9, $s_\eta \geq 1$ implies that $\lim_{\rho\to\eta^-} T(\rho) = \infty$ for all $\alpha > 0$. If S_α continues to the left eventually, by continuity, we have that $\lim_{\rho\to\eta^-} T(\rho) < \infty$, which is a contradiction. Hence S_α continues to the right eventually for all $\alpha > 0$. This proves part (iii). Similarly, part (iv) can be proved. We omit it here.

The proof of Lemma 3.12 is complete. □

4 Proofs of main results

Proof of Theorem 2.1. By (3.2), (3.4) and Lemma 3.3, the bifurcation curve S_α is a continuous curve on the $(\lambda, \|u\|_\infty)$-plane for all fixed $\alpha > 0$. Moreover, parts (i) and (ii)(a) follow by (3.4) and Lemma 3.8. Part (ii) (b) follows by (3.4) and Lemma 3.9. Part (iii) follows by (3.2), (3.4)–(3.6) and Lemma 3.2 (i).

We next prove part (iv). To prove that $S_{\alpha,2}$ lies on the left hand side of (possibly coincides with) $S_{\alpha,1}$ on the $(\lambda, \|u\|_\infty)$-plane, by (3.3)–(3.4), it suffices to prove that $\lambda_1(\rho) \geq \lambda_2(\rho)$ for $0 < \rho < \eta$, where $\lambda_1(\rho)$ and $\lambda_2(\rho)$ are defined in (3.3) for (1.1) with $f = f_1$ and $f = f_2$ respectively. We prove this by contradiction. For a fixed $\rho \in (0,\eta)$, suppose on the contrary that $\lambda_1(\rho) < \lambda_2(\rho)$ and denote positive solutions $u_1(x)$ and $u_2(x)$ of (1.1) with $f = f_1$ and $f = f_2$ respectively. Since $u_1(0) = \rho = u_2(0)$, $u_1'(0) = u_2'(0) = 0$ and $u_1''(0) = -\lambda_1(\rho)f_1(\rho) > -\lambda_2(\rho)f_2(\rho) = u_2''(0)$, there exists $x_0 \in (0,1)$ such that $u_1(x) > u_2(x)$ and $u_1'(x) > u_2'(x)$ for $0 < x < x_0$. We claim that $u_1(x) > u_2(x)$ and $u_1'(x) > u_2'(x)$ on $[0,1]$. If not, there exists $x_1 \in (x_0,1)$ such that $u_1'(x_1) = u_2'(x_1)$ and $u_1(x_1) > u_2(x_1)$. Multiplying both side of $u_i''(x) + \lambda_i f_i(u_i(x)) = 0$ by $u_i'(x)$, $i = 1,2$, and integrating from 0 to x, we obtain that

$$\int_0^x u_i''(t)u_i'(t)dt = -\int_0^x \lambda_i f_i(u_i(t))u_i'(t)dt, \qquad i = 1,2.$$

Since $u_1'(0) = u_2'(0) = 0$, we obtain that

$$\frac{[u_i'(x)]^2}{2} = -\lambda_i \int_{u_i(0)}^{u_i(x)} f_i(u)du = \lambda_i \int_{u_i(x)}^{\rho} f_i(u)du, \qquad i = 1,2.$$

Then, since $u_1'(x_1) = u_2'(x_1)$, we see that

$$\lambda_1 \int_{u_1(x_1)}^{\rho} f_1(u)du = \lambda_2 \int_{u_2(x_1)}^{\rho} f_2(u)du.$$

So we observe that

$$\begin{aligned}
\int_{u_2(x_1)}^{\rho} f_2(u)du &= \frac{\lambda_1}{\lambda_2}\int_{u_1(x_1)}^{\rho} f_1(u)du \\
&< \int_{u_1(x_1)}^{\rho} f_1(u)du \quad (\text{since } \lambda_1(\rho) < \lambda_2(\rho)) \\
&\leq \int_{u_1(x_1)}^{\rho} f_2(u)du \quad (\text{since } f_1(u) \leq f_2(u) \text{ on } (0,\eta)) \\
&\leq \int_{u_2(x_1)}^{\rho} f_2(u)du \quad (\text{since } u_1(x_1) > u_2(x_1) \text{ and } f_2(u) > 0 \text{ on } (0,\eta)),
\end{aligned}$$

which is a contradiction. So $u_1(x) > u_2(x)$ and $u_1'(x) > u_2'(x)$ on $[0,1]$. Finally by the boundary condition at $x = 1$ in (1.1), we see that

$$u_1'(1) = -\alpha u_1(1) < -\alpha u_2(1) = u_2'(1),$$

which is again a contradiction. Thus $\lambda_1(\rho) \geq \lambda_2(\rho)$ for $0 < \rho < \eta$. This proves that $S_{\alpha,2}$ lies on the left hand side of $S_{\alpha,1}$ on the $(\lambda, \|u\|_\infty)$-plane. The rest of part (iv) follows by Lemma 3.2 (ii).

The proof of Theorem 2.1 is complete. □

Proof of Theorem 2.3. Suppose that $\theta'(u) = f(u) - uf'(u)$ changes sign exactly k times in the interval $(a, b) \subset (0, \eta)$, where $0 < a = p_{-1} < p_0 < p_1 < p_2 < \cdots < p_k = b \leq \eta$ such that $\theta'(u)$ has the same sign in the interval (p_{i-1}, p_i), $i = 0, 1, 2, ..., k$, and has different signs in every pair of adjacent intervals. Then, by Lemma 3.4, there exist positive numbers $\beta_0, \beta_1, \ldots, \beta_k$ small enough, such that $T'(p_i) > 0$ if $\theta'(u)$ has a local maximum at p_i (or $T'(p_i) < 0$ if $\theta'(u)$ has a local minimum at p_i) for $0 < \alpha < \beta_i$, $i = 0, 1, 2, \ldots, k$. Define $\alpha_i = \min\{\beta_0, \beta_1, \ldots, \beta_i\}$. So if $0 < \alpha \leq \alpha_i$ for $i \in \{1, 2, \ldots, k\}$, then $T(\alpha)$ has at least i critical points on $(0, \eta)$. Then, by (3.4) and the Intermediate Value Theorem, the bifurcation curve S_α has at least i turning points on the $(\lambda, \|u\|_\infty)$-plane for $0 < \alpha \leq \alpha_i$, $i = 0, 1, 2, \ldots, k$.

The proof of Theorem 2.3 is complete. □

Proof of Theorem 2.4. **(I)** Assume that $\theta'(u) = f(u) - uf'(u) > 0$ almost everywhere for $u > 0$. We see that, for every $\alpha,\rho > 0$, $\theta'(u) > 0$ almost everywhere in the interval $(\frac{\rho}{\alpha+1}, \rho)$. Then by Lemma 3.4, we have that $T'(\rho) > 0$ for every $\alpha,\rho > 0$. Then by (3.3)–(3.4), we obtain that the bifurcation curve S_α is strictly increasing on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$. Similarly, assuming that $\theta'(u) = f(u) - uf'(u) < 0$ almost everywhere for $u > 0$, we can prove that the bifurcation curve S_α is strictly decreasing on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$. So part (i) holds.

