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Existence of Solutions for Fractional Differential Equations with Multi-point Boundary Conditions at Resonance on a Half-line

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Abstract

In this paper, we investigate the existence of solutions for multi-point boundary value problems at resonance concerning fractional differential equation on a half-line. Our analysis relies on the coincidence degree of Mawhin. As an application, an example is presented to illustrate the main results.

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1 Introduction

Fractional differential equations have been of great interest recently. This is because of the intensive development of the theory of fractional calculus itself as well as its applications. Apart from diverse areas of mathematics, fractional differential equations arise in a variety of different areas such as rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, and many other branches of science (see [1, 9, 10, 11, 16, 17, 25] and references cited therein). The research of fractional differential equations on boundary value problems, as one of the focal topics, has attained a great deal of attention from many researchers (see [18, 19, 20, 21, 29, 34]).

In this paper, we study the existence of solutions for the fractional differential equation at resonance with multi-point boundary value problem on a half-line:

$$D_{0+}^{\alpha}x(t) = f(t, x(t), D_{0+}^{\alpha-1}x(t)),\ t \in (0, +\infty), \tag{1.1}$$

$$x(0) = 0, \lim_{t\to+\infty} D_{0+}^{\alpha-1}x(t) = \beta x(\eta), \tag{1.2}$$

where $1 < \alpha \le 2$, $\eta > 0$, $f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an S-Carathéodory function, and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. Moreover, we need the following condition

$$\Gamma(\alpha) = \beta\eta^{\alpha-1}. \tag{1.3}$$

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The condition (1.3) is critical since the fractional differential operator in (1.1) has a nontrivial kernel. Boundary value problems with such critical conditions are so-called problems at resonance. Boundary value problems for differential equations of integer order at resonance have been studied by many authors (see [3, 4, 5, 6, 7, 12, 13, 14, 22, 23, 24, 26, 27] and references cited therein).

More recently, various types of multi-point boundary value problems for fractional differential equations at resonance on a bounded domain have been analyzed by Kosmatov [8], Jiang [35], Bai [30], Bai and Zhang [31, 32].

However, to our knowledge, it is rare for work to be done on the solutions of fractional differential equations at resonance on a half-line. In this paper, our goal is to fill this gap in the literature.

The layout of this paper is as follow. In Section 2, we provide some necessary background. In particular, we shall introduce some lemmas and definitions related with problem (1.1) and (1.2). In Section 3, the main results of problem (1.1) and (1.2) will be stated and proved. Finally, one example is also included to illustrate the main results.

2 Background materials and preliminaries

In this section, to establish the existence of solutions, we present some necessary background and lemmas. These definitions and lemmas can be found in the recent literature [1, 2, 15, 28, 36].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s$$

provided the right side is pointwise defined on $(0, +\infty)$, and we have

$$I_{0+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \alpha > 0, \mu > -1.$$

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = D^n I_{0+}^{n-\alpha} f(t)$$

where $n = [\alpha] + 1, D^n = \frac{\mathrm{d}^n}{\mathrm{d}t^n}, t > 0$, and we have for $\lambda > -1$

$$D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}.$$

Lemma 2.1. Assume that $u \in C(0, +\infty) \cap L_{loc}(0, +\infty)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, +\infty) \cap L_{loc}(0, +\infty)$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-N},$$

for some $c_i \in \mathbb{R}, i = 1, \ldots, N$, where N is the smallest integer greater than or equal to α.

We now give the background from the coincidence degree theory.

Definition 2.3. Let X and Y be normed spaces. A linear operator $L:\mathrm{dom}(L)\subset X\to Y$ is said to be a Fredholm operator of index zero provided that

(i) $\mathrm{Im}L$ is closed subset of Y, and

(ii) dim $\mathrm{Ker}L$=codim $\mathrm{Im}L<+\infty$.

From definition2.3 it follows that there exist continuous projects $P: X\to X$ and $Q: Y\to Y$ such that $\mathrm{Im}P=\mathrm{Ker}L$, $\mathrm{Ker}Q=\mathrm{Im}L$, $X=\mathrm{Ker}L\oplus\mathrm{Ker}P$, $Y=\mathrm{Im}L\oplus\mathrm{Im}Q$ and $L|_{\mathrm{Dom}L\cap\mathrm{Ker}P}:\mathrm{Dom}L\cap\mathrm{Ker}P\to\mathrm{Im}L$ is invertible. We denote the inverse that map by $K_p:\mathrm{Im}L\to\mathrm{Dom}L\cap\mathrm{Ker}P$. The generalized inverse of L denoted by $K_{P,Q}:Y\to\mathrm{Dom}L\cap\mathrm{Ker}P$ is defined by $K_{P,Q}=K_P(I-Q)$.

Definition 2.4. Let $L:\mathrm{Dom(L)}\subset X\to Y$ be a Fredholm operator, E be a metric space, and $N:E\to Y$ be an operator. We say that N is L-compact on E if $QN:E\to Y$ and $K_{P,Q}N:E\to X$ are compact on E.

Definition 2.5. $f:[0,+\infty)\times\mathbb{R}^2\to\mathbb{R}$ is said to be an S-Carathéodory function if and only if

(i) for each $(u,v)\in\mathbb{R}^2$, $t\mapsto f(t,u,v)$ is measurable on $[0,+\infty)$;

(ii) for a.e. $t\in[0,+\infty)$, $(u,v)\mapsto f(t,u,v)$ is continuous on $\mathbb{R}^2$;

(iii) for each $r>0$, there exists $\varphi_r(t)\in L^1[0,+\infty)\cap C[0,+\infty)$ satisfying $\sup_{t\geq0}|\varphi_r(t)|<+\infty$, $\varphi_r(t)>0$, $t\in(0,+\infty)$ such that

$$\max\{|u|,|v|\}\leq r \text{ impiles } |f(t,(1+t^{\alpha-1})u,v)|\leq\varphi_r(t),\text{ a.e. } t\in[0,+\infty).$$

Since the Arzela-Ascoli theorem fails to work in the space C_∞, we need a modified compactness criterion to prove that N is L-compact($C_\infty=\{x\in C[0,+\infty),\lim_{t\to+\infty}x(t)\text{ exists}\}$).

