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Gronwall–Bellman estimates for linear Volterra-type inequalities with delay

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Abstract. We obtain some estimates of Gronwall–Bellman-type for Volterra's inequalities, which have applications to the study of the stability properties of the solutions to some linear functional differential equations.

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1 Introduction

The Gronwall–Bellman estimates are a very useful tool in order to study the stability properties of solutions to differential equations. In particular, they provide explicit estimates of the solutions to certain implicit inequalities, which are frequently related to the study of differential equations. For a real interval $I := [t_0, T)$ and a real constant $c \geq 0$, the well-known Gronwall Lemma is stated as follows.

Theorem 1.1 ([8, Theorem 1.2.2])**.** *Let f be a nonnegative continuous function defined on I. If x is a nonnegative continuous function defined on I and satisfies*

$$x(t) \leq c + \int_{\alpha}^{t} f(s)x(s)ds, \qquad t \in I, \tag{1.1}$$

then the following estimate holds:

$$x(t) \leq c \exp\left(\int_{\alpha}^{t} f(s)ds\right), \qquad t \in I.$$

Every x in the framework of Theorem 1.1 is called a solution to (1.1). There have been several generalizations of these type of results since the works [2, 4]. We refer the reader to references [3, 8]. Nevertheless, for the aim of this work, we will only highlight two of them.

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The first one deals with the addition of a new variable in the definition of f, so that f becomes a kernel k in some $\Delta := \{(t,s) \in \mathbb{R}^2 : t_0 \leq s \leq t < T\}$. Norbury and Stuart [7] obtained an explicit estimate involving the exponential of the integral of the kernel by considering that it is nondecreasing in the first variable, as we state below.

Theorem 1.2 ([8, Theorem 1.4.2 (i)]). *Let $k(t,s)$ be a nonnegative continuous function defined on Δ, such that $k(\cdot,s)$ is nondecreasing for each $s \in I$. If x is a nonnegative continuous function defined on I and satisfies*

$$x(t) \leq c + \int_\alpha^t k(t,s)x(s)ds, \qquad t \in I,$$

then the following estimate holds:

$$x(t) \leq c \exp\left(\int_\alpha^t k(t,s)ds\right), \qquad t \in I.$$

The other generalization deals with the concept of delay by considering in (1.1) an integral between t_0 and $\phi(t)$, for a certain function $\phi : [t_0, T) \to [t_0, T)$ such that $\phi(t) \leq t$, for any $t \in [t_0, T)$. Although one can find some results in this direction (see Chapter 1 in [3], which has a result by Ahmedov, Jakubov and Veisov [1]), we will focus on the one proposed by Győri and Horváth [5]. In this work, the authors use the characteristic equation (and a certain inequality coming from this equation) that arises in the study of the nonautonomous linear delay differential equations in order to find the following result.

Theorem 1.3 ([5, Theorem 2.2]). *Let f be a nonnegative locally integrable function. Assume that $r \geq 0$ and $\tau : [t_0, T) \to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

If x is a nonnegative Borel measurable and locally bounded function defined on $[t_0 - r, T)$ such that

$$x(t) \leq c + \int_{t_0}^t f(u)x(u - \tau(u))du, \qquad t_0 \leq t < T,$$

then

$$x(t) \leq K \exp\left(\int_{t_0}^t \gamma(s)ds\right), \qquad t_0 \leq t < T,$$

where the function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ is locally integrable and satisfies the characteristic inequality

$$\gamma(t) \geq f(t) \exp\left(-\int_{t-\tau(t)}^t \gamma(s)ds\right), \qquad t_0 \leq t < T$$

and

$$K := \max\left\{c \exp\left(\int_{t_0 - r}^{t_0} \gamma(s)ds\right), \sup_{t_0 - r \leq s \leq t_0} x(s) \exp\left(\int_s^{t_0} \gamma(w)\,dw\right)\right\}.$$

This previous result is also interesting because it does not require the continuity of the functions involved in it.

The main objective of this work is to provide an extension of Theorems 1.2 and 1.3 at the same time. We will provide some results that follow the main lines in [5] by introducing a new variable as it is done in Theorem 1.2 with respect to Theorem 1.1, within the following integral inequality with delay

$$x(t) \leq c + \int_{t_0}^t k(t,u)x(u - \tau(u))du, \qquad t_0 \leq t < T, \tag{1.2}$$

where τ, k are functions endowed with some suitable measurability and integrability properties. In order to obtain estimates for the solutions to the previous integral inequality, we make a revision of some of the procedures and proofs in [5], and derive a generalized characteristic inequality involving the kernel k of the inequality, just by using the identity

$$\int_{t_0}^{t} k(t,u)du = \int_{t_0}^{t} \left[k(u,u) + \int_{t_0}^{u} D_1 k(u,s)ds \right] du, \qquad t_0 \le t < T,$$

where $D_1 k$ means the partial derivative of k with respect to the first variable.

Our results generalize Theorem 1.3, since, if we take the particular case of $k(t,u) = f(u)$, the imposed hypotheses on k are consistent with the hypotheses in [5] over the function f. Theorem 1.2 is also generalized, not only by not considering the continuity of the involved functions, but also by introducing the delay. We also discuss the sharpness of the estimates for the solutions to (1.2) that our results give.

Finally, we provide an example of a functional differential equation whose stability properties can be derived from Theorem 3.1. For more applications of generalized characteristic inequalities arising from delay differential equations, we refer to [6].

2 Preliminaries

In this section we sum up all the employed notation, set the definitions and recall some needed results.