(II) Assume that f satisfies (C1$^+$), (C2$^+$) and (D1) with positive $p^* < p < \eta$ and

$$\int_0^p s\left[f(s) - sf'(s)\right] ds < 0.$$

Since f satisfies (C1$^+$) and (C2$^+$) and by Lemma 3.12 (i), (iii), the bifurcation curve S_α continues to the right both initially and eventually on the $(\lambda, \|u\|_\infty)$-plane. So, to prove that S_α is S-shaped on the $(\lambda, \|u\|_\infty)$-plane, it is sufficient to prove that S_α has at least two turning points on the $(\lambda, \|u\|_\infty)$-plane. First, conditions (C1$^+$) and (C2$^+$) imply that $\theta'(u) > 0$ for $u > 0$ small enough and u large enough. These and Lemma 3.4 imply that $T'(\rho) > 0$ for $\rho > 0$ small enough and ρ large enough. Since f satisfies (D1) with positive $p^* < p < \eta$ and $\int_0^p s\left[f(s) - sf'(s)\right] ds < 0$ and by Lemma 3.10, we have that $T'(p) < 0$. Thus $T'(\rho)$ changes sign at least twice on $(0, \infty)$. So by (3.4), S_α has at least two turning points on the $(\lambda, \|u\|_\infty)$-plane and the bifurcation curve S_α is S-shaped on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$. Similarly, assuming that f satisfies (C1$^-$), (C2$^-$) and (D2) with positive $p^* < p < \eta$ and $\int_0^p s\left[f(s) - sf'(s)\right] ds < 0$, we can prove that the bifurcation curve S_α is reversed S-shaped on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$. So part (ii) holds.

(III) Assume that f satisfies (C1$^+$), (C2$^+$), (D1) with positive $p^* < p < \eta$, the bifurcation curve $\bar{S}$ of (1.2) is S-shaped on the $(\lambda, \|u\|_\infty)$-plane and $0 < p^* < \|u_{\lambda_1}\|_\infty < p < \|u_{\lambda_2}\|_\infty$; see Fig. 2.3. Again, since f satisfies (C1$^+$) and (C2$^+$) and by Lemma 3.12 (i), (iii), the bifurcation curve S_α continues to the right both initially and eventually on the $(\lambda, \|u\|_\infty)$-plane. So, to prove that the bifurcation curve S_α is S-shaped on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$, it is sufficient to prove that S_α has at least two turning points on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$. Since f satisfies (D1) and by Lemma 3.11 (i), letting $k_\rho \equiv \frac{m_\rho}{\alpha}$, we have that

$$\left.\frac{\partial\Delta(\rho,k,\alpha)}{\partial k}\right|_{k=k_\rho} + \frac{2}{B(k_\rho)}\left[\frac{f(k_\rho)}{4D(k_\rho)}B(k_\rho) + \frac{1}{2}\theta'(k_\rho)\right]\Delta(\rho,k_\rho,\alpha) < 0$$
$$\text{for } \rho = p > p^* > k = k_\rho > 0.$$

By Lemma 3.2 (i), k_ρ is a strictly decreasing function of $\alpha > 0$. So we have that $\frac{\partial\alpha}{\partial k_\rho} = \left(\frac{\partial k_\rho}{\partial\alpha}\right)^{-1} \leq 0$. In the next equation

$$\frac{\partial\alpha}{\partial k_\rho}\left.\frac{\partial\Delta(\rho,k,\alpha)}{\partial\alpha}\right|_{k=k_\rho} + \frac{2}{B(k_\rho)}\left[\frac{f(k_\rho)}{4D(k_\rho)}B(k_\rho) + \frac{1}{2}\theta'(k_\rho)\right]\Delta(\rho,k_\rho,\alpha) < 0$$
$$\text{for } \rho = p > p^* > k = k_\rho > 0,$$

the coefficient of $\frac{\partial\Delta(\rho,k,\alpha)}{\partial\alpha}\big|_{\rho=p,k=k_\rho}$ is nonpositive, and the coefficient of $\Delta(\rho,k_\rho,\alpha)\big|_{\rho=p}$ is positive for $0 < k = k_\rho < p^*$ since $\frac{f(k_\rho)}{4D(k_\rho)}B(k_\rho) > 0$ and $\theta'(k_\rho) > 0$ for $0 < k = k_\rho < p^*$, see Fig. 2.2. Thus $\Delta(\rho = p, k_\rho, \alpha)$ can only change sign once from "$-$" to "$+$" for α increasing from 0^+ to ∞. Since $\bar{S}$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane and f satisfies (D1) with $\|u_{\lambda_2}\|_\infty > p > \|u_{\lambda_1}\|_\infty$ and by (3.6) and Remark 3.5, we have that $\bar{T}'(p) < 0$, and $\theta'(p) = 0$ and $\theta(p)$ is a local minimum. Then by continuity, we have that $T'(p) < 0$ for α large enough. In addition, by Lemma 3.4, we have that $T'(p) < 0$ for $\alpha > 0$ small enough. Thus $T'(p)$ $(= \frac{\Delta(\rho=p)}{\Phi(\rho=p)}) < 0$ holds for all $\alpha > 0$, otherwise it contradicts that $\Delta(\rho = p, k_\rho, \alpha)$ can only change sign once from "$-$" to "$+$" for α increasing from 0^+ to ∞. Together with conditions ($C1^+$) and ($C2^+$), we see that $T'(\rho)$ changes sign at least twice on $(0,\infty)$ for all $\alpha > 0$. So by (3.4), S_α is S-shaped on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$. The rest of part (iii) in this case follow easily. Similarly, assuming that f satisfies ($C1^-$), ($C2^-$), (D2) with positive $p^* < p < \eta$, $\bar{S}$ is reversed S-shaped on the $(\lambda, \|u\|_\infty)$-plane and $\|u_{\lambda_2}\|_\infty > p > \|u_{\lambda_1}\|_\infty > p^*$, we obtain that S_α is reversed S-shaped on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$. The rest of part (iii) in this case follow easily. So part (iii) holds.

(IV) Assume that f satisfies ($C1^-$) and ($C2^+$). Since f satisfies ($C1^-$) and ($C2^+$) and by Lemma 3.4, for all $\alpha > 0$, we have that $T'(\rho) < 0$ for small $\rho > 0$ and $T'(\rho) > 0$ for ρ large enough. Thus $T'(\rho)$ changes sign at least once on $(0,\infty)$ for all $\alpha > 0$. Then by (3.4), the bifurcation curve S_α is $\subset$-shaped on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$. Similarly, assuming that f satisfies ($C1^+$) and ($C2^-$), we can prove that the bifurcation curve S_α is reversed $\subset$-shaped on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$. So part (iv) holds.

The proof of Theorem 2.4 is complete. □

Proof of Theorem 2.5. **(I)** Since f satisfies ($C1^+$) and ($C2^+$) and by Lemma 3.4, we have that $T'(\rho) > 0$ for $\rho > 0$ small enough and ρ large enough. Since $\theta'(\tilde{p}) = f(\tilde{p}) - \tilde{p}f'(\tilde{p}) < 0$, by Lemma 3.4, we have that $T'(\tilde{p}) < 0$ for $\alpha > 0$ small enough. Hence, by (3.4), the bifurcation curve S_α is S-shaped on the $(\lambda, \|u\|_\infty)$-plane for $\alpha > 0$ small enough. So part (i) holds.