Lemma 2.6.[28] Let $M\subset C_\infty=\{x\in C[0,+\infty),\lim_{t\to+\infty}x(t)\text{ exists}\}$. Then M is relatively compact if the following conditions hold:

(i) all functions from M are uniformly bounded;

(ii) all functions from M are equicontinuous on any compact interval of $[0,+\infty)$;

(iii) all functions from M are equiconvergent at infinity, that is, for any given $\varepsilon>0$, there exists a $T=T(\varepsilon)>0$ such that $|f(t_1)-f(t_2)|<\varepsilon$, for all $t_1,t_2>T$ and $f\in M$.

In this paper, we use the space X,Y defined by

$$X=\{x\in C[0,+\infty):\lim_{t\to+\infty}x(t)/(1+t^{\alpha-1})\text{ exists},\lim_{t\to+\infty}D_{0+}^{\alpha-1}x(t)\text{ exists}\},$$

$$Y=\{y\in C[0,+\infty):y\in L^1[0,+\infty),\ \sup_{t\geq0}|y(t)|<+\infty\},$$

with the norm $\|x\|_X=\max\{\|x\|_0,\|D_{0+}^{\alpha-1}x\|_\infty\}$ and $\|y\|_Y=\max\{\|y\|_{L^1},\|y\|_\infty\}$ respectively, where $\|\cdot\|_\infty$ is the supremum norm on $[0,+\infty)$ and $\|x\|_0=\sup_{t\geq0}|x(t)|/(1+t^{\alpha-1})$, $\|y\|_{L^1}=\int_0^{+\infty}|y(s)|\mathrm{d}s$. By the standard arguments, we can prove that $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ are Banach spaces.

Define L be the linear operator from $\mathrm{Dom}L\cap X$ to Y with

$$\mathrm{Dom}L=\{x\in X:D_{0+}^{\alpha}x\in L^1[0,+\infty)\cap C[0,+\infty),x\text{ satisfies (1.2)}\}$$

and

$$Lx=D_{0+}^{\alpha}x,x\in\mathrm{Dom}L.$$

We define $N : X \to Y$ by setting

$$(Nx)(t) = f(t, x(t), D_{0+}^{\alpha-1}x(t)).$$

Lemma 2.7. The operator $L : \mathrm{Dom(L)} \subset X \to Y$ is a Fredholm operator of index zero. Furthermore, the linear projector operator $Q : Y \to Y$ can be defined by

$$(Qy)(t) = w(t)\left(\int_0^{+\infty} y(s)\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}y(s)\mathrm{d}s\right)$$

where $w(t) \in Y$ satisfy $w(t) > 0$ and $\int_0^{+\infty} w(s)\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}w(s)\mathrm{d}s = 1$ and the linear operator $K_p : \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P$ can be written by

$$K_P y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s,\ t \in [0, +\infty),$$

also

$$\|K_P y\|_X \leq \frac{1}{\Gamma(\alpha)}\|y\|_{L^1},\ \text{for all } y \in \mathrm{Im}L.$$

Proof. It is clear that $\mathrm{Ker}L = \{x = ct^{\alpha-1} : c \in \mathbb{R}\}$. Now we show that

$$\mathrm{Im}L = \{y \in Y : \int_0^{+\infty} y(s)\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}y(s)\mathrm{d}s = 0\}. \tag{2.1}$$

If $y \in \mathrm{Im}L$, then there exists a function $x \in \mathrm{Dom}L$ such that $y(t) = D_{0+}^{\alpha}x(t)$. By Lemma2.1,

$$I_{0+}^{\alpha}y(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

By virtue of the boundary condition (1.2), we have

$$\lim_{t\to+\infty} D_{0+}^{\alpha-1}I_{0+}^{\alpha}y(t) = \beta I_{0+}^{\alpha}y(\eta)$$

and therefore

$$\int_0^{+\infty} y(s)\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}y(s)\mathrm{d}s = 0. \tag{2.2}$$

On the other hand, suppose $y \in Y$ and satisfies (2.2), let $x(t) = I_{0+}^{\alpha}y(t)$, then $x(0) = 0, D_{0+}^{\alpha-1}x(t) = \int_0^t y(s)\mathrm{d}s$. Thus $\lim_{t\to+\infty} D_{0+}^{\alpha-1}x(t) = \beta x(\eta)$. Then $x \in \mathrm{Dom}L$ and $D_{0+}^{\alpha}x(t) = y(t)$. That is to say, (2.1) holds.

For $y \in Y$, taking the projector

$$(Qy)(t) = w(t)\left(\int_0^{+\infty} y(s)\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}y(s)\mathrm{d}s\right).$$

It is easy to verify that the operator Q is a projector. Letting $y_1 = y - Qy$. Then $y_1 \in \mathrm{Im}L$(since $\int_0^{+\infty} y_1(s)\mathrm{d}s - (\beta/\Gamma(\alpha))\int_0^{\eta}(\eta-s)^{\alpha-1}y_1(s)\mathrm{d}s = 0$). Hence $Y = \mathrm{Im}L + \mathrm{Im}Q$. From $y \in \mathrm{Im}Q$, there exists a constant $c \in \mathbb{R}$, such that $y = cw(t)$, and from $y \in \mathrm{Im}L$, we obtain

$$c = \int_0^{+\infty} cw(s)\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}cw(s)\mathrm{d}s = 0,$$

which implies $\mathrm{Im}L \cap \mathrm{Im}Q = \{0\}$ and $Y = \mathrm{Im}L \oplus \mathrm{Im}Q$. Thus we have

$$\dim \mathrm{Ker}L = \dim \mathrm{Im}Q = \mathrm{codim}\ \mathrm{Im}L = 1.$$

This implies that L is a Fredholm operator of index zero.