Let $\mathbb{N}$ and $\mathbb{R}_+$ denote, respectively, the sets of nonnegative integer and nonnegative real numbers. Let $t_0 \in \mathbb{R}$ and $T \in \mathbb{R} \cup \{\infty\}$ be such that $t_0 < T$. Let $c, r \in \mathbb{R}_+$, $\Delta := \{(t,u) \in \mathbb{R}^2 : t_0 \le u \le t < T\}$, $D := \{(t,u) \in \Delta : t = u\}$ and $\Delta_\eta := \{(t,u) \in \Delta : t \le \eta\}$, $\eta \in [t_0, T)$ (see Figure 2.1). For a given function $k : \Delta \to \mathbb{R}$, we define the function $k_D : [t_0, T) \to \mathbb{R}$ as

$$k_D(u) = k(u,u), \qquad u \in [t_0, T).$$

It is obvious that if $k(\Delta) \subset \mathbb{R}_+$, then $k_D([t_0, T)) \subset \mathbb{R}_+$.

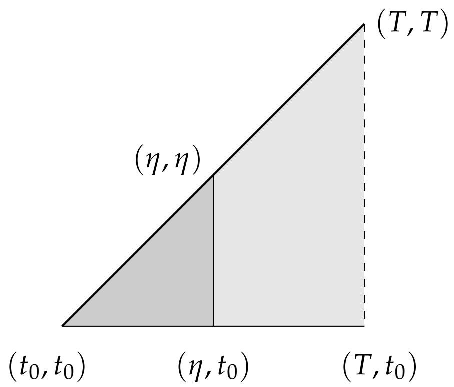

Figure 2.1: The set Δ. The subset D is represented with a thick line. The subset Δ_η, for some $\eta \in [t_0, T)$, is represented with darker colour.

Following the definitions in [5], measurability will refer to Lebesgue measurability, integrable will mean here Lebesgue integrable, thus we say that a function $f : [t^*, T) \to \mathbb{R}$, where $t^* \in \mathbb{R}$ is such that $t^* < T$, is locally integrable if it is integrable over $[t^*, \hat{t}]$, for every $\hat{t} \in [t^*, T)$.

Analogously, f is locally bounded if it is bounded on $[t^*, \hat{t}]$, for every $\hat{t} \in [t^*, T)$. We recall from [9] that the function $\varphi : I \to \mathbb{R}$, with I a real compact interval, is called absolutely continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |\varphi(\beta_i) - \varphi(\alpha_i)| < \varepsilon$$

for any $n \in \mathbb{Z}$, $n \geq 1$ and any disjoint set of intervals $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \subset I$, whose lengths satisfy that

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta.$$

Analogously, we will say that the function $f : [t^*, T) \to \mathbb{R}$ is locally absolutely continuous if it is absolutely continuous on each $[t^*, t]$, for every $t \in [t^*, T)$.

Finally, for each locally integrable mapping $h : [t_0 - r, T) \to \mathbb{R}_+$, we define the function $q_h : [t_0 - r, T) \to \mathbb{R}_+$ as

$$q_h(t) = \int_{t_0}^{t} h(s)ds, \qquad t_0 - r \leq t < T.$$

In order to proof the results of the following section, we need some previous results. The first one is the Fundamental Theorem of Calculus for Lebesgue integration, which can be found in a large list of references such as [9].

Theorem 2.1. *Let I be an interval of the real line and $\varphi : I \to \mathbb{R}$ be an absolutely continuous function. Then, φ' exists almost everywhere (a.e.), it is integrable and*

$$\varphi(b) = \varphi(a) + \int_{a}^{b} \varphi'(s)ds, \qquad a, b \in I.$$

In the following, the existence of derivatives of φ in the whole interval will not be guaranteed but, provided the hypotheses of the Fundamental Theorem of Calculus hold, we will refer to φ' as some function defined in the same interval as φ, which can be extended to the remaining set of measure zero by defining it as needed.

Lemma 2.2. *Let $\zeta : \Delta \to \mathbb{R}$ be such that $\zeta(\cdot, u) : [u, T) \to \mathbb{R}$ is locally absolutely continuous for each fixed $u \in [t_0, T)$. Moreover, suppose that $\zeta_D : [t_0, T) \to \mathbb{R}$ is locally integrable and that $D_1\zeta : \Delta \to \mathbb{R}$ is integrable on each Δ_η, for all $\eta \in [t_0, T)$. Then, the following identity holds:*

$$\int_{t_0}^{t} \zeta(t, u)du = \int_{t_0}^{t} \left[\zeta(u, u) + \int_{t_0}^{u} D_1\zeta(u, s)ds \right] du, \qquad t_0 \leq t < T.$$

Proof. For each $u \in [t_0, T)$, the function $\zeta(\cdot, u)$ is locally absolutely continuous and, by Theorem 2.1, $D_1\zeta$ exists a.e. in Δ. Hence, for a given $(t, u) \in \Delta$, we have

$$\zeta(t, u) = \zeta(u, u) + \int_{u}^{t} D_1\zeta(s, u)ds. \tag{2.1}$$

Note that the integrability of $D_1\zeta$ on each Δ_η, $\eta \in [t_0, T)$, implies that

$$y(u) := \int_{u}^{t} D_1\zeta(s, u)ds$$

is integrable in u. Then, if we integrate (2.1) between t_0 and t, we reach to

$$\int_{t_0}^{t} \zeta(t, u)du = \int_{t_0}^{t} \left[\zeta(u, u) + \int_{u}^{t} D_1\zeta(s, u)ds \right] du. \tag{2.2}$$

Using the integrability of $D_1\zeta$ on each Δ_η, $\eta \in [t_0, T)$, we can change the order of variables of integration in (2.2) and obtain

$$\int_{t_0}^{t} \zeta(t,u)du = \int_{t_0}^{t} \zeta(u,u)du + \int_{t_0}^{t}\int_{t_0}^{s} D_1\zeta(s,u)duds. \tag{2.3}$$

□

3 Volterra-type inequalities with delay dependence

In this section, we will present the principal results of this work. Theorems 3.1 and 3.4 provide the core of it, while Theorems 3.6 and 3.8 show that this type of results work in a more general framework.