(II) Since f satisfies ($C1^-$) and ($C2^-$) and by Lemma 3.4 we have that $T'(\rho) < 0$ for $\rho > 0$ small enough and ρ large enough. Since $\theta'(\tilde{p}) = f(\tilde{p}) - \tilde{p}f'(\tilde{p}) > 0$, by Lemma 3.4, we have that $T'(\tilde{p}) > 0$ for $\alpha > 0$ small enough. Hence, by (3.4), the bifurcation curve S_α is reversed S-shaped on the $(\lambda, \|u\|_\infty)$-plane for $\alpha > 0$ small enough. So part (ii) holds.

(III) Assume that the bifurcation curve $\bar{S}$ of (1.2) is strictly increasing on the $(\lambda, \|u\|_\infty)$-plane and (1.2) has no degenerate solutions. Then by (3.6), we have that $\bar{T}'(\rho) > 0$ for all $\rho > 0$. In addition, assume that f satisfies (C1$^+$) with some $\beta_1 \in (0, \rho_0) \subset (0, \eta)$. Then since $\theta'(u) = f(u) - uf'(u)$ and by Lemma 3.4, $T'(\rho) > 0$ on $(0, \beta_1)$. For every $\rho_0 \in (\beta_1, \eta)$, since $T'(\rho)$ is continuous in both α and ρ and $\lim_{\alpha\to\infty} T'(\rho) = \bar{T}'(\rho)$ for fixed $\rho \in [\beta_1, \rho_0]$ by applying (3.12)–(3.14) and Remark 3.5, and since $[\beta_1, \rho_0]$ is a compact set, we obtain that there exists $\alpha^* = \alpha^*(\rho_0) > 0$ such that, for $\alpha \geq \alpha^*$, $T'(\rho) > 0$ on $[\beta_1, \rho_0]$; the detail of the proof is omitted here. Thus, for $\alpha \geq \alpha^*$, $T'(\rho) > 0$ on $(0, \rho_0]$. So by (3.4), on the $(\lambda, \|u\|_\infty)$-plane, for $\alpha \geq \alpha^*$, S_α is strictly increasing in the striped region $\{(\lambda, \|u\|_\infty) : \lambda > 0 \text{ and } \|u\|_\infty \in (0, \rho_0]\}$. In addition, assume that there exists a constant $\bar{\rho}_0 \in (\beta_1, \eta)$ such that (2.4) holds. Then by Lemma 3.4, $T'(\rho) > 0$ on $[\bar{\rho}_0, \eta)$. So by (3.4), on the $(\lambda, \|u\|_\infty)$-plane, for all $\alpha > 0$, S_α is strictly increasing in the striped region $\{(\lambda, \|u\|_\infty) : \lambda > 0 \text{ and } \|u\|_\infty \in [\bar{\rho}_0, \eta)\}$. By previous results and choosing $\rho_0 = \bar{\rho}_0 \in (\beta_1, \eta)$, we have that, for $\alpha \geq \alpha^* = \alpha^*(\bar{\rho}_0)$, S_α is strictly increasing on the $(\lambda, \|u\|_\infty)$-plane.

Assume that $f(u)$ satisfies (C1$^-$) with some $\beta_1 \in (0, \eta)$, the bifurcation curve $\bar{S}$ of (1.2) is strictly decreasing on the $(\lambda, \|u\|_\infty)$-plane, (1.2) has no degenerate solution, and there exists a constant $\bar{\rho}_0 \in (\beta_1, \eta)$ such that (2.5) holds. Then, applying similar arguments in above, we can prove that, there exists $\alpha^* > 0$ such that, for $\alpha \geq \alpha^*$, S_α is strictly decreasing on the $(\lambda, \|u\|_\infty)$-plane.

The proof of Theorem 2.5 is complete. □

Proof of Theorem 2.6. **(I)** Consider that $f(u) = \exp\left(\frac{u}{1+\varepsilon u}\right)$ and $0 < \varepsilon < \tilde{\varepsilon} \approx 0.2457$. We obtain that the bifurcation curve $\bar{S}$ of (1.2), (1.6) is exactly S-shaped on the $(\lambda, \|u\|_\infty)$-plane by Theorem 1.1. In addition, f satisfies (C1$^+$) and (C2$^+$). Then by Theorem 2.4 (iii), to complete the proof of part (i) that the bifurcation curve S_α is S-shaped on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$, it is sufficient to prove that f satisfies (D1) and $\bar{S}$ satisfies condition (3b). We compute that

$$\theta'(u) = f(u) - uf'(u) = \frac{\varepsilon^2 u^2 + (2\varepsilon - 1)u + 1}{(\varepsilon u + 1)^2} f(u) \tag{4.1}$$

which has two positive zeros $p_1 = \frac{1-2\varepsilon-\sqrt{1-4\varepsilon}}{2\varepsilon^2} < p_2 = \frac{1-2\varepsilon+\sqrt{1-4\varepsilon}}{2\varepsilon^2}$ for $0 < \varepsilon < \varepsilon^* = 1/4 = 0.25$. We have that

$$\theta'(u) \begin{cases} > 0 & \text{for } u \in (0, p_1) \cup (p_2, \infty), \\ = 0 & \text{for } u = p_1 \text{ and } u = p_2, \\ < 0 & \text{for } u \in (p_1, p_2). \end{cases}$$

So f satisfies (D1) with $p^* \equiv p_1 < p \equiv p_2$. In addition, by [7, Theorem 1.2 and Figure 1.2], we obtain that $\bar{S}$ satisfies condition (3b). This completes the proof of part (i).

(II) First, for $0.2457 \approx \tilde{\varepsilon} \leq \varepsilon < \varepsilon^* = 1/4 = 0.25$, the assertion that S_α is S-shaped for $0 < \alpha \leq \alpha_*(\varepsilon)$ follows by Theorem 2.5 (i). To find the function $\alpha_*(\varepsilon)$ explicitly, we note that $\theta'(u) < 0$ for $u \in (p_1, p_2)$. Then by Lemma 3.4, we have that $T'(\rho) < 0$ if $\left(\frac{\rho}{\alpha+1}, \rho\right) \subseteq (p_1, p_2)$. Letting $\frac{\rho}{\alpha+1} = p_1$ and $\rho = p_2$, and solving for α, we obtain that

$$\alpha_*(\varepsilon) \equiv \frac{p_2}{p_1} - 1 = \frac{2\sqrt{1-4\varepsilon}}{1 - 2\varepsilon - \sqrt{1-4\varepsilon}} > 0,$$

and it is a strictly decreasing function of $\varepsilon \in (\tilde{\varepsilon}, \varepsilon^*)$ and satisfies (2.6) by simple computations.

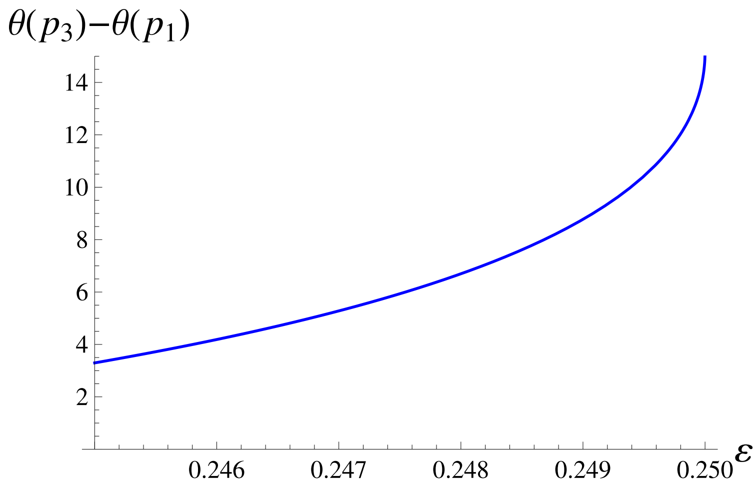

Figure 4.1: Graph of $\theta(p_3) - \theta(p_1)$ for $0.2457 \approx \tilde{\varepsilon} < \varepsilon < \varepsilon^* = 1/4 = 0.25$.