Taking $P : X \to X$ as follows:

$$(Px)(t) = \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1} x(0) t^{\alpha-1},$$

then the generalized inverse $K_P : \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P$ can be written by

$$K_P y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \mathrm{d}s,\ t \in [0, +\infty).$$

In fact, for $y \in \mathrm{Im}L$, we have

$$(LK_P)y(t) = D_{0+}^{\alpha}((K_P y)(t)) = y(t),$$

and for $x \in \mathrm{Dom}L \cap \mathrm{Ker}P$, we know

$$(K_P L)x(t) = (K_P) D_{0+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2},$$

for some $c_1, c_2 \in \mathbb{R}$. In view of $x \in \mathrm{Dom}L \cap \mathrm{Ker}P$, $D_{0+}^{\alpha-1} x(0) = 0$ and (1.2), we obtain $c_1 = c_2 = 0$. Therefore

$$(K_P L)x(t) = x(t).$$

This shows that $K_P = (L|_{\mathrm{Dom}L \cap \mathrm{Ker}P})^{-1}$.

Again from the definition of K_P, we have

$$\|K_P y\|_0 = \sup_{t \geq 0} \frac{1}{\Gamma(\alpha)} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}} y(s) \mathrm{d}s \right| \leq \frac{1}{\Gamma(\alpha)} \|y\|_{L^1}, \tag{2.3}$$

and

$$\|D_{0+}^{\alpha-1} K_P y\|_\infty = \sup_{t \geq 0} |D_{0+}^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \mathrm{d}s| = \sup_{t \geq 0} | \int_0^t y(s) \mathrm{d}s| \leq \|y\|_{L^1}. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\|K_p y\|_X \leq \frac{1}{\Gamma(\alpha)} \|y\|_{L^1}.$$

This completes the proof of Lemma 2.7.

Lemma 2.8. Let f be an S-Carathéodory function, then N is L-compact.

Proof. Obviously, QN and $K_P(I-Q)N$ are continuous. So we only need to prove the compactness, i.e. QN and $K_P(I-Q)N$ maps bounded sets into relatively compact ones.

Suppose $U \subset X$ is a bounded set. Then there exists $r > 0$ such that $\|x\|_X \leq r$, for all $x \in U$. Because f is an S-Carathéodory function, there exists $\varphi_r(t) \in L^1[0, +\infty) \cap C[0, +\infty)$ satisfying $\sup_{t \geq 0} |\varphi_r(t)| < +\infty,\ \varphi_r(t) > 0,\ t \in (0, +\infty)$ such that

$$|f(t, x(t), D_{0+}^{\alpha-1} x(t))| = |f(t, (1+t^{\alpha-1}) \frac{x(t)}{1+t^{\alpha-1}}, D_{0+}^{\alpha-1} x(t))| \leq \varphi_r(t),\ \text{a.e. } t \in [0, +\infty).$$

Then for any $x \in U$,

$$
\begin{aligned}
\|QNx\|_{L^1} &= \int_0^{+\infty} \Big| w(t)(\int_0^{+\infty} f(s,x(s),D_{0+}^{\alpha-1}x(s))\,\mathrm{d}s \\
&\qquad - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s,x(s),D_{0+}^{\alpha-1}x(s))\mathrm{d}s)\Big|\,\mathrm{d}t \\
&\le \int_0^{+\infty}|w(t)|\mathrm{d}t\left(\int_0^{+\infty}\varphi_r(s)\mathrm{d}s + \frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\eta}\varphi_r(s)\mathrm{d}s\right) \\
&\le \left(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\|w\|_{L^1}\|\varphi_r\|_{L^1},
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
\|QNx\|_{\infty} &= \sup_{t\ge 0}\Big| w(t)(\int_0^{+\infty} f(s,x(s),D_{0+}^{\alpha-1}x(s))\,\mathrm{d}s \\
&\qquad - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s,x(s),D_{0+}^{\alpha-1}x(s))\mathrm{d}t)\Big| \\
&\le \left(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\|w\|_{\infty}\|\varphi_r\|_{L^1}.
\end{aligned}
\tag{2.6}
$$

It follows from (2.5) and (2.6) that $\|QNx\|_Y = \max\{\|QNx\|_{L^1}, \|QNx\|_{\infty}\} \le (1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)})\|w\|_Y\|\varphi_r\|_{L^1}$. Noting that $\mathrm{Im}Q \simeq \mathbb{R}$, we have QN is compact.