Theorem 3.1. *Let $k : \Delta \to \mathbb{R}_+$ be such that $k(\cdot, u) : [u, T) \to \mathbb{R}$ is locally absolutely continuous for each fixed $u \in [t_0, T)$. Moreover, suppose that $k_D : [t_0, T) \to \mathbb{R}_+$ is locally integrable and that $D_1k : \Delta \to \mathbb{R}_+$ is integrable on each Δ_η, for all $\eta \in [t_0, T)$. Assume that $\tau : [t_0, T) \to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \le t - \tau(t), \qquad t_0 \le t < T.$$

If $x : [t_0 - r, T) \to \mathbb{R}_+$ is Borel measurable and locally bounded such that

$$x(t) \le c + \int_{t_0}^{t} k(t,u)x(u-\tau(u))du, \qquad t_0 \le t < T, \tag{3.1}$$

then

$$x(t) \le K \exp\left(\int_{t_0}^{t} \gamma(s)ds\right), \qquad t_0 \le t < T, \tag{3.2}$$

where the function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ is locally integrable and satisfies the characteristic inequality

$$\gamma(t) \ge k(t,t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma(s)ds\right) + \int_{t_0}^{t} D_1k(t,s)\exp\left(-\int_{s-\tau(s)}^{t} \gamma(w)dw\right)ds, \tag{3.3}$$

for $t_0 \le t < T$ a.e., and

$$K := \max\left\{c\exp\left(\int_{t_0-r}^{t_0} \gamma(s)ds\right), \sup_{t_0-r\le s\le t_0} x(s)\exp\left(\int_{s}^{t_0} \gamma(w)\,dw\right)\right\}.$$

Proof. First, we consider a continuous x in $[t_0, T)$.

Let $y : [t_0 - r, T) \to \mathbb{R}_+$ be defined as

$$y(t) = x(t)\exp\left(-\int_{t_0-r}^{t} \gamma(s)ds\right), \qquad t_0 - r \le t < T.$$

Then, (3.1) implies that

$$y(t) \le \exp\left(-\int_{t_0-r}^{t} \gamma(s)ds\right)\left(c + \int_{t_0}^{t} k(t,u)y(u-\tau(u))\exp\left(\int_{t_0-r}^{u-\tau(u)} \gamma(s)ds\right)du\right),$$

for each $t \in [t_0, T)$. Let $R : [t_0, T) \to \mathbb{R}_+$ be defined as

$$R(t) = y(t-\tau(t))\exp\left(\int_{t_0-r}^{t-\tau(t)} \gamma(s)ds\right),$$

so we can write

$$y(t) \le \exp\left(-\int_{t_0-r}^{t} \gamma(s)ds\right)\left(c + \int_{t_0}^{t} k(t,u)R(u)du\right), \qquad t_0 \le t < T.$$

Now we are going to apply Lemma 2.2 to the function $\zeta(t,u) := k(t,u)R(u)$. First, $\zeta(\cdot,u)$ is locally absolutely continuous due to the hypotheses in k. The function R is measurable and locally bounded and combined with the local integrability of k_D, we obtain that ζ_D is locally integrable. By a same reasoning, using the previous properties of R and the hypotheses in k, one can conclude that $D_1\zeta$ is integrable in each Δ_η, $\eta \in [t_0, T)$. Then,

$$y(t) \le \exp\left(-\int_{t_0-r}^{t} \gamma(s)ds\right)\left(c + \int_{t_0}^{t}\left[k(u,u)R(u) + \int_{t_0}^{u} D_1k(u,s)R(s)ds\right]du\right),$$

for any $t \in [t_0, T)$. Taking

$$L := \max\left\{c, \sup_{u\in[t_0-r,t_0]} x(u)\exp\left(-\int_{t_0-r}^{u}\gamma(s)ds\right)\right\},$$

we arrive, for an $L_1 > L$ arbitrarily chosen, to the following inequality:

$$y(t) \le L < L_1, \qquad t_0 - r \le t \le t_0.$$

Because of the continuity of y, there exists an $\varepsilon > 0$ such that $y(t) < L_1$ for $t \in [t_0 - r, t_0 + \varepsilon]$. Suppose that there exists a $t_1 > t_0 + \varepsilon$ such that $y(t_1) = L_1$, which can be chosen as the first element satisfying this condition. Then,

$$\begin{aligned} L_1 \le{}& c\exp\left(-\int_{t_0-r}^{t_1}\gamma(s)ds\right) + L_1\exp\left(-\int_{t_0}^{t_1}\gamma(s)ds\right) \\ &\times \int_{t_0}^{t_1}\left(k(u,u)e^{q_\gamma(u-\tau(u))} + \int_{t_0}^{u} D_1k(u,s)e^{q_\gamma(s-\tau(s))}ds\right)du. \end{aligned} \tag{3.4}$$