Secondly, for $0.2457 \approx \tilde{\varepsilon} < \varepsilon \leq \varepsilon^* = 1/4 = 0.25$, we choose $p_3 \equiv 3p_1 = \frac{3(1-2\varepsilon-\sqrt{1-4\varepsilon})}{2\varepsilon^2}$ and we obtain that

$$\theta(p_3) = \theta(3p_1) > \theta(p_1) > 0,$$

see Fig. 4.1. Thus

$$0 < \theta(u) < \theta(p_3) < \theta(\bar{u}_1) < \theta(\bar{u}_2) \ \text{ for } 0 < u < p_3 < \bar{u}_1 < \bar{u}_2 < \infty,$$

see Fig. 2.2 (I). Hence, for all $\alpha > 0$, $T'(\rho) > 0$ on $[p_3(\varepsilon^*), \infty) = [12, , \infty) \subset [p_3(\varepsilon), \infty)$ by Lemma 3.4 and since $p_3(\varepsilon)$ is strictly increasing on $(\tilde{\varepsilon}, \varepsilon^*]$. So on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is strictly increasing and (1.2), (1.6) has no degenerate solution in the striped region $\{(\lambda, \|u\|_\infty) : \lambda > 0 \text{ and } \|u\|_\infty \in [12, \infty)\}$. For $0.2457 \approx \tilde{\varepsilon} < \varepsilon < \varepsilon^* = 1/4 = 0.25$, on the $(\lambda, \|u\|_\infty)$-plane, by Theorem 1.1 (iii), the bifurcation curve $\bar{S}$ of (1.2), (1.6) is strictly increasing on the $(\lambda, \|u\|_\infty)$-plane, and (1.2), (1.6) has no degenerate solution. So by Theorem 2.5 (iii), for $\rho_0 \equiv p_3(\varepsilon^*) = 12$, there exists $\alpha^* = \alpha^*(\rho_0, \varepsilon) = \alpha^*(12, \varepsilon) > 0$ such that, for $\alpha \geq \alpha^*$, S_α is strictly increasing in the striped region $\{(\lambda, \|u\|_\infty) : \lambda > 0 \text{ and } \|u\|_\infty \in (0, \rho_0] = (0, 12]\}$. By above, we obtain that, for $0.2457 \approx \tilde{\varepsilon} < \varepsilon < \varepsilon^* = 1/4 = 0.25$, S_α is strictly increasing on the $(\lambda, \|u\|_\infty)$-plane for $\alpha \geq \alpha^*(12, \varepsilon)$.

The proof of part (ii) is complete.

(III) We see that, for $\varepsilon \geq \varepsilon^* = 1/4 = 0.25$, the function $\theta'(u) = f(u) - uf'(u)$ in (4.1) is positive almost everywhere on $(0, \infty)$. Hence by Theorem 2.4 (i), on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve S_α is strictly increasing for all $\alpha > 0$. So part (iii) holds.

The proof of Theorem 2.6 is complete. □

Proof of Theorem 2.8. **(I)** For $f(u) = u(1 - \sin u) + u^p$ with $p \geq 1$, we compute that

$$\theta(u) = 2F(u) - uf(u) = \frac{1-p}{p+1}u^{p+1} + 2u\cos u + (u^2 - 2)\sin u, \tag{4.2}$$

$$\theta'(u) = u^2\left[\cos u - (p-1)u^{p-2}\right].$$

If $p = 1$, for any $n \in \mathbb{N}$, then by (4.2), we have that

$$\theta(u) = 2u\cos u + (u^2 - 2)\sin u \begin{cases} > \theta(\frac{3+4n}{2}\pi) & \text{for } u \in \left(0, \frac{3+4n}{2}\pi\right), \\ < \theta(\frac{1+4n}{2}\pi) & \text{for } u \in \left(0, \frac{1+4n}{2}\pi\right). \end{cases}$$

So by Lemma 3.4, we have that $T'(\frac{3+4n}{2}\pi) < 0$ and $T'(\frac{1+4n}{2}\pi) > 0$ for each $\alpha > 0$. So $T(\rho)$ has infinitely many critical points on $(0, \infty)$ for all $\alpha > 0$. Thus, by (3.4), the bifurcation curve S_α has infinitely many turning points on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$. So part (i) holds.

(II) If $1 < p < 2$, $\theta(u)$ has infinitely many local maxima and infinitely many local minima on $(0, \infty)$. So part (ii) holds by applying Theorem 2.3.

(III) If $p = 2$, $\theta'(u) = u^2 [\cos u - 1]$ is negative almost everywhere on $(0, \infty)$. So by Theorem 2.4 (i), S_α is strictly decreasing on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$.

(IV) If $p > 2$, $\theta'(u) = u^2 \left[\cos u - (p-1)u^{p-2}\right]$ is positive for small $u > 0$ and is negative for u large enough. Thus $f(u)$ satisfies (C1$^-$) and (C2$^+$). So by Theorem 2.4 (iv), S_α is $\subset$-shaped on the $(\lambda, \|u\|_\infty)$-plane for all $\alpha > 0$.

The proof of Theorem 2.8 is complete. □

Appendix A

Proof of Lemma 3.2 (iii). We simply prove (3.9). The proof of (3.8) is similar. By Lemma 3.2 (ii), we obtain that $\lim_{\rho\to\infty} m_\rho = \infty$ and

$$\frac{\alpha}{\alpha+1} \le \lim_{\rho\to\infty} \frac{m_\rho}{\rho} \le \alpha.$$

By (2.2), for two arbitrary positive numbers $\gamma_1 < \gamma_2$, we compute that

$$\begin{aligned}\lim_{\rho\to\infty} \frac{f(\gamma_1\rho)}{f(\frac{\gamma_2\rho}{\alpha+1})} &= \lim_{\rho\to\infty} \frac{\frac{f(\gamma_1\rho)}{(\gamma_1\rho)^{s_\infty}}}{\frac{f(\frac{\gamma_2\rho}{\alpha+1})}{(\frac{\gamma_2\rho}{\alpha+1})^{s_\infty}}} \frac{(\gamma_1\rho)^{s_\infty}}{(\frac{\gamma_2\rho}{\alpha+1})^{s_\infty}} = \frac{\lim_{\rho\to\infty} \frac{f(\gamma_1\rho)}{(\gamma_1\rho)^{s_\infty}}}{\lim_{\rho\to\infty} \frac{f(\frac{\gamma_2\rho}{\alpha+1})}{(\frac{\gamma_2\rho}{\alpha+1})^{s_\infty}}} \left(\frac{\gamma_1}{\gamma_2}\right)^{s_\infty} (\alpha+1)^{s_\infty} \\ &= \frac{L_\infty}{L_\infty} \left(\frac{\gamma_1}{\gamma_2}\right)^{s_\infty} (\alpha+1)^{s_\infty} = \left(\frac{\gamma_1}{\gamma_2}\right)^{s_\infty} (\alpha+1)^{s_\infty}. \end{aligned} \tag{A.1}$$

