Partial periodic oscillation: an interesting phenomenon for a system of three coupled unbalanced damped Duffing oscillators with delays

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Abstract. In this paper, a system of two coupled damped Duffing resonators driven by a van der Pol oscillator with delays is studied. Some sufficient conditions to ensure the periodic and partial periodic oscillations for the system are established. Computer simulation is given to demonstrate our result.

Keywords: unbalanced damped Duffing system, time delay, periodic and partial periodic oscillation.

2010 Mathematics Subject Classification: 34K11.

1 Introduction

The dynamics of isolated and coupled Duffing oscillators or Duffing–van der Pol oscillators with or without time delays is an important topic of research in different fields of science and engineering. For example, the magneto-elastic mechanical systems [5], large amplitude oscillation of centrifugal governor systems [30], nonlinear vibration of beams and plates [14], electroencephalogram signals model [3], micro-electro-mechanical systems resonators [11], fluid flow and gas flow induced vibration [23], a weak signal detection method [35], are modeled by the nonlinear Duffing equations or Duffing–van der Pol equations. Many researchers have studied various Duffing systems [4,9,12,16,17,22,29,37]. Recently, the study of nonlinear dynamics of micro-electro-mechanical systems (MEMS) and nano-electro-mechanical systems (NEMS) has grown rapidly over the last decades. It is known that the fundamental study of coupled nonlinear oscillators is very important in understanding the emergent behavior of complex dynamical systems in MEMS or NEMS. Analysis of simple cases as the building blocks in MEMS or NEMS can gain insight into larger complicated systems. In 2009, Karabalin *et al.* have discussed a system of two coupled nonlinear nano-electro-mechanical resonators

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using a structure of doubly clamped beams with a shared mechanical ledge. The authors modeled the behavior of the two strongly interacting nonlinear resonators by a coupled equations of motion for the beam as follows [8]:

$$\begin{cases} x_1''(t) + \gamma_1 x_1'(t) + \omega_1^2 x_1(t) + \alpha_1 x_1^3(t) + D(x_1(t) - x_2(t)) = g_{D1}(t), \\ x_2''(t) + \gamma_2 x_2'(t) + \omega_2^2 x_2(t) + \alpha_2 x_2^3(t) + D(x_2(t) - x_1(t)) = g_{D2}(t). \end{cases} \quad (1.1)$$

By using the standard methods of secular perturbation theory, the complex nonlinear behavior of the system has been demonstrated. Nonlinear response of one oscillation can be modified by driven the second oscillation. Complicated frequency-sweep response curves are found when both oscillations are driven into their strongly nonlinear range. The nonlinear behavior of coupled equations can be understood, controlled, and exploited. In order to understand the emergent behavior of complex dynamical systems and develop novel NEMS devices, Wei *et al*. [19,26] have investigated the dynamics of a periodically driven Duffing resonator coupled to a van der Pol oscillator using the standard two time scales approach. The motion of the coupled dynamical system is described by the following equations:

$$\begin{cases} x_1''(t) + \epsilon\mu_1 x_1'(t) + x_1(t) + \epsilon\alpha x_1^3(t) = \epsilon\beta(x_2(t) - x_1(t)) + \epsilon F\cos(\Omega\tau), \\ x_2''(t) + \epsilon\mu_2(x_2^2(t) - 1)x_2'(t) + x_2(t) = \epsilon\beta(x_1(t) - x_2(t)). \end{cases} \quad (1.2)$$

Due to the difference of order of magnitude about the coupling stiffness and other parameters, however, it is not easy to investigate the effects of the coupling stiffness on the steady state response using the multiple time scales analysis method. Therefore, Leung *et al*. have discussed the following damped Duffing resonator driven by a van der Pol oscillator [13]:

$$\begin{cases} u_1''(t) - \varepsilon_1 u_1'(t) + \Omega_1^2 u_1(t) + k_1 u_1^3(t) - k_c(u_2(t) - u_1(t)) = 0, \\ u_2''(t) - \varepsilon_2(u_2^2(t) - 1)u_2'(t) + \Omega_2^2 u_2(t) - k_c(u_1(t) - u_2(t)) = 0. \end{cases} \quad (1.3)$$

By solving nonlinear algebraic equations, highly accurate bifurcation frequencies for various parameters are provided. The effects of the nonlinear damping, coupling stiffness on the angular frequency and amplitude of steady state response are studied. The obtained results were in good agreement with respect to the numerical integration solutions. Rand and Wong have considered a system of four coupled phase-only oscillators. The qualitative dynamics is depended upon a parameter representing coupling strength. This work has been used to MEMS artificial intelligence decision-making devices [18].