By using the characteristic inequality (3.3) in (3.4), we reach to

$$L_1 \le c\exp\left(-\int_{t_0-r}^{t_1}\gamma(s)ds\right) + L_1\exp\left(-\int_{t_0}^{t_1}\gamma(s)ds\right)\int_{t_0}^{t_1}\gamma(u)e^{q_\gamma(u)}du.$$

By calculating the integral, we get

$$L_1 \le c\exp\left(-\int_{t_0-r}^{t_1}\gamma(s)ds\right) + L_1\left(1 - \exp\left(-\int_{t_0}^{t_1}\gamma(s)ds\right)\right).$$

Then, rewriting terms, we obtain, by the choices of L and L_1, that

$$L_1 \le L_1 + \exp\left(-\int_{t_0}^{t_1}\gamma(s)ds\right)\left(c\exp\left(-\int_{t_0-r}^{t_0}\gamma(s)ds\right) - L_1\right) < L_1,$$

which is a contradiction. We can assure that $y(t) \le L$, for $t \in [t_0 - r, T)$. Now,

$$\begin{aligned} x(t) &= y(t)\exp\left(\int_{t_0-r}^{t}\gamma(s)ds\right) \\ &\le L\exp\left(\int_{t_0-r}^{t}\gamma(s)ds\right) = K\exp\left(\int_{t_0}^{t}\gamma(s)ds\right), \qquad t_0 \le t < T. \end{aligned}$$

To prove the result for the general case of x, we proceed in the same way as in Theorem 2.2 [5]. $\square$

Remark 3.2. The assumptions of Theorem 1.3 (Theorem 2.2 [5]) are generalized by the ones in the previous theorem. If $k(t,u) = f(u)$, for some locally integrable and nonnegative function f, then this kernel satisfies all the hypotheses written in Theorem 3.1. First, as k does not depend on the first variable, $k(\cdot,u)$ is trivially locally absolutely continuous, for all $u \in [t_0, T)$, and $D_1 k \equiv 0$ in Δ. Moreover, k_D is locally integrable because $k_D(u) = k(u,u) = f(u)$, which is locally integrable by hypothesis. As a final comment, in Theorem 3.1 inequality (3.3) is not imposed to be satisfied in every point of $[t_0, T)$.

As a consequence of Theorem 3.1, we derive the next result, which provides a simple estimate for the solutions of (3.1) through the use of k.

Corollary 3.3. *Let all the hypotheses of Theorem 3.1 hold. Then,*

$$x(t) \leq K \exp\left(\int_{t_0}^{t} k(t,s)ds\right), \qquad t_0 \leq t < T,$$

where K is the same as in Theorem 3.1.

Proof. By applying Lemma 2.2 to function $\zeta = k$, we get

$$\int_{t_0}^{t} k(t,u)du = \int_{t_0}^{t} \left[k(u,u) + \int_{t_0}^{u} D_1 k(u,s)ds\right] du, \qquad t_0 \leq t < T.$$

Now, one can conclude by choosing

$$\gamma(t) = k(t,t) + \int_{t_0}^{t} D_1 k(t,s)ds, \qquad t_0 \leq t < T,$$

and

$$\gamma(t) = 0, \qquad t_0 - r \leq t < t_0.$$

Indeed, it is obvious that γ satisfies (3.3). □

The previous result provides an estimate for the solutions of the inequality (3.1) in terms of the function $k(t,u)$, but it is not necessarily the best estimate. The next result states which is the sharpest one and how it is characterized.

Theorem 3.4. *Let all the hypotheses of Theorem 3.1 hold. Then, the following assertions are valid.*

1. *There exists a unique locally integrable function $\hat{\gamma} : [t_0 - r, T) \to \mathbb{R}_+$ satisfying (3.3) with equality, for every $t \in [t_0, T)$, and*

$$\hat{\gamma}(t) = 0, \qquad t \in [t_0 - r, t_0).$$

2. *Any $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ satisfying (3.3) also satisfies*

$$\hat{\gamma}(t) \leq \gamma(t), \qquad t_0 \leq t < T.$$

3. *The function $\hat{x} : [t_0 - r, T) \to \mathbb{R}_+$ defined as*

$$\hat{x}(t) = c \exp\left(\int_{t_0}^{t} \hat{\gamma}(s)ds\right),$$

is the unique solution to the integral equation

$$x(t) = c + \int_{t_0}^{t} k(t,u)x(u - \tau(u))du, \qquad t_0 \leq t < T,$$

with

$$x(t) = c, \qquad t_0 - r \leq t \leq t_0.$$

Proof. The proof is analogous to that of Theorem 2.7 [5]. We will only remark some points. To prove assertion 1, we first extend the function k to

$$\tilde{D} := \{(t,u) \in \mathbb{R}^2 : t_0 - r \leq u \leq t < T\}$$

by defining

$$k(t,u) = 0, \quad (t,u) \in \tilde{D}, \quad u \in [t_0 - r, t_0).$$

Then, we choose a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of functions with domain $[t_0 - r, T)$ defined as

$$\gamma_0(t) = k(t,t) + \int_{t_0}^{t} D_1 k(t,s)ds, \qquad t_0 - r \leq t < T,$$

and then, by recurrence, for $n \in \mathbb{N}$,

$$\gamma_{n+1}(t) = k(t,t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma_n(s)ds\right) + \int_{t_0}^{t} D_1 k(t,u)\exp\left(-\int_{u-\tau(u)}^{t} \gamma_n(s)ds\right) du,$$

for each $t_0 \leq t < T$ and

$$\gamma_{n+1}(t) = 0, \qquad t_0 - r \leq t < t_0.$$

Following the same reasoning of the proof of Theorem 2.7 in [5], one can assure that