By (2.2) again, letting $R_\infty(u) \equiv \frac{f(u)}{L_\infty u^{s_\infty}} - 1$, we can rewrite $f(u)$ as $f(u) = L_\infty u^{s_\infty} [1 + R_\infty(u)]$, in which $\lim_{u\to\infty} R_\infty(u) = 0$. So given $\varepsilon > 0$ such that $1 - \varepsilon < 2^{s_\infty}(1+\varepsilon)$, there exists $M > 0$ such that $|R_\infty(u)| < \varepsilon$ for $u \ge M$. Now we consider $\rho \ge 2(\alpha+1) M$. We have that $\frac{m_\rho}{\alpha} \ge \frac{\rho}{\alpha+1} \ge 2M$. In addition, if $u \in (\frac{m_\rho}{\alpha}, \rho)$, then

$$M \le \frac{\rho}{2(\alpha+1)} < \frac{u}{2} < u < 2u < 2\rho.$$

Hence, for $u \in (\frac{m_\rho}{\alpha}, \rho)$,

$$\begin{aligned} f(\frac{\rho}{2(\alpha+1)}) &= L_\infty \left[\frac{\rho}{2(\alpha+1)}\right]^{s_\infty} \left[1 + R_\infty(\frac{\rho}{2(\alpha+1)})\right] \le L_\infty \left(\frac{u}{2}\right)^{s_\infty} (1+\varepsilon) \\ &\le L_\infty u^{s_\infty}(1-\varepsilon) \le f(u) = L_\infty u^{s_\infty} [1 + R_\infty(u)] \le L_\infty u^{s_\infty}(1+\varepsilon) \\ &\le L_\infty (2u)^{s_\infty}(1-\varepsilon) \le L_\infty (2\rho)^{s_\infty} [1 + R_\infty(2\rho)] = f(2\rho). \end{aligned}$$

By the Mean Value Theorem and (A.1) with $\gamma_1 = 1/2 < 2 = \gamma_2$, there exist $\rho_1 \in (\frac{m_\rho}{\alpha}, \rho)$ and $\rho_2 \in (u, \rho)$, such that

$$\begin{aligned} \lim_{\rho\to\infty} \frac{F(\rho) - F\left(\frac{m_\rho}{\alpha}\right)}{F(\rho) - F(u)} &= \lim_{\rho\to\infty} \frac{f(\rho_1)\left(\rho - \frac{m_\rho}{\alpha}\right)}{f(\rho_2)(\rho - u)} \le \lim_{\rho\to\infty} \frac{f(2\rho)\left(\rho - \frac{m_\rho}{\alpha}\right)}{f\left(\frac{\rho}{2(\alpha+1)}\right)(\rho - u)} \\ &\le 4^{s_\infty} (\alpha+1)^{s_\infty} \lim_{\rho\to\infty} \frac{\rho - \frac{m_\rho}{\alpha}}{\rho - u}. \end{aligned}$$

So we observe that

$$\begin{aligned}
\lim_{\rho\to\infty}\frac{m_\rho}{\rho} &= \lim_{\rho\to\infty}\int_{\frac{m_\rho}{\alpha}}^{\rho}\frac{\sqrt{F(\rho)-F\left(\frac{m_\rho}{\alpha}\right)}}{\rho\sqrt{F(\rho)-F(u)}}du \quad \text{(by (3.2))}\\
&\le 2^{s_\infty}(\alpha+1)^{\frac{s_\infty}{2}}\lim_{\rho\to\infty}\left(\frac{\sqrt{\rho-\frac{m_\rho}{\alpha}}}{\rho}\int_{\frac{m_\rho}{\alpha}}^{\rho}\sqrt{\frac{1}{\rho-u}}du\right)\\
&= 2^{s_\infty+1}(\alpha+1)^{\frac{s_\infty}{2}}\lim_{\rho\to\infty}\frac{\rho-\frac{m_\rho}{\alpha}}{\rho}\\
&= 2^{s_\infty+1}(\alpha+1)^{\frac{s_\infty}{2}}\lim_{\rho\to\infty}\left(1-\frac{m_\rho}{\alpha\rho}\right).
\end{aligned}$$

It follows that

$$\lim_{\rho\to\infty}\frac{m_\rho}{\rho}\le\frac{\alpha}{1+\frac{\alpha}{2^{s_\infty+1}(\alpha+1)^{\frac{s_\infty}{2}}}}<\alpha.$$

So (3.9) holds.

The proof of Lemma 3.2 (iii) is complete. □

Appendix B

Proof of Lemma 3.3. For fixed $\alpha>0$, since m_ρ is defined by the equation $H(m_\rho,\rho)=G(m_\rho,\rho)$, by the Implicit Function Theorem, m_ρ is a C^k function in ρ if $H(m,\rho)-G(m,\rho)$ is a C^k function in ρ. Since $f\in C^2(0,\eta)$ and by (3.1), we have that $H(m,\rho)-G(m,\rho)$ is C^3 in ρ and thus m_ρ is C^3 in ρ as well. We consider m_ρ as a function of α and ρ, and denote $\frac{\partial}{\partial\rho}m_\rho$ by m_ρ'. We then rewrite (3.2) as

$$T(\rho)=\rho\int_{\frac{m_\rho}{\alpha\rho}}^{1}\frac{ds}{\sqrt{F(\rho)-F(s\rho)}}=\frac{m_\rho}{\sqrt{F(\rho)-F(\frac{m_\rho}{\alpha})}}.$$

Differentiating the above equation with respect to ρ, we have that

$$\begin{aligned}
T'(\rho) &= \int_{\frac{m_\rho}{\alpha\rho}}^{1}\frac{ds}{\sqrt{F(\rho)-F(s\rho)}}-\frac{\rho}{2}\int_{\frac{m_\rho}{\alpha\rho}}^{1}\frac{f(\rho)-sf(s\rho)}{[F(\rho)-F(s\rho)]^{\frac{3}{2}}}ds-\rho\frac{\frac{m_\rho'}{\alpha\rho}-\frac{m_\rho}{\alpha\rho^2}}{\sqrt{F(\rho)-F\left(\frac{m_\rho}{\alpha}\right)}}\\
&= \frac{m_\rho'}{\sqrt{F(\rho)-F\left(\frac{m_\rho}{\alpha}\right)}}-\frac{m_\rho}{2}\frac{f(\rho)-\frac{m_\rho'}{\alpha}f\left(\frac{m_\rho}{\alpha}\right)}{\left[F(\rho)-F\left(\frac{m_\rho}{\alpha}\right)\right]^{\frac{3}{2}}}.
\end{aligned}$$

Using notations in (3.11), we obtain that

$$T'(\rho)=\frac{1}{2}\int_{\frac{m_\rho}{\alpha\rho}}^{1}\frac{\theta(\rho)-\theta(s\rho)}{D^{\frac{3}{2}}(s\rho)}ds+\frac{\frac{m_\rho}{\alpha\rho}-\frac{m_\rho'}{\alpha}}{\sqrt{D\left(\frac{m_\rho}{\alpha}\right)}}=\frac{m_\rho'}{\sqrt{D\left(\frac{m_\rho}{\alpha}\right)}}-\frac{m_\rho}{2}\frac{f(\rho)-\frac{m_\rho'}{\alpha}f\left(\frac{m_\rho}{\alpha}\right)}{D^{\frac{3}{2}}\left(\frac{m_\rho}{\alpha}\right)}. \tag{B.1}$$

Multiplying both sides of above RHS equation by $\sqrt{D(\frac{m_\rho}{\alpha})}$ and gathering terms containing m'_ρ, we obtain that

$$\frac{1}{2}P+\frac{m_\rho}{\alpha\rho}+\frac{m_\rho f(\rho)}{2D\left(\frac{m_\rho}{\alpha}\right)}=\left[1+\frac{1}{\alpha}+\frac{\frac{m_\rho}{\alpha}f\left(\frac{m_\rho}{\alpha}\right)}{2D\left(\frac{m_\rho}{\alpha}\right)}\right]m'_\rho.$$

Clearly, the coefficient of m'_ρ is always positive. Dividing both sides by the coefficient of m'_ρ, we obtain (3.10).