Furthermore, for any $x \in U$ we have

$$
\begin{aligned}
\left|\frac{(K_{P,Q}x)(t)}{1+t^{\alpha-1}}\right| &= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}\Big| f(s,x(s),D_{0+}^{\alpha-1}x(s)) \\
&\qquad - w(s)\left(\int_0^{+\infty} f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\,\mathrm{d}\tau\right. \\
&\qquad \left. -\frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\tau)^{\alpha-1}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right)\Big|\,\mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t\varphi_r(s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t w(s)\mathrm{d}s\left(\int_0^{+\infty}\varphi_r(\tau)\mathrm{d}\tau\right. \\
&\qquad \left. + \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\tau)^{\alpha-1}\varphi_r(\tau)\mathrm{d}\tau\right) \\
&\le \frac{1}{\Gamma(\alpha)}\|\varphi_r\|_{L^1} + \frac{1}{\Gamma(\alpha)}\left(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\|w\|_{L^1}\|\varphi_r\|_{L^1},
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
|D_{0+}^{\alpha-1}(K_{P,Q}x)(t)| &= \Big|\int_0^t (f(s,x(s),D_{0+}^{\alpha-1}x(s)) - w(s)\left(\int_0^{+\infty} f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right. \\
&\qquad \left. -\frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\tau)^{\alpha-1}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\Big| \\
&\le \|\varphi_r\|_{L^1} + \left(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)}\right)\|w\|_{L^1}\|\varphi_r\|_{L^1}.
\end{aligned}
\tag{2.8}
$$

It follows from (2.7) and (2.8) that $K_{P,Q}U$ is uniformly bounded. Meanwhile, for any $t_1, t_2 \in [0, T]$ with T is a positive constant

$$
\begin{aligned}
&\left|\frac{(K_{P,Q}x)(t_1)}{1+t_1^{\alpha-1}} - \frac{(K_{P,Q}x)(t_2)}{1+t_2^{\alpha-1}}\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\Big| f(s,x(s),D_{0+}^{\alpha-1}x(s)) \\
&\qquad - w(s)\left(\int_0^{+\infty} f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right. \\
&\qquad \left. -\frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\tau)^{\alpha-1}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\Big| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\Big| f(s,x(s),D_{0+}^{\alpha-1}x(s)) \\
&\qquad - w(s)\left(\int_0^{+\infty} f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right. \\
&\qquad \left. -\frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\tau)^{\alpha-1}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\Big| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\Big|\varphi_r(s) + w(s)\left(\int_0^{+\infty}\varphi_r(\tau)\mathrm{d}\tau\right. \\
&\qquad \left. -\frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\tau)^{\alpha-1}\varphi_r(\tau)\mathrm{d}\tau\right)\mathrm{d}s\Big| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\Big|\varphi_r(s) + w(s)\left(\int_0^{+\infty}\varphi_r(\tau)\mathrm{d}\tau\right. \\
&\qquad \left. -\frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\tau)^{\alpha-1}\varphi_r(\tau)\mathrm{d}\tau\right)\mathrm{d}s\Big| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\left|\varphi_r(s) + w(s)\|\varphi_r\|_{L^1}(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)})\right|\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\left|\varphi_r(s) + w(s)\|\varphi_r\|_{L^1}(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)})\right|\mathrm{d}s \\
&\to 0 \quad \text{as } t_1 \to t_2
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
&|D_{0+}^{\alpha-1}(K_{P,Q}x)(t_1) - D_{0+}^{\alpha-1}(K_{P,Q}x)(t_2)| \\
&= \Big|\int_{t_1}^{t_2} f(s,x(s),D_{0+}^{\alpha-1}x(s)) - w(s)\left(\int_0^{+\infty}\varphi_r(\tau)\mathrm{d}\tau\right. \\
&\qquad \left. -\frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\tau)^{\alpha-1}\varphi_r(\tau)\mathrm{d}\tau\right)\mathrm{d}s\Big| \\
&\leq \left|\int_{t_1}^{t_2}\varphi_r(s) + w(s)\|\varphi_r\|_{L^1}(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)})\mathrm{d}s\right| \\
&\to 0 \quad \text{as } t_1 \to t_2.
\end{aligned}
\tag{2.10}
$$

It follows from (2.9) and (2.10) that $K_{P,Q}U$ is equicontinuous. From Lemma 2.6, we can see that if $K_{P,Q}U/(1+t^{\alpha-1})$, $D_{0+}^{\alpha-1}K_{P,Q}U$ are equiconvergent at infinity, then $K_{P,Q}U$ is relatively compact in X. In fact, considering that the following estimate

$$\frac{1}{\Gamma(\alpha)}\int_L^{+\infty}\varphi_r(s)+w(s)\|\varphi_r\|_{L^1}(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)})\mathrm{d}s<\frac{\varepsilon}{3}, \tag{2.11}$$

holding for $\varepsilon>0$ and some $L>0$, we have

$$\lim_{t\to\infty}\sup_{s\in[0,L]}|g(t,s)-1|\le\lim_{t\to\infty}g(t,L)=0,$$

where $g(t,s)=\frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}$, $s\in[0,L], t\in[L,+\infty)$. Thus, there exists $T>L$ such that for $t_1,t_2\ge T$,

$$\begin{aligned}&\sup_{s\in[0,L]}|g(t_1,s)-g(t_2,s)|\\&\le\sup_{s\in[0,L]}|g(t_1,s)-1|+\sup_{s\in[0,L]}|g(t_2,s)-1|\\&<\frac{\varepsilon}{3}\left(\|\varphi_r\|_{L^1}(1+\|w\|_{L^1}(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)}))\right)^{-1}.\end{aligned} \tag{2.12}$$