$$0 \leq \gamma_{2k+1}(t) \leq \gamma_{2k+3}(t) \leq \cdots \leq \gamma_{2k+4}(t) \leq \gamma_{2k+2}(t) \leq \gamma_0(t),$$

for any $k \in \mathbb{N}$ and $t \in [t_0 - r, T)$. Then, by the Theorem of Dominated Convergence (see [9]), there exist

$$\gamma_{\text{up}} = \lim_{k\to\infty} \gamma_{2k},$$

and

$$\gamma_{\text{low}} = \lim_{k\to\infty} \gamma_{2k+1},$$

and these limits satisfy

$$\gamma_{\text{up}}(t) = k(t,t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma_{\text{low}}(s)ds\right) + \int_{t_0}^{t} D_1 k(t,u)\exp\left(-\int_{u-\tau(u)}^{t} \gamma_{\text{low}}(s)ds\right) du, \quad (3.5)$$

and

$$\gamma_{\text{low}}(t) = k(t,t)\exp\left(-\int_{t-\tau(t)}^{t} \gamma_{\text{up}}(s)ds\right) + \int_{t_0}^{t} D_1 k(t,u)\exp\left(-\int_{u-\tau(u)}^{t} \gamma_{\text{up}}(s)ds\right) du.$$

for $t_0 \leq t < T$ and

$$\gamma_{\text{up}}(t) = \gamma_{\text{low}}(t) = 0, \qquad t_0 - r \leq t < t_0.$$

By using that $|e^{-x} - e^{-y}| \leq |x - y|$, for $x, y \geq 0$, we get to

$$\begin{aligned} 0 &\leq \gamma_{\text{up}}(t) - \gamma_{\text{low}}(t) \\ &= k(t,t)\int_{t-\tau(t)}^{t} [\gamma_{\text{up}}(s) - \gamma_{\text{low}}(s)]ds + \int_{t_0}^{t} D_1 k(t,u)\int_{u-\tau(u)}^{t} [\gamma_{\text{up}}(s) - \gamma_{\text{low}}(s)]dsdu, \end{aligned}$$

for any $t \in [t_0, T)$. By using the nonnegativity of $\gamma_{\text{up}}(t) - \gamma_{\text{low}}(t)$, we can assure that

$$0 \leq \gamma_{\text{up}}(t) - \gamma_{\text{low}}(t) \leq \left(k(t,t) + \int_{t_0}^{t} D_1 k(t,u)du\right)\int_{t_0 - r}^{t} [\gamma_{\text{up}}(s) - \gamma_{\text{low}}(s)]ds.$$

By applying a simple Gronwall–Bellman inequality result (for example, see Theorem 1.3.2 [8]), we conclude that $\hat{\gamma} := \gamma_{\text{up}} = \gamma_{\text{low}}$ on $[t_0 - r, T)$. It is clear that $\hat{\gamma}$ satisfies (3.3) with equality because of the expresion (3.5). The uniqueness of $\hat{\gamma}$ follows analogously to the procedure in [5], by using the same Gronwall–Bellman type result.

The second assertion can be proved in the same way as in [5].

It only remains to complete the proof of assertion 3. Indeed,

$$c + \int_{t_0}^{t} k(t,u)\hat{x}(u - \tau(u))du = c + \int_{t_0}^{t} k(t,u)c \exp\left(\int_{t_0}^{u-\tau(u)} \hat{\gamma}(s)ds\right) du, \tag{3.6}$$

and the last expression in (3.6) is equal to

$$c + c\int_{t_0}^{t} \left[k(u,u)e^{q_{\hat{\gamma}}(u-\tau(u))} + \int_{t_0}^{u} D_1k(u,s)e^{q_{\hat{\gamma}}(s-\tau(s))}ds\right] du.$$

Using the characteristic inequality with equality in this case, the right-hand expression of (3.6) is also equal to

$$c\left(1 + \int_{t_0}^{t} \gamma(u)e^{q_{\hat{\gamma}}(u)}du\right) = ce^{q_{\hat{\gamma}}(t)} = c \exp\left(\int_{t_0}^{t} \hat{\gamma}(s)ds\right) = \hat{x}(t).$$

In order to prove the uniqueness, one can reproduce similar arguments as those used in the proof of the first assertion, reaching an inequality of the type of (3.1), to which we can apply Theorem 3.1. □

In the following example we provide a family of functions k for which we can find the sharpest estimate.

Example 3.5. Let us consider

$$x(t) \le c + a(t)\int_{1}^{\sqrt{t}} \frac{1}{\sqrt{u}}x(u)du, \qquad t \ge 1. \tag{3.7}$$

Then, after the substitution $u = \sqrt{z}$ and renaming the variable, we obtain

$$x(t) \le c + a(t)\int_{1}^{t} \frac{1}{2u^{\frac{3}{4}}}x(u - (u - \sqrt{u}))du, \qquad t \ge 1.$$

In Example 3.5 (a) [5], the function $\gamma(t) = \frac{1}{2t}$ provides the best estimate one could give for the solutions of (3.7) with $a(t) = 1$. This remains true when

$$a(t) = 1 - K \exp\left(-\int_{1}^{t} \frac{1}{2(s - \sqrt{s})}ds\right),$$

for any $K \in [0,1]$. This can be proved by checking that (3.3) holds with equality a.e. on $[1, \infty)$ and, then, Theorem 3.4 can be applied.

Now, we show some generalizations of Theorem 3.1. The first one is related to the solutions to another inequality of the type of (3.1), which has another integral term (without "delay") on the right-hand side.