In (B.1), plugging in (3.10), we obtain that

$$\begin{aligned}
T'(\rho)&=\frac{1}{\sqrt{D\left(\frac{m_\rho}{\alpha}\right)}}\left[\frac{1}{2}P+\frac{m_\rho}{\alpha\rho}-\frac{1}{\alpha}\frac{\left(2\frac{m_\rho}{\alpha\rho}+P\right)D\left(\frac{m_\rho}{\alpha}\right)+m_\rho f(\rho)}{2\left(\frac{1}{\alpha}+1\right)D\left(\frac{m_\rho}{\alpha}\right)+\frac{m_\rho}{\alpha}f\left(\frac{m_\rho}{\alpha}\right)}\right]\\
&=\frac{1}{\sqrt{D\left(\frac{m_\rho}{\alpha}\right)}\left[2\left(\frac{1}{\alpha}+1\right)D\left(\frac{m_\rho}{\alpha}\right)+\frac{m_\rho}{\alpha}f\left(\frac{m_\rho}{\alpha}\right)\right]}\\
&\quad\times\left\{\left(\frac{1}{2}P+\frac{m_\rho}{\alpha\rho}\right)\left[2\left(\frac{1}{\alpha}+1\right)D\left(\frac{m_\rho}{\alpha}\right)+\frac{m_\rho}{\alpha}f\left(\frac{m_\rho}{\alpha}\right)\right]\right.\\
&\qquad\left.-\frac{1}{\alpha}\left[\left(2\frac{m_\rho}{\alpha\rho}+P\right)D\left(\frac{m_\rho}{\alpha}\right)+m_\rho f(\rho)\right]\right\}.
\end{aligned}$$

The denominator is exactly equal to $\Phi(\rho)$ as defined in (3.14), and the numerator

$$\begin{aligned}
\Delta(\rho)&=\left(\frac{1}{2}P+\frac{m_\rho}{\alpha\rho}\right)\left[2D\left(\frac{m_\rho}{\alpha}\right)+\frac{m_\rho}{\alpha}f\left(\frac{m_\rho}{\alpha}\right)\right]+\frac{2}{\alpha}D\left(\frac{m_\rho}{\alpha}\right)\left(\frac{1}{2}P+\frac{m_\rho}{\alpha\rho}\right)\\
&\quad-\frac{1}{\alpha}\left[\left(2\frac{m_\rho}{\alpha\rho}+P\right)D\left(\frac{m_\rho}{\alpha}\right)+m_\rho f(\rho)\right]\\
&=\left(\frac{1}{2}P+\frac{m_\rho}{\alpha\rho}\right)\left[2D\left(\frac{m_\rho}{\alpha}\right)+\frac{m_\rho}{\alpha}f\left(\frac{m_\rho}{\alpha}\right)\right]-\frac{m_\rho}{\alpha}f(\rho)\\
&=P\left[D\left(\frac{m_\rho}{\alpha}\right)+\frac{m_\rho}{2\alpha}f\left(\frac{m_\rho}{\alpha}\right)\right]+\frac{m_\rho}{\alpha\rho}\left[2F(\rho)-2F\left(\frac{m_\rho}{\alpha}\right)+\frac{m_\rho}{\alpha}f\left(\frac{m_\rho}{\alpha}\right)-\rho f(\rho)\right]\\
&=P\left[D\left(\frac{m_\rho}{\alpha}\right)+\frac{m_\rho}{2\alpha}f\left(\frac{m_\rho}{\alpha}\right)\right]+\frac{m_\rho}{\alpha\rho}\left[\theta(\rho)-\theta\left(\frac{m_\rho}{\alpha}\right)\right]
\end{aligned}$$

is exactly equal to (3.13) as well.

The proof of Lemma 3.3 is complete. □

Appendix C

Proof of Lemma 3.9. **(I)** Suppose that $s_\eta\geq 1$. We first claim that $\lim_{\rho\to\eta^-}\frac{m_\rho}{\alpha}=\eta$. Then the denominator of RHS of (3.2) is zero and $\lim_{\rho\to\eta^-}T(\rho)=\infty$. We could suppose for a contradiction that $\liminf_{\rho\to\eta^-}\frac{m_\rho}{\alpha}=M<\eta$. By definitions of s_η and L_η, there exists $\delta\in(0,\eta-M)$ such that, for $\eta-\delta<u<\eta$,

$$\frac{f(u)}{|u-\eta|^{s_\eta}}<\frac{3}{2}L_\eta$$

and

$$F(\eta)-F(s)=\int_s^\eta f(t)dt<\int_s^\eta \frac{3}{2}L_\eta\,|t-\eta|^{s_\eta}\,dt=\int_s^\eta \frac{3}{2}L_\eta(\eta-t)^{s_\eta}dt=\frac{3}{2(s_\eta+1)}L_\eta(\eta-s)^{s_\eta+1}.$$

Then, taking limit inferior to both sides of (3.2), we have that

$$\begin{aligned}
\liminf_{\rho\to\eta^-} LHS &= \liminf_{\rho\to\eta^-}\int_{\frac{m_\rho}{\alpha}}^{\rho}\frac{ds}{\sqrt{F(\rho)-F(s)}}\\
&\geq \liminf_{\rho\to\eta^-}\int_{\eta-\delta}^{\rho}\frac{ds}{\sqrt{F(\eta)-F(s)}}\quad (\text{since } F(\eta)\geq F(\rho) \text{ and } \eta-\delta\geq\frac{m_\rho}{\alpha})\\
&\geq \liminf_{\rho\to\eta^-}\int_{\eta-\delta}^{\rho}\frac{ds}{\sqrt{\frac{3}{2(s_\eta+1)}L_\eta(\eta-s)^{s_\eta+1}}}=\sqrt{\frac{2(s_\eta+1)}{3L_\eta}}\liminf_{\rho\to\eta^-}\int_{\eta-\delta}^{\rho}(\eta-s)^{-\frac{s_\eta+1}{2}}ds\\
&=\sqrt{\frac{2(s_\eta+1)}{3L_\eta}}\int_{\eta-\delta}^{\eta}(\eta-s)^{-\frac{s_\eta+1}{2}}ds,
\end{aligned}$$

which diverges to infinity since $s_\eta\geq 1$. But

$$\liminf_{\rho\to\eta^-} RHS=\liminf_{\rho\to\eta^-}\frac{m_\rho}{\sqrt{F(\eta)-F(\frac{m_\rho}{\alpha})}}=\frac{\alpha M}{\sqrt{F(\eta)-F(M)}}=\frac{\alpha M}{\sqrt{\int_M^\eta f(t)dt}}$$

is finite, which is a contradiction. So $\liminf_{\rho\to\eta^-}\frac{m_\rho}{\alpha}=\eta$. By Lemma 3.2 (ii), it implies $\lim_{\rho\to\eta^-}\frac{m_\rho}{\alpha}=\eta$ and hence $\lim_{\rho\to\eta^-}T(\rho)=\infty$.