Therefore, it follows from (2.11) and (2.12) that for $t_1,t_2\ge T$, we get

$$\begin{aligned}&\left|\frac{(K_{P,Q}x)(t_1)}{1+t_1^{\alpha-1}}-\frac{(K_{P,Q}x)(t_2)}{1+t_2^{\alpha-1}}\right|\\
&\le\frac{1}{\Gamma(\alpha)}\int_0^L\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|\Bigg|f(s,x(s),D_{0+}^{\alpha-1}x(s))\\
&\qquad-w(s)\left(\int_0^{+\infty}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right.\\
&\qquad\left.-\frac{\beta}{\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\Bigg|\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_L^{t_1}\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\Bigg|f(s,x(s),D_{0+}^{\alpha-1}x(s))-w(s)\left(\int_0^{+\infty}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right.\\
&\qquad\left.-\frac{\beta}{\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\Bigg|\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_L^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\Bigg|f(s,x(s),D_{0+}^{\alpha-1}x(s))-w(s)\left(\int_0^{+\infty}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right.\\
&\qquad\left.-\frac{\beta}{\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\Bigg|\\
&\le\frac{\varepsilon}{3}\left(\int_0^L|f(s,x(s),D_{0+}^{\alpha-1}x(s))|+w(s)\left|\left(\int_0^{+\infty}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right.\right.\right.\\
&\quad\left.\left.\left.-\frac{\beta}{\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}f(\tau,x(\tau),D_{0+}^{\alpha-1}x(\tau))\mathrm{d}\tau\right)\right|\mathrm{d}s\right)\left(\|\varphi_r\|_{L^1}(1+\|w\|_{L^1}(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)}))\right)^{-1}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_L^{t_1}\varphi_r(s)+w(s)\|\varphi_r\|_{L^1}(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)})\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_L^{t_2}\varphi_r(s)+w(s)\|\varphi_r\|_{L^1}(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)})\mathrm{d}s\\
&\le\varepsilon\end{aligned}$$

and

$$\begin{aligned}
&|D_{0+}^{\alpha-1}(K_{P,Q}x)(t_1)-D_{0+}^{\alpha-1}(K_{P,Q}x)(t_2)|\\
&\leq \int_{\min\{t_1,t_2\}}^{\max\{t_1,t_2\}}\left|f(s,x(s),D_{0+}^{\alpha-1}x(s))-w(s)\left(\int_0^{+\infty}\varphi_r(\tau)\mathrm{d}\tau\right.\right.\\
&\qquad\left.\left.-\frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-\tau)^{\alpha-1}\varphi_r(\tau)\mathrm{d}\tau\right)\right|\mathrm{d}s\\
&\leq \int_L^{+\infty}\varphi_r(s)+w(s)\|\varphi_r\|_{L^1}(1+\frac{\beta\eta^{\alpha-1}}{\Gamma(\alpha)})\mathrm{d}s\\
&\leq \varepsilon.
\end{aligned}$$

So we complete the proof.

The following fixed point theorem due to Mawhin is fundamental in the proofs of our main results.

Lemma 2.9.[36] Let $\Omega\subset X$ be open and bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:

(i) $Lx\neq\lambda Nx$ for every $(x,\lambda)\in(\mathrm{Dom}L\backslash\mathrm{Ker}L)\cap\partial\Omega)\times(0,1)$;

(ii) $Nx\notin\mathrm{Im}L$ for every $x\in\mathrm{Ker}L\cap\partial\Omega$;

(iii) $\deg(JQN|_{KerL\cap\partial\Omega},\Omega\cap\mathrm{Ker}L,0)\neq 0$, with $Q:Y\to Y$ a continuous projector such that Ker$Q$=ImL and J :Im$Q\to$Ker$L$ is an isomorphism.

Then the equation $Lx=Nx$ has at least one solution in $\mathrm{Dom}L\cap\overline{\Omega}$.

3 Main results

In this section, we present our main results and prove them.

Theorem 3.1. Assume that f is an S-Carathéodory function and the following conditions are satisfied:

(H_1) There exist functions $a,b,c\in L^1[0,+\infty)$ such that

$$|f(t,u,v)|\leq a(t)|u|+b(t)|v|+c(t),\ \text{a.e. } t\in[0,+\infty)\text{ and all }(u,v)\in\mathbb{R}^2 \tag{3.1}$$

and

$$\|a\|_1=\int_0^{+\infty}a(s)(1+t^{\alpha-1})\mathrm{d}s<+\infty;$$

(H_2) There exists constant $A>0$ such that for $x\in\mathrm{Dom}L$, if $|D_{0+}^{\alpha-1}x(t)|>A$ for all $t\in[0,+\infty)$, then

$$\int_0^{+\infty}f(s,x(s),D_{0+}^{\alpha-1}x(s))\mathrm{d}s-\frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s,x(s),D_{0+}^{\alpha-1}x(s))\mathrm{d}s\neq 0;$$

(H_3) There exists constant $B>0$ such that for all $c\in\mathbb{R}$ with $|c|>B$, either

$$c\left(\int_0^{+\infty}f(s,cs^{\alpha-1},c\Gamma(\alpha))\mathrm{d}s-\frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s,cs^{\alpha-1},c\Gamma(\alpha))\mathrm{d}s\right)<0 \tag{3.2}$$

or else

$$c\left(\int_0^{+\infty} f(s, cs^{\alpha-1}, c\Gamma(\alpha))\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s, cs^{\alpha-1}, c\Gamma(\alpha))\mathrm{d}s\right) > 0. \tag{3.3}$$

Then the BVP (1.1), (1.2) has at least one solution provided that

$$\frac{2}{\Gamma(\alpha)}\|a\|_1 + \frac{2}{\Gamma(\alpha)}\|b\|_{L^1} < 1.$$

Proof. We construct an open bounded set $\Omega \subset X$ that satisfies the assumption of Lemma 2.9. Let

$$\Omega_1 = \{x \in \mathrm{Dom}L\backslash\mathrm{Ker}L : Lx = \lambda Nx, \text{for some } \lambda \in [0,1]\}.$$