Theorem 3.6. *Let $k, g : \Delta \to \mathbb{R}_+$ be such that $k(\cdot, u), g(\cdot, u) : [u, T) \to \mathbb{R}$ are locally absolutely continuous for each fixed $u \in [t_0, T)$. Moreover, let $k_D, g_D : [t_0, T) \to \mathbb{R}_+$ be locally integrable and such that $D_1 k, D_1 g : \Delta \to \mathbb{R}_+$ are integrable on each Δ_η, for all $\eta \in [t_0, T)$. Assume that $\tau : [t_0, T) \to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

If $x : [t_0 - r, T) \to \mathbb{R}_+$ is Borel measurable and locally bounded such that

$$x(t) \leq c + \int_{t_0}^{t} g(t,u)x(u)du + \int_{t_0}^{t} k(t,u)x(u - \tau(u))du, \qquad t_0 \leq t < T, \tag{3.8}$$

then

$$x(t) \leq K \exp\left(\int_{t_0}^{t} \gamma(s)ds\right), \qquad t_0 \leq t < T, \tag{3.9}$$

where the function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ is locally integrable and satisfies the characteristic inequality

$$\begin{aligned} \gamma(t) \geq k(t,t) \exp &\left(-\int_{t-\tau(t)}^{t} [\gamma(w) + g(t,w)]\, dw\right) \\ &+ \int_{t_0}^{t} D_1 k(t,s) \exp\left(-\int_{s-\tau(s)}^{t} [\gamma(w) + g(s,w)] dw\right) ds, \end{aligned} \tag{3.10}$$

for $t_0 \leq t < T$ a.e., and K is equal to

$$\max\left\{ c \exp\left(\int_{t_0-r}^{t_0} \gamma(s)ds\right), \sup_{t_0 - r \leq s \leq t_0} x(s)\ \exp\left(\int_{s}^{t_0} \gamma(w)\, dw\right) \right\}. \tag{3.11}$$

Proof. The key point of the proof will be to manipulate the inequality (3.8) in order to reach an expression which satisfies Theorem 3.1.

In the appendix (Lemma 6.1) we prove that

$$\begin{aligned} x(t) \leq\ & c \exp\left(\int_{t_0}^{t} g(t,u)du\right) \\ &+ \exp\left(\int_{t_0}^{t} g(t,u)du\right) \int_{t_0}^{t} k(t,u)x(u - \tau(u)) \exp\left(-\int_{t_0}^{u} g(u,s)ds\right) du, \end{aligned} \tag{3.12}$$

for each $t \in [t_0, T)$. Then, by defining

$$G(t) := g(t,t) + \int_{t_0}^{t} D_1 g(t,s)ds, \tag{3.13}$$

and

$$y(t) := x(t) \exp\left(-\int_{t_0}^{t} G(s)ds\right),$$

it is possible to rewrite (3.12) as

$$y(t) \leq c + \exp\left(-\int_{t_0}^{t} G(s)ds\right) \int_{t_0}^{t} k(t,u)y(u - \tau(u)) \exp\left(\int_{t_0}^{u-\tau(u)} G(s)ds + \int_{u}^{t} G(s)ds\right) du,$$

or equivalently,

$$y(t) \leq c + \int_{t_0}^{t} k(t,u)y(u - \tau(u)) \exp\left(-\int_{u-\tau(u)}^{u} G(s)ds\right) du.$$

Now, by using analogous arguments as in the beginning of the proof for Theorem 3.1, the function

$$\psi(t,u) := k(t,u)\exp\left(-\int_{u-\tau(u)}^{u} G(s)ds\right) = k(t,u)\exp\left(-\int_{u-\tau(u)}^{u} g(t,s)ds\right)$$

can play the role of k in Theorem 3.1 and

$$y(t) \leq K\exp\left(\int_{t_0}^{t}\gamma(s)ds\right), \qquad t_0 \leq t < T,$$

where $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ satisfies

$$\gamma(t) \geq \psi(t,t)\exp\left(-\int_{t-\tau(t)}^{t}\gamma(w)dw\right) + \int_{t_0}^{t} D_1\psi(t,s)\exp\left(-\int_{s-\tau(s)}^{t}\gamma(w)dw\right)ds$$

and K is as (3.11). □

Remark 3.7. In the proof of the previous theorem, by some calculations, we have used the thesis of Theorem 3.1. Therefore, by a parallel way, the assertions of Theorem 3.4 can be adapted to this case, and one can see the optimality properties of the unique function $\hat{\gamma}$ that satisfies (3.10) with equality.

Finally, as done in [5], it is possible to substitute c by a positive, measurable and nondecreasing function with the goal of obtaining an analogous result to the previous ones.

Theorem 3.8. *Let $p : [t_0, T) \to \mathbb{R}_+$ be a measurable, positive and (monotone) nondecreasing function. Let $k, g : \Delta \to \mathbb{R}_+$ be such that $k(\cdot, u), g(\cdot, u) : [u, T) \to \mathbb{R}$ are locally absolutely continuous for each fixed $u \in [t_0, T)$. Moreover, suppose that $k_D, g_D : [t_0, T) \to \mathbb{R}_+$ are locally integrable and $D_1k, D_1g : \Delta \to \mathbb{R}_+$ are integrable on each Δ_η, for all $\eta \in [t_0, T)$. Assume that $\tau : [t_0, T) \to \mathbb{R}_+$ is a measurable function such that*

$$t_0 - r \leq t - \tau(t), \qquad t_0 \leq t < T.$$

If $x : [t_0 - r, T) \to \mathbb{R}_+$ is Borel measurable and locally bounded such that