(II) Suppose that $0<s_\eta<1$. Since $\frac{1}{\sqrt{F(\eta)-F(s)}}>0$ for $0<s<\eta$ and

$$\begin{aligned}
G(m,\eta)&=\int_{\frac{m}{\alpha}}^{\eta}\frac{ds}{\sqrt{F(\eta)-F(s)}}\leq\int_0^\eta\frac{ds}{\sqrt{F(\eta)-F(s)}}\\
&=\int_0^{\eta-\delta}\frac{ds}{\sqrt{F(\eta)-F(s)}}+\int_{\eta-\delta}^{\eta}\frac{ds}{\sqrt{F(\eta)-F(s)}},
\end{aligned}$$

where δ is chosen such that $\frac{f(u)}{|u-\eta|^{s_\eta}}>\frac{1}{2}L_\eta$ for $\eta-\delta<u<\eta$. The integral $\int_0^{\eta-\delta}\frac{ds}{\sqrt{F(\eta)-F(s)}}$ is a finite number and

$$\int_{\eta-\delta}^{\eta}\frac{ds}{\sqrt{F(\eta)-F(s)}}\leq\sqrt{\frac{2}{L_\eta}}\int_{\eta-\delta}^{\eta}(\eta-s)^{-\frac{s_\eta+1}{2}}ds$$

is also finite for $0<s_\eta<1$. So we have that $G(m,\eta)$ is well-defined for $0<s_\eta<1$. Then by Lemma 3.1, m_η can be uniquely defined and hence $\lim_{\rho\to\eta^-}T(\rho)=G(m_\eta,\eta)$ is a finite positive number.

The proof of Lemma 3.9 is complete. □

Appendix D

Proof of Lemma 3.10. Assuming that f satisfies condition (D1) with positive $p^*<p<\eta$ and $\int_0^p s\,[f(s)-sf'(s)]\,ds<0$, we show that, in (3.12), $T'(p)=\frac{\Delta(p)}{\Phi(p)}<0$ for $\rho=p$ and for all $\alpha > 0$ by showing that $\Delta(p) < 0$ since $\Phi(p) > 0$. Since f satisfies (D1), there exists $\bar{p} \in (0, p^*) \subset (0, p)$ such that $\theta(p) = \theta(\bar{p})$; see Fig. 2.2 (I). Clearly, we have that

$$\theta(p) - \theta(u) \begin{cases} > 0 & \text{for } 0 < u < \bar{p}, \\ = 0 & \text{for } u = \bar{p}, \\ < 0 & \text{for } \bar{p} < u < p. \end{cases} \tag{D.1}$$

First, assume that $0 < \frac{m_p}{\alpha} < \bar{p}$. We first observe that

$$\begin{aligned}
&\int_{\frac{m_p}{\alpha}}^{p} [\theta(p) - \theta(s)]\, ds \\
&\quad = -\frac{m_p}{\alpha}\left[\theta(p) - \theta\left(\frac{m_p}{\alpha}\right)\right] + \int_{\frac{m_p}{\alpha}}^{p} s\theta'(s) ds \quad \text{(by integration by parts)} \\
&\quad < -\frac{m_p}{\alpha}\left[\theta(p) - \theta\left(\frac{m_p}{\alpha}\right)\right] + \int_{0}^{p} s\theta'(s) ds \quad (\text{since } \theta'(s) = f(s) - sf'(s) > 0 \text{ on } (0, \tfrac{m_p}{\alpha})) \\
&\quad = -\frac{m_p}{\alpha}\left[\theta(p) - \theta\left(\frac{m_p}{\alpha}\right)\right] + \int_{0}^{p} s\left[f(s) - sf'(s)\right] ds < 0.
\end{aligned} \tag{D.2}$$

In addition, we observe that

$$\begin{aligned}
&\int_{\frac{m_p}{\alpha}}^{p} \frac{\sqrt{D(\frac{m_p}{\alpha})}}{D^{\frac{3}{2}}(s)} [\theta(p) - \theta(s)]\, ds \\
&\quad = \int_{\frac{m_p}{\alpha}}^{\bar{p}} \frac{\sqrt{D(\frac{m_p}{\alpha})}}{D^{\frac{3}{2}}(s)} [\theta(p) - \theta(s)]\, ds + \int_{\bar{p}}^{p} \frac{\sqrt{D(\frac{m_p}{\alpha})}}{D^{\frac{3}{2}}(s)} [\theta(p) - \theta(s)]\, ds \\
&\quad < \frac{\sqrt{D(\frac{m_p}{\alpha})}}{D^{\frac{3}{2}}(\bar{p})} \int_{\frac{m_p}{\alpha}}^{\bar{p}} [\theta(p) - \theta(s)]\, ds + \frac{\sqrt{D(\frac{m_p}{\alpha})}}{D^{\frac{3}{2}}(\bar{p})} \int_{\bar{p}}^{p} [\theta(p) - \theta(s)]\, ds \\
&\quad\quad \text{(by (D.1) and since } D(u) \text{ is strictly decreasing in } u) \\
&\quad < \frac{\sqrt{D(\frac{m_p}{\alpha})}}{D^{\frac{3}{2}}(\frac{m_p}{\alpha})} \int_{\frac{m_p}{\alpha}}^{p} [\theta(p) - \theta(s)]\, ds \quad (\text{since } D(u) \text{ is strictly decreasing in } u) \\
&\quad = \frac{1}{D(\frac{m_p}{\alpha})} \int_{\frac{m_p}{\alpha}}^{p} [\theta(p) - \theta(s)]\, ds < 0.
\end{aligned} \tag{D.3}$$

By (3.13) and (3.11), we further observe that

$$\begin{aligned}
\Delta(p) &= \frac{1}{2p}\left[\frac{m_p}{\alpha} f\left(\frac{m_p}{\alpha}\right) + 2D\left(\frac{m_p}{\alpha}\right)\right] \left\{\int_{\frac{m_p}{\alpha}}^{p} \frac{\sqrt{D(\frac{m_p}{\alpha})}}{D^{\frac{3}{2}}(s)} [\theta(p) - \theta(s)]\, ds\right\} + \frac{m_p}{\alpha p}\left[\theta(p) - \theta\left(\frac{m_p}{\alpha}\right)\right] \\
&< \frac{1}{2p} \frac{2D(\frac{m_p}{\alpha})}{D(\frac{m_p}{\alpha})} \left\{\int_{\frac{m_p}{\alpha}}^{p} [\theta(p) - \theta(s)]\, ds\right\} + \frac{m_p}{\alpha p}\left[\theta(p) - \theta\left(\frac{m_p}{\alpha}\right)\right] \\
&\quad \text{(by (D.3) and (D.2) and since } \tfrac{m_p}{\alpha} f(\tfrac{m_p}{\alpha}) > 0) \\
&= \frac{1}{p} \int_{\frac{m_p}{\alpha}}^{p} [\theta(p) - \theta(s)]\, ds + \frac{m_p}{\alpha p}\left[\theta(p) - \theta\left(\frac{m_p}{\alpha}\right)\right]
\end{aligned}$$

$$= -\frac{m_p}{\alpha p}\left[\theta(p) - \theta\left(\frac{m_p}{\alpha}\right)\right] + \frac{1}{p}\int_{\frac{m_p}{\alpha}}^{p} s\theta'(s)ds + \frac{m_p}{\alpha p}\left[\theta(p) - \theta\left(\frac{m_p}{\alpha}\right)\right]$$