For $x \in \Omega_1$, we have $x \notin \mathrm{Ker}L$, $\lambda \neq 0$ and $Nx \in \mathrm{Im}L$. Note that $\mathrm{Ker}Q = \mathrm{Im}L$ and, thus,

$$\int_0^{+\infty} f(s, x(s), D_{0+}^{\alpha-1}x(s))\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s, x(s), D_{0+}^{\alpha-1}x(s))\mathrm{d}s = 0$$

since $QNx = 0$. It follows from (H_2) that there exists $t_0 \in [0, +\infty)$ such that $|D_{0+}^{\alpha-1}x(t_0)| \leq A$. In view of $D_{0+}^{\alpha-1}x(0) = D_{0+}^{\alpha-1}x(t_0) - \int_0^{t_0} D_{0+}^{\alpha}x(t)\mathrm{d}t$, we have

$$|D_{0+}^{\alpha-1}x(0)| \leq A + \|D_{0+}^{\alpha}x\|_{L^1} = A + \|Lx\|_{L^1} \leq A + \|Nx\|_{L^1}. \tag{3.4}$$

Again for $x \in \Omega_1$, $x \in \mathrm{Dom}L\backslash\mathrm{Ker}L$, then $(I-P)x \in \mathrm{Dom}L \cap \mathrm{Ker}P$ and $LPx = 0$, thus from Lemma 2.7, we have

$$\begin{aligned} \|(I-P)x\|_X = \|K_P L(I-P)x\|_X &\leq \frac{1}{\Gamma(\alpha)}\|L(I-P)x\|_{L^1} \\ &= \frac{1}{\Gamma(\alpha)}\|Lx\|_{L^1} \leq \frac{1}{\Gamma(\alpha)}\|Nx\|_{L^1}. \end{aligned} \tag{3.5}$$

From (3.4), (3.5), we have

$$\|x\|_X \leq \|Px\|_X + \|(I-P)x\|_X \leq \frac{A}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha)}\|Nx\|_{L^1}. \tag{3.6}$$

If (3.6) holds, from (3.1), we have

$$\|x\|_X \leq \frac{A}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha)}(\|a\|_1\|x\|_0 + \|b\|_{L^1}\|D_{0+}^{\alpha-1}x\|_\infty + \|c\|_{L^1}). \tag{3.7}$$

Thus from $\|x\|_0 \leq \|x\|_X$ and (3.7), we have

$$\|x\|_0 \leq \frac{1}{1 - \frac{2}{\Gamma(\alpha)}\|a\|_1}\left(\frac{A}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha)}(\|b\|_{L^1}\|D_{0+}^{\alpha-1}x\|_\infty + \|c\|_{L^1})\right). \tag{3.8}$$

Again from (3.7), (3.8) and $\|D_{0+}^{\alpha-1}x\|_\infty \leq \|x\|_X$ we obtain

$$\|D_{0+}^{\alpha-1}x\|_\infty \leq \frac{1}{1 - \frac{2}{\Gamma(\alpha)}\|a\|_1 - \frac{2}{\Gamma(\alpha)}\|b\|_{L^1}}\left(\frac{A}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha)}\|c\|_{L^1}\right). \tag{3.9}$$

It follows from (3.8) and (3.9) that, there exists an $M > 0$ such that $\|x\|_X \leq M$ for all $x \in \Omega_1$, that is, Ω_1 is bounded.

Let

$$\Omega_2 = \{x \in \mathrm{Ker}L : Nx \in \mathrm{Im}L\}.$$

For $x \in \Omega_2$, $x \in \mathrm{Ker}L$ implies that x can be expressed by $x = ct^{\alpha-1}$, where c is an arbitrary constant, and $QNx = 0$, thus

$$w(t)\left(\int_0^{+\infty} f(s, cs^{\alpha-1}, c\Gamma(\alpha))\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s, cs^{\alpha-1}, c\Gamma(\alpha))\mathrm{d}s\right) = 0.$$

It follows from (H_2) that, we get $\|x\|_X \leq \frac{1}{\Gamma(\alpha)}|c| \leq \frac{1}{\Gamma(\alpha)}A$. So Ω_2 is bounded too.

We define the isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$ by

$$J(cw(t)) = ct^{\alpha-1},\ c \in \mathbb{R}.$$

If the first part of (H_3) is fulfilled, we set

$$\Omega_3 = \{x \in \mathrm{Ker}L : -\lambda J^{-1}x + (1-\lambda)QNx = 0\}.$$

For every $x = ct^{\alpha-1} \in \Omega_3$, one has $\lambda J^{-1}x = (1-\lambda)QNx$,

$$\lambda cw(t) = (1-\lambda)w(t)\left(\int_0^{+\infty} f(s, cs^{\alpha-1}, c\Gamma(\alpha))\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s, cs^{\alpha-1}, c\Gamma(\alpha))\mathrm{d}s\right).$$

If $\lambda = 1$, then $c = 0$ and, if $|c| > B$, in view of (3.2), one has

$$\lambda c^2 w(t) = (1-\lambda)w(t)c\left(\int_0^{+\infty} f(s, cs^{\alpha-1}, c\Gamma(\alpha))\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s, cs^{\alpha-1}, c\Gamma(\alpha))\mathrm{d}s\right) < 0,$$

which contradicts $\lambda c^2 w(t) > 0$. If the other part of (H_3) is satisfied, we take

$$\Omega_3 = \{x \in \mathrm{Ker}L : \lambda J^{-1}x + (1-\lambda)QNx = 0\}$$

and, again, obtain a contradiction. Thus, in ether case $\|x\|_X \leq (\frac{1}{\Gamma(\alpha)}+1)|c| \leq (\frac{1}{\Gamma(\alpha)}+1)B$, that is, Ω_3 is bounded.

In what follows, we shall prove that all conditions of Lemma 2.9 are satisfied. Set Ω be a bounded open subset of X such that $\cup_{i=1}^{3}\overline{\Omega}_i \subset \Omega$. we know that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω, we have

(i)$Lx \neq \lambda Nx$ for every $(x, \lambda) \in (\mathrm{Dom}L\backslash\mathrm{Ker}L) \cap \partial\Omega) \times (0, 1)$;

(ii) $Nx \notin \mathrm{Im}L$, for every $x \in \mathrm{Ker}L \cap \partial\Omega$.