It is known that time delay is ubiquitous in many physical systems, for example due to finite switching speeds of amplifiers in electronic circuits, finite signal propagation times in networks and circuits, and so on. Recently, many researchers have studied the dynamical behavior of various isolated and coupled time delay systems [6,20,27,28,33,34,36]. Zhang *et al*. have investigated three coupled van der Pol oscillators with delay as follows [34]:

$$\begin{cases} x_1''(t) + x_1(t) - \varepsilon_1(1 - x_1^2(t))x_1'(t) = k[x_2(t - \tau) - x_1(t - \tau)] + k[x_3(t - \tau) - x_1(t - \tau)], \\ x_2''(t) + x_2(t) - \varepsilon_1(1 - x_2^2(t))x_2'(t) = k[x_3(t - \tau) - x_2(t - \tau)] + k[x_1(t - \tau) - x_2(t - \tau)], \\ x_3''(t) + x_3(t) - \varepsilon_1(1 - x_3^2(t))x_3'(t) = k[x_1(t - \tau) - x_3(t - \tau)] + k[x_2(t - \tau) - x_3(t - \tau)]. \end{cases}$$
$$(1.4)$$

By using a symmetric Hopf bifurcation theory, the Hopf bifurcations at zero point appear as the delay increases and the existence of multiple periodic solutions are also established.

Luo and Huang have studied a discontinuous dynamics of a non-linear, self-excited, friction-induced, periodically forced oscillator [15]. Tchakui and Woafo have discussed the dynamics of three unidirectionally coupled autonomous Duffing oscillators and application to inch-worm piezoelectric motors [24]. Verichev *et al.* have investigated the dynamics of a "flexible-rotor/limited-power-excitation-source" system [25]. The vibration of the mass unbalance of the rotating component in a power plant has been studied by Kim *et al.* [10]. For dynamic analysis of a system of Van der Pol–Duffing oscillators with delay coupled, Zang *et al.* have investigated the existence of Hopf bifurcation and the bifurcation periodic solution [31]. Rusinek *et al.* have discussed the dynamics of a time delayed Duffing oscillator [21]. Motivated by the above research work, in this paper we shall extend Leung's system to the following model:

$$
\begin{cases}
x_1'' + \varepsilon_1 x_1' + \Omega_1^2 x_1 + k_1 x_1^3 = p_1[x_2(t - \tilde{\tau}_2) - x_1(t - \tilde{\tau}_1)] + q_1[x_3(t - \tilde{\tau}_3) - x_1(t - \tilde{\tau}_1)], \\
x_2'' + \varepsilon_2 x_2' + \Omega_2^2 x_2 + k_2 x_2^3 = p_2[x_3(t - \tilde{\tau}_3) - x_2(t - \tilde{\tau}_2)] + q_2[x_1(t - \tilde{\tau}_1) - x_2(t - \tilde{\tau}_2)], \\
x_3'' + \varepsilon_3(x_3^2 - 1)x_3' + \Omega_3^2 x_3 = p_3[x_1(t - \tilde{\tau}_1) - x_3(t - \tilde{\tau}_3)] + q_3[x_2(t - \tilde{\tau}_2) - x_3(t - \tilde{\tau}_3)],
\end{cases}
\tag{1.5}
$$

where $x_i = x_i(t)$ represents coordinate, ε_i, Ω_i ($i = 1, 2, 3$), k_j ($j = 1, 2$) are the damping coefficient, linear frequency and nonlinear stiffness of the Duffing resonator respectively. p_i, q_i ($i = 1, 2, 3$) are the coupling linear stiffness between the three resonators. It is well known that the Duffing oscillator is a nonlinear second order differential equation. The equation describes the motion of a damped oscillator with a complex potential than in simple harmonic motion. The Duffing oscillator is an example of a dynamical system that exhibits chaotic behavior. In system (1.5), the first two Duffing oscillators are coupled and driven by a van der Pol oscillator, in which the system appeared a partial oscillation under some restrictive conditions. It is an interesting phenomenon. By means of mathematical analysis method, some sufficient conditions to ensure the periodic and partial periodic oscillations of system (1.5) were obtained. Numerical simulation is provided to support our result. It should be emphasized that if the constants $\varepsilon_i, \Omega_i, p_i, q_i, \tilde{\tau}_i$ ($i = 1, 2, 3$), k_j ($j = 1, 2$) are different values, then the method of Hopf bifurcation is very hard to deal with system (1.5). This is due to the complexity of finding the bifurcating parameter.

2 Preliminaries

Let $\tau_1 = \tilde{\tau}_1, \tau_3 = \tilde{\tau}_2, \tau_5 = \tilde{\tau}_3$. It is convenient to write (1.5) as an equivalent six-dimensional first order system

$$
\begin{cases}
u_1' = u_2, \\
u_2' = -\varepsilon_1 u_2 - \Omega_1^2 u_1 - k_1 u_1^3 + p_1[u_3(t - \tau_3) - u_1(t - \tau_1)] + q_1[u_5(t - \tau_5) - u_1(t - \tau_1)], \\
u_3' = u_4, \\
u_4' = -\varepsilon_2 u_4 - \Omega_2^2 u_3 - k_2 u_3^3 + p_2[u_5(t - \tau_5) - u_3(t - \tau_3)] + q_2[u_1(t - \tau_1) - u_3(t - \tau_3)], \\
u_5' = u_6, \\
u_6' = -\varepsilon_3(u_5^2 - 1)u_6 - \Omega_3^2 u_5 + p_3[u_1(t - \tau_1) - u_5(t - \tau_5)] + q_3[u_3(t - \tau_3) - u_5(t - \tau_5)],
\end{cases}
\tag{2.1}
$$