(by integration by parts)

$$= \frac{1}{p}\int_{\frac{m_p}{\alpha}}^{p} s\theta'(s)ds = \frac{1}{p}\int_{\frac{m_p}{\alpha}}^{p} s\left[f(s) - sf'(s)\right]ds < \frac{1}{p}\int_{0}^{p} s\left[f(s) - sf'(s)\right]ds < 0$$

by (D.1) and since $\int_0^p s\left[f(s) - sf'(s)\right]ds < 0$. So, for $0 < \frac{m_p}{\alpha} < \bar{p}$, we have $\Delta(p) < 0$ for all $\alpha > 0$. Secondly, assume that $\bar{p} \le \frac{m_p}{\alpha} < p$. Since $\theta(p) - \theta(s) < 0$ for $\frac{m_p}{\alpha} < s < p$ by (D.1), and by (3.13), we obtain $\Delta(p) < 0$ for all $\alpha > 0$. The proof of part (i) is complete. Similarly, part (ii) can be proved. We omit it here.

The proof of Lemma 3.10 is complete. □

Appendix E

Proof of Lemma 3.11. We compute the numerator of $T'(\rho)$ in (3.12)

$$\begin{aligned}\Delta(\rho) = \Delta(\rho,k,\alpha) &= P\left[D(k) + \frac{k}{2}f(k)\right] + \frac{k}{\rho}\left[\theta(\rho) - \theta(k)\right] \quad \text{(by (3.13))}\\ &= \left(\frac{k}{\rho} + \frac{1}{2}P\right)\left[kf(k) + 2D(k)\right] - kf(\rho) \quad \text{(by (3.11))}\\ &= \left(\frac{k}{\rho} + \frac{1}{2}P\right)B(k) - kf(\rho),\end{aligned}$$

where

$$P = \frac{1}{\rho}\int_k^{\rho} \frac{\sqrt{D(k)}}{D^{\frac{3}{2}}(s)}\left[\theta(\rho) - \theta(s)\right]ds.$$

We observe that

$$\frac{\partial}{\partial k}D(k) = \frac{\partial}{\partial k}\left[F(\rho) - F(k)\right] = -f(k),$$

$$\frac{\partial B(k)}{\partial k} = f(k) + kf'(k) - 2f(k) = kf'(k) - f(k) = -\theta'(k), \tag{E.1}$$

and

$$\begin{aligned}\frac{\partial P}{\partial k} &= -\frac{1}{\rho}\frac{\sqrt{D(k)}}{D^{\frac{3}{2}}(k)}\left[\theta(\rho) - \theta(k)\right] + \frac{1}{\rho}\frac{\frac{\partial}{\partial k}D(k)}{2\sqrt{D(k)}}\int_k^{\rho}\frac{1}{D^{\frac{3}{2}}(s)}\left[\theta(\rho) - \theta(s)\right]ds\\ &= -\frac{\theta(\rho) - \theta(k)}{\rho D(k)} - \frac{f(k)}{2D(k)}P.\end{aligned} \tag{E.2}$$

Differentiating the above expression of $\Delta(\rho,k,\alpha)$ with respect to k $(= \frac{m}{\alpha})$, we then compute that

$$\begin{aligned}&\frac{\partial\Delta(\rho,k,\alpha)}{\partial k}\\ &= \left(\frac{1}{\rho} + \frac{1}{2}\frac{\partial P}{\partial k}\right)B(k) + \left(\frac{k}{\rho} + \frac{1}{2}P\right)\frac{\partial B(k)}{\partial k} - f(\rho)\\ &= \left[\frac{1}{\rho} - \frac{1}{2}\frac{\theta(\rho) - \theta(k)}{\rho D(k)} - \frac{f(k)}{4D(k)}P\right]B(k) - \left(\frac{k}{\rho} + \frac{1}{2}P\right)\theta'(k) - f(\rho) \quad \text{(by (E.1) and (E.2))}\\ &= \left[-\frac{f(k)}{4D(k)}B(k) - \frac{1}{2}\theta'(k)\right]P + \left[\frac{1}{\rho} - \frac{1}{2}\frac{\theta(\rho) - \theta(k)}{\rho D(k)}\right]B(k) - \frac{k}{\rho}\theta'(k) - f(\rho).\end{aligned} \tag{E.3}$$

Adding the term $\frac{2}{B(k)}\left[\frac{f(k)}{4D(k)}B(k)+\frac{1}{2}\theta'(k)\right]\Delta(\rho,k,\alpha)$ to (E.3) to eliminate P, we have that

$$\begin{aligned}
&\frac{\partial\Delta(\rho,k,\alpha)}{\partial k}+\frac{2}{B(k)}\left[\frac{f(k)}{4D(k)}B(k)+\frac{1}{2}\theta'(k)\right]\Delta(\rho,k,\alpha)\\
&=\left(\frac{1}{\rho}-\frac{\theta(\rho)-\theta(k)}{2\rho D(k)}\right)B(k)-\frac{k}{\rho}\theta'(k)-f(\rho)+\frac{2}{B(k)}\left[\frac{f(k)}{4D(k)}B(k)+\frac{1}{2}\theta'(k)\right]\left[\frac{k}{\rho}B(k)-kf(\rho)\right]\\
&=-\frac{\theta(\rho)-\theta(k)}{2\rho D(k)}B(k)-\frac{k}{\rho}\theta'(k)+\frac{\theta(\rho)-\theta(k)}{\rho}+\frac{1}{B(k)}\left[\frac{f(k)}{2D(k)}B(k)+\theta'(k)\right]\frac{k}{\rho}\left[\theta(\rho)-\theta(k)\right]\\
&=\frac{1}{\rho}\left[-\frac{B(k)}{2D(k)}+1+\frac{kf(k)}{2D(k)}\right]\left[\theta(\rho)-\theta(k)\right]+\left\{-\frac{k}{\rho}+\frac{1}{B(k)}\frac{k}{\rho}\left[\theta(\rho)-\theta(k)\right]\right\}\theta'(k)\\
&=\frac{k}{\rho}\left[\frac{\theta(\rho)-\theta(k)}{B(k)}-1\right]\theta'(k)\quad\left(\text{since }-\tfrac{B(k)}{2D(k)}+1+\tfrac{kf(k)}{2D(k)}=0\text{ by (3.23)}\right)\\
&=-\frac{kf(\rho)}{B(k)}\theta'(k)\begin{cases}<0 & \text{for }\rho=p>p^*>k>0,\text{ if } f \text{ satisfies (D1) with } p^*<p<\eta,\\ >0 & \text{for }\rho=p>p^*>k>0,\text{ if } f \text{ satisfies (D2) with } p^*<p<\eta,\end{cases}
\end{aligned}$$

see Fig. 2.2 (I)–(II).

The proof of Lemma 3.11 is complete. □

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