At last we will prove that (iii) of Lemma 2.9 is satisfied. To this end, let

$$H(x, \lambda) = \pm\lambda Idx + (1-\lambda)JQNx,$$

where Id is the identical operator. By virtue of the definition of Ω, we know $\Omega \supset \overline{\Omega}_3$, thus $H(x, \lambda) \neq 0$ for $x \in \mathrm{Ker}L \cap \partial\Omega$, then by homotopy property of degree, we get

$$\begin{aligned}
\deg(JQN|_{\mathrm{Ker}L\cap\partial\Omega}, \Omega \cap \mathrm{Ker}L, 0) &= \deg(H(\cdot, 0), \Omega \cap \mathrm{Ker}L, 0)\\
&= \deg(H(\cdot, 1), \Omega \cap \mathrm{Ker}L, 0)\\
&= \deg(\pm\mathrm{Id}, \Omega \cap \mathrm{Ker}L, 0)\\
&= \pm 1 \neq 0.
\end{aligned}$$

So, the third assumption of Lemma 2.9 is fulfilled and $Lx = Nx$ has at least one solution in $\mathrm{Dom}L \cap \overline{\Omega}$. The proof is complete.

Corollary 3.2. Assume that f is an S-Carathéodory function and the conditions (H_1) and (H_3) in Theorem 3.1 are satisfied with (H_2) replaced with

($\mathrm{H}_2^{'}$): There exist functions $l, m \in L^1[0,+\infty)$ with $l(t), m(t) \geq 0$ and $l(t) \not\equiv 0$ such that

$$f(t,u,v) \geq l(t)|v| - m(t).$$

Then the BVP (1.1) and (1.2) has at least one solution provided that

$$\frac{2}{\Gamma(\alpha)}\|a\|_1 + \frac{2}{\Gamma(\alpha)}\|b\|_{L^1} < 1.$$

Proof. We only need to prove that the hypothesis ($\mathrm{H}_2^{'}$) implies the hypothesis (H_2) in Theorem 3.1.

In fact, setting

$$A = 2\|m\|_{L^1}\left(\int_0^{\eta}(1-\frac{\beta}{\Gamma(\alpha)}(\eta-s)^{\alpha-1})l(s)\mathrm{d}s + \int_{\eta}^{+\infty} l(s)\mathrm{d}s\right)^{-1},$$

we have

$$\begin{aligned}
&\int_0^{+\infty} f(s,x(s),D_{0+}^{\alpha-1}x(s))\mathrm{d}s - \frac{\beta}{\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}f(s,x(s),D_{0+}^{\alpha-1}x(s))\mathrm{d}s \\
&= \int_0^{\eta}(1-\frac{\beta}{\Gamma(\alpha)}(\eta-s)^{\alpha-1})f(s,x(s),D_{0+}^{\alpha-1}x(s))\mathrm{d}s + \int_{\eta}^{+\infty} f(s,x(s),D_{0+}^{\alpha-1}x(s))\mathrm{d}s \\
&> \int_0^{\eta}(1-\frac{\beta}{\Gamma(\alpha)}(\eta-s)^{\alpha-1})(l(s)A - m(s))\mathrm{d}s + \int_{\eta}^{+\infty}(l(s)A - m(s))\mathrm{d}s \\
&= 2\|m\|_{L^1} - \int_0^{\eta}(1-\frac{\beta}{\Gamma(\alpha)}(\eta-s)^{\alpha-1})(m(s)\mathrm{d}s - \int_{\eta}^{+\infty} m(s)\mathrm{d}s \geq 0.
\end{aligned}$$

Therefore, the hypothesis (H_2) in Theorem 3.1 is satisfied and the conclusion of the corollary follows from Theorem 3.1.

To illustrate how our main results can be used in practice we present an example.

Example 3.1. Consider the following boundary value problem

$$D_{0+}^{\frac{3}{2}}x(t) = \frac{\sqrt{\pi}}{8}e^{-t}(3\sin(x^2(t) + D_{0+}^{\frac{1}{2}}x(t)) + D_{0+}^{\frac{1}{2}}x(t)), \tag{3.10}$$

$$x(0) = 0, \quad \lim_{t\to+\infty} D_{0+}^{\frac{1}{2}}x(t) = \sqrt{\pi}x(\frac{1}{4}). \tag{3.11}$$

Conclusion. The BVP (3.10), (3.11) has at least one solution in $X = \{x \in C[0,+\infty) : \lim_{t\to+\infty} x(t)/(1+\sqrt{t}) \text{ exists}, \lim_{t\to+\infty} D_{0+}^{\frac{1}{2}}x(t) \text{ exists}\}$.

Proof. Let $\alpha = \frac{3}{2}, \beta = \sqrt{\pi}, \eta = \frac{1}{4}$, it is easily to see $\Gamma(\alpha) = \beta\eta^{\frac{1}{2}}$, that is, the BVP (3.10), (3.11) is a resonance problem. And let $f(t,u,v) = \frac{\sqrt{\pi}}{8}e^{-t}(3\sin(u^2+v)+v)$, then we have

$$|f(t,u,v)| \leq \frac{\sqrt{\pi}}{8}e^{-t}(3+|v|).$$

Setting $a(t) = 0, b(t) = \frac{\sqrt{\pi}}{8}e^{-t}, c(t) = \frac{3\sqrt{\pi}}{8}e^{-t}$. Obviously ($H_2$) is fulfilled as well as the conditions

$$\frac{2}{\Gamma(\frac{3}{2})}\|a\|_1 + \frac{2}{\Gamma(\frac{3}{2})}\|b\|_{L^1} = \frac{1}{2} < 1.$$

Now taking $A = 4$, for any $x \in X$, assuming $|D_{0+}^{\frac{1}{2}}x(t)| > 4$ holds for $t \in [0, +\infty)$, from the condition of (H_2), we have either $D_{0+}^{\frac{1}{2}}x(t) > A$ or $D_{0+}^{\frac{1}{2}}x(t) < -A$ holds for $t \in [0, +\infty)$.