$$x(t) \leq p(t) + \int_{t_0}^{t} g(t,u)x(u)du + \int_{t_0}^{t} k(t,u)x(u-\tau(u))du, \qquad t_0 \leq t < T, \tag{3.14}$$

then

$$x(t) \leq Kp(t)\exp\left(\int_{t_0}^{t}\gamma(s)ds\right), \qquad t_0 \leq t < T, \tag{3.15}$$

where the function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ is locally integrable and satisfies the characteristic inequality

$$\begin{aligned}\gamma(t) \geq\ & k(t,t)\exp\left(-\int_{t-\tau(t)}^{t}[\gamma(w) + g(t,w)dw]\right) \\ & + \int_{t_0}^{t} D_1k(t,s)\exp\left(-\int_{s-\tau(s)}^{t}[\gamma(w)dw + g(s,w)]dw\right)ds,\end{aligned} \tag{3.16}$$

for $t_0 \leq t < T$ a.e., and

$$K := \max\left\{\exp\left(\int_{t_0-r}^{t_0}\gamma(s)ds\right), \sup_{t_0 - r \leq s \leq t_0}\frac{x(s)}{p(s)}\exp\left(\int_{s}^{t_0}\gamma(w)\,dw\right)\right\}.$$

Proof. The proof follows by dividing by $p(t)$ in (3.14) and applying Theorem 3.6 with $c = 1$ and the function

$$z(t) := \frac{x(t)}{p(t)}$$

playing the role of x in that result. □

4 An application to functional differential equations

The results of the previous sections can be employed to study some types of functional differential equations. In particular, some equations with an integral term can be included in our framework. For example, consider the functional differential equation

$$x'(t) = -\frac{2}{t}x(t) - \frac{1}{4t^2}x\left(\frac{1}{2}\sqrt{t}\right) + \int_1^t \frac{1}{4t^2 s}x\left(\frac{1}{2}\sqrt{s}\right)ds, \qquad t \geq 1. \tag{4.1}$$

We will see that all solutions to (4.1) tend to 0 as $t \to \infty$. By integrating and applying analogous arguments to those of the Appendix, we get

$$\begin{aligned} x(t) &= x(1)\exp\left(-\int_1^t \frac{2}{s}ds\right) \\ &\quad + \exp\left(-\int_1^t \frac{2}{s}ds\right)\int_1^t \left[-\frac{1}{4s^2}x\left(\frac{1}{2}\sqrt{s}\right) + \frac{1}{4s^2 u}\int_1^s x\left(\frac{1}{2}\sqrt{u}\right)du\right]\exp\left(\int_1^s \frac{2}{w}dw\right)ds. \end{aligned}$$

This leads to

$$x(t) = \frac{x(1)}{t^2} + \frac{1}{t^2}\int_1^t \left[-\frac{1}{4}x\left(\frac{1}{2}\sqrt{s}\right) + \int_1^s \frac{1}{4u}x\left(\frac{1}{2}\sqrt{u}\right)du\right]ds. \tag{4.2}$$

If we define

$$y(t) := t^2 x(t),$$

we can substitute in (4.2) and obtain

$$y(t) = y(1) + \int_1^t \left[-\frac{1}{s}y\left(\frac{1}{2}\sqrt{s}\right) + \int_1^s \frac{1}{u^2}y\left(\frac{1}{2}\sqrt{u}\right)du\right]ds.$$

Then,

$$|y(t)| \leq |y(1)| + \int_1^t \left[\frac{1}{s}\left|y\left(\frac{1}{2}\sqrt{s}\right)\right| + \int_1^s \frac{1}{u^2}\left|y\left(\frac{1}{2}\sqrt{u}\right)\right|du\right]ds. \tag{4.3}$$

By applying Lemma 2.2, we can write (4.3) as

$$|y(t)| \leq |y(1)| + \int_1^t \frac{t}{s^2}\left|y\left(\frac{1}{2}\sqrt{s}\right)\right|ds,$$

where $\zeta = k(t,s)|y(\frac{1}{2}\sqrt{s})|$, for

$$k(t,s) = \frac{t}{s^2},$$

which is included in the framework of Theorem 3.1. Now, we can choose any locally integrable function $\gamma : [\frac{1}{2}, \infty) \to \mathbb{R}_+$ such that

$$\gamma(t) := \frac{1}{t}, \qquad t \geq 1.$$

It can be checked by some calculations that the generalized characteristic inequality (3.3) holds for this case, i.e.,

$$\frac{1}{t} \geq \frac{1}{t}\exp\left(-\int_{\frac{1}{2}\sqrt{t}}^t \frac{1}{s}ds\right) + \int_1^t \frac{1}{u^2}\exp\left(-\int_{\frac{1}{2}\sqrt{u}}^t \frac{1}{s}ds\right)du.$$

Then, we can assure that there exists some $K \in \mathbb{R}_+$ such that

$$|y(t)| \leq K \exp\left(\int_1^t \gamma(s)ds\right) = K \exp\left(\int_1^t \frac{1}{s}ds\right) = Kt, \qquad t \geq 1.$$

Then,

$$|x(t)| \leq \frac{K}{t}, \qquad t \geq 1,$$

which obviously leads to

$$\lim_{t\to\infty} x(t) = 0.$$

The classical estimate that Corollary 3.3 provides is

$$|y(t)| \leq K' \exp\left(\int_1^t k(t,s)ds\right) = K' \exp\left(\int_1^t \frac{t}{s^2}ds\right) = K'e^{t-1}, \qquad t \geq 1,$$

for some $K' \in \mathbb{R}_+$. The previous estimate is not good enough to assure global attractivity of 0, so this example shows the importance of results like Theorem 2.2 [5] or Theorem 3.1. The reason is they provide more functions (than the classical estimates with $f(u)$ and $k(t,u)$, respectively) from which we can "build" estimates for the solutions to certain integral inequalities.