If $D_{0+}^{\frac{1}{2}}x(t) > A$ holds, for $t \in [0, +\infty)$, then

$$\begin{aligned} QNx = w(t)\Bigg(&\int_0^{+\infty} \frac{\sqrt{\pi}}{8}e^{-s}(3\sin(x^2(s) + D_{0+}^{\frac{1}{2}}x(s)) + D_{0+}^{\frac{1}{2}}x(s))\mathrm{d}s \\ &-2\int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{\frac{1}{2}}\frac{\sqrt{\pi}}{8}e^{-s}(3\sin(x^2(s) + D_{0+}^{\frac{1}{2}}x(s)) + D_{0+}^{\frac{1}{2}}x(s))\mathrm{d}s\Bigg) \\ > w(t)&\int_{\frac{1}{4}}^{+\infty} 8e^{-s}\mathrm{d}s(A-3) > 0, \end{aligned}$$

and if $D_{0+}^{\frac{1}{2}}x(t) < -A$ holds, for $t \in [0, +\infty)$, then

$$\begin{aligned} QNx = w(t)\Bigg(&\int_0^{+\infty} \frac{\sqrt{\pi}}{8}e^{-s}(3\sin(x^2(s) + D_{0+}^{\frac{1}{2}}x(s)) + D_{0+}^{\frac{1}{2}}x(s))\mathrm{d}s \\ &-2\int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{\frac{1}{2}}\frac{\sqrt{\pi}}{8}e^{-s}(3\sin(x^2(s) + D_{0+}^{\frac{1}{2}}x(s)) + D_{0+}^{\frac{1}{2}}x(s))\mathrm{d}s\Bigg) \\ < w(t)&\int_{\frac{1}{4}}^{+\infty} 8e^{-s}\mathrm{d}s(3-A) < 0. \end{aligned}$$

Thus the condition (H_1) holds.

It is easy to see that for all $c > 0$, we have

$$\begin{aligned} c\Bigg(&\int_0^{+\infty} \frac{\sqrt{\pi}}{8}e^{-s}(3\sin(c^2s + c\frac{\sqrt{\pi}}{2}) + c\frac{\sqrt{\pi}}{2})\mathrm{d}s \\ &-2\int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{\frac{1}{2}}\frac{\sqrt{\pi}}{8}e^{-s}(3\sin(c^2s + c\frac{\sqrt{\pi}}{2}) + c\frac{\sqrt{\pi}}{2})\mathrm{d}s\Bigg) \\ &> c\left(\frac{\sqrt{\pi}}{2}c - 3\right)\int_{\frac{1}{4}}^{+\infty} \frac{\sqrt{\pi}}{8}e^{-s}\mathrm{d}s \end{aligned}$$

and for all $c < 0$, we have

$$\begin{aligned} c\Bigg(&\int_0^{+\infty} \frac{\sqrt{\pi}}{8}e^{-s}(3\sin(c^2s + c\frac{\sqrt{\pi}}{2}) + c\frac{\sqrt{\pi}}{2})\mathrm{d}s \\ &-2\int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{\frac{1}{2}}\frac{\sqrt{\pi}}{8}e^{-s}(3\sin(c^2s + c\frac{\sqrt{\pi}}{2}) + c\frac{\sqrt{\pi}}{2})\mathrm{d}s\Bigg) \\ &> c\left(\frac{\sqrt{\pi}}{2}c + 3\right)\int_{\frac{1}{4}}^{+\infty} \frac{\sqrt{\pi}}{8}e^{-s}\mathrm{d}s. \end{aligned}$$

Hence the second inequality in (H_3) becomes

$$
\begin{aligned}
& c\left(\int_0^{+\infty} \frac{\sqrt{\pi}}{8} e^{-s}(3\sin(c^2 s + c\frac{\sqrt{\pi}}{2}) + c\frac{\sqrt{\pi}}{2})\mathrm{d}s \right.\\
& \qquad\qquad \left. -2\int_0^{\frac{1}{4}} (\frac{1}{4}-s)^{\frac{1}{2}} \frac{\sqrt{\pi}}{8} e^{-s}(3\sin(c^2 s + c\frac{\sqrt{\pi}}{2}) + c\frac{\sqrt{\pi}}{2})\mathrm{d}s\right)\\
& \qquad > \min\{c\left(\frac{\sqrt{\pi}}{2}c - 3\right), c\left(\frac{\sqrt{\pi}}{2}c + 3\right)\} \int_{\frac{1}{4}}^{+\infty} \frac{\sqrt{\pi}}{8} e^{-s}\mathrm{d}s > 0
\end{aligned}
$$

and is satisfied for all c with $|c| > \frac{6}{\sqrt{\pi}}$. So the condition (H_3) holds. Thus, Theorem 3.1 implies BVP (3.10) and (3.11) has at least one solution in $X = \{x \in C[0,+\infty) : \lim_{t\to+\infty} x(t)/(1+\sqrt{t}) \text{ exists}, \lim_{t\to+\infty} D_{0+}^{\frac{1}{2}} x(t) \text{ exists}\}$.

Remark 1. Example implies that there is a large number of functions that satisfy the conditions of Theorem 3.1. In addition, the conditions of Theorem 3.1 also easy to check.

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