5 Conclusion

In this work, we have presented some generalizations of the results of [5] and some classical results as Theorems 1.1 and 1.2. Theorems 3.1 and 3.4 represent the central part of the exposed contents and provide, respectively, estimates for the solutions to the integral inequality (3.1) and their sharpness properties. Finally, we have proposed an example of a functional differential equation (with an integral term) whose stability properties are derived from the aforementioned results.

6 Appendix

In this appendix, we give the proof of a certain step (related to auxiliar estimates) in the proof of Theorem 3.6.

Lemma 6.1. *Let the hypotheses of Theorem 3.6 hold. Then,*

$$\begin{aligned} x(t) \leq{}& c\exp\left(\int_{t_0}^t g(t,u)du\right) \\ &+ \exp\left(\int_{t_0}^t g(t,u)du\right)\int_{t_0}^t k(t,u)x(u-\tau(u))\exp\left(-\int_{t_0}^u g(u,w)dw\right)du, \end{aligned} \tag{6.1}$$

for each $t_0 \leq t < T$.

Proof. Define z as the right-hand side of (3.8). This function is absolutely continuous on each $[t_0, t^*]$. Then, z is differentiable almost everywhere, i.e., on some set Ω_{t^*}, with $[t_0, t^*] \backslash \Omega_{t^*}$ having zero measure. Besides,

$$\begin{aligned} z'(t) ={}& g(t,t)x(t) + \int_{t_0}^t D_1 g(t,u)x(u)du \\ &+ k(t,t)x(t-\tau(t)) + \int_{t_0}^t D_1 k(t,u)x(u-\tau(u))du, \qquad t \in \Omega_{t^*}, \end{aligned}$$

and using $x(t) \le z(t)$ for $t \in [t_0, t^*]$, then

$$\begin{aligned} z'(t) \le g(t,t)z(t) + \int_{t_0}^{t} D_1 g(t,u) z(u) du \\ + k(t,t)x(t-\tau(t)) + \int_{t_0}^{t} D_1 k(t,u) x(u-\tau(u)) du, \qquad t \in \Omega_{t^*}. \end{aligned} \tag{6.2}$$

As $z(u) \le z(t)$, for $u \in [t_0, t]$ a.e., we can assure that

$$\begin{aligned} \left[z(t) \exp\left(-\int_{t_0}^{t} g(t,u) du \right) \right]' \le \left[z'(t) - g(t,t)z(t) - \int_{t_0}^{t} D_1 g(t,u) z(u) du \right] \\ \times \exp\left(-\int_{t_0}^{t} g(t,u) du \right), \qquad t \in \Omega_{t^*}. \end{aligned} \tag{6.3}$$

Then, by using (6.2) and (6.3), we get

$$\begin{aligned} & \left[z(t) \exp\left(-\int_{t_0}^{t} g(t,u) du \right) \right]' \\ & \le \left[k(t,t)x(t-\tau(t)) + \int_{t_0}^{t} D_1 k(t,u) x(u-\tau(u)) du \right] \\ & \quad \times \exp\left(-\int_{t_0}^{t} g(t,u) du \right), \qquad t \in \Omega_{t^*}. \end{aligned} \tag{6.4}$$

By integrating (6.4) between t_0 and t^*, we reach to

$$\begin{aligned} & z(t) \exp\left(-\int_{t_0}^{t} g(t,u) du \right) - z(t_0) \\ & \le \int_{t_0}^{t} \left[k(s,s)x(s-\tau(s)) + \int_{t_0}^{s} D_1 k(s,u) x(u-\tau(u)) du \right] \\ & \quad \times \exp\left(-\int_{t_0}^{s} g(s,w) dw \right) ds, \qquad t \in [t_0, t^*]. \end{aligned} \tag{6.5}$$

Now, if we use the nonnegativity of g and $D_1 g$, then the right-hand side of (6.5) is less than or equal to

$$\begin{aligned} & \int_{t_0}^{t} k(s,s)x(s-\tau(s)) \exp\left(-\int_{t_0}^{s} g(s,w) dw \right) ds \\ & + \int_{t_0}^{t} \int_{t_0}^{s} D_1 k(s,u) x(u-\tau(u)) \exp\left(-\int_{t_0}^{u} g(u,w) dw \right) du ds. \end{aligned} \tag{6.6}$$

Then, by using Lemma 2.2 with ζ being defined as

$$\zeta(t,u) = k(t,u)x(u-\tau(u)) \exp\left(-\int_{t_0}^{u} g(u,w) dw \right),$$

(6.6) is equivalent to

$$\int_{t_0}^{t} k(t,u)x(u-\tau(u)) \exp\left(-\int_{t_0}^{u} g(u,w) dw \right) du.$$

Following this previous reasoning, (6.5) leads to

$$z(t) \leq c \exp\left(\int_{t_0}^{t} g(t,u)du\right) + \exp\left(\int_{t_0}^{t} g(t,u)du\right) \int_{t_0}^{t} k(t,u)x(u-\tau(u)) \exp\left(-\int_{t_0}^{u} g(u,w)dw\right) du,$$

where we have used $z(t_0) = c$. Now the result follows from the definition of z ($x(t) \leq z(t)$, for any $t \in [t_0, T)$) and from the fact that t^* was arbitrarily chosen in $[t_0, T)$. □

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