where $u_i = u_i(t)$ ($i = 1, 2, \ldots, 6$). The matrix form of system (2.1) is as follows:

$$
U'(t) = AU(t) + BU(t - \tau) + P(U(t))
\tag{2.2}
$$

where $U(t) = [u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t)]^T$, $U(t - \tau) = [u_1(t - \tau_1), 0, u_3(t - \tau_3), 0, u_5(t - \tau_5), 0]^T$,

$$A = (a_{ij})_{6\times 6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\Omega_1^2 & -\varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\Omega_2^2 & -\varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\Omega_3^2 & \varepsilon_3 \end{pmatrix},$$

$$B = (b_{ij})_{6\times 6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ l_1 & 0 & p_1 & 0 & q_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ q_2 & 0 & l_2 & 0 & p_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ p_3 & 0 & q_3 & 0 & l_3 & 0 \end{pmatrix}, \qquad P(U(t)) = \begin{pmatrix} 0 \\ -k_1 u_1^3 \\ 0 \\ -k_2 u_3^3 \\ 0 \\ -\varepsilon_3 u_5^2 u_6 \end{pmatrix},$$

where $l_i = -p_i - q_i$ $(i = 1, 2, 3)$. Obviously, the linearized system of (2.2) is the following:

$$U'(t) = AU(t) + BU(t - \tau). \tag{2.3}$$

Definition 2.1. A solution of system (2.1) is called oscillatory if the solution is neither eventually positive nor eventually negative.

Definition 2.2. An oscillatory solution of system (2.1) is called partial oscillation if there is at least one component of the solution is non-oscillatory.

Lemma 2.3. *Assume that system (2.1) has a unique equilibrium point and all solutions are bounded. If the unique equilibrium point of system (2.1) is unstable, then system (2.1) generates a limit cycle. In other words, there exists a periodic oscillatory solution of system (2.1).*

Proof. See [1] and the appendix of [2]. □

Lemma 2.4. *For selected parameter values Ω_i, p_i, q_i $(i = 1, 2, 3)$, if M is a nonsingular matrix, then system (2.1) has a unique equilibrium point. where*

$$M = (c_{ij})_{3\times 3} = \begin{pmatrix} -\Omega_1^2 + p_1 + q_1 & -p_1 & -q_1 \\ -q_2 & -\Omega_2^2 + p_2 + q_2 & -p_2 \\ -p_3 & -q_3 & -\Omega_3^2 + p_3 + q_3 \end{pmatrix}.$$

Proof. An equilibrium point $u^* = (u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*)^T$ of system (2.1) is a constant solution of the following algebraic equation

$$\begin{cases} u_2^* = 0, \\ -\varepsilon_1 u_2^* - \Omega_1^2 u_1^* - k_1(u_1^*)^3 - p_1[u_3^* - u_1^*] - q_1[u_5^* - u_1^*] = 0, \\ u_4^* = 0, \\ -\varepsilon_2 u_4^* - \Omega_2^2 u_3^* - k_2(u_3^*)^3 - p_2[u_5^* - u_3^*] - q_2[u_1^* - u_3^*] = 0, \\ u_6^* = 0, \\ -\varepsilon_3[(u_5^*)^2 - 1]u_6^* - \Omega_3^2 u_5^* - p_3[u_1^* - u_5^*] - q_3[u_3^* - u_5^*] = 0. \end{cases} \tag{2.4}$$

Since $u_2^* = 0, u_4^* = 0, u_6^* = 0$, from (2.4) we have

$$\begin{cases} (-\Omega_1^2 + p_1 + q_1)u_1^* - p_1 u_3^* - q_1 u_5^* = k_1(u_1^*)^3, \\ -q_2 u_1^* + (-\Omega_2^2 + p_2 + q_2)u_3^* - p_2 u_5^* = k_2(u_3^*)^3, \\ -p_3 u_1^* - q_3 u_3^* + (-\Omega_3^2 + p_3 + q_3)u_5^* = 0. \end{cases} \quad (2.5)$$

We first consider the homogeneous system associated with system (2.5) as follows:

$$\begin{cases} (-\Omega_1^2 + p_1 + q_1)u_1^* - p_1 u_3^* - q_1 u_5^* = 0, \\ -q_2 u_1^* + (-\Omega_2^2 + p_2 + q_2)u_3^* - p_2 u_5^* = 0, \\ -p_3 u_1^* - q_3 u_3^* + (-\Omega_3^2 + p_3 + q_3)u_5^* = 0. \end{cases} \quad (2.6)$$

Since M is a non-singular matrix, the determinant of the coefficient matrix of system (2.6) does not equal to zero. According to the algebraic basic theorem, system (2.6) implies that $u_1^* = 0, u_3^* = 0, u_5^* = 0$. In other words, system (2.6) has a unique zero point.

We see that the third equation of system (2.5) is the same as the third equation of system (2.6). Note that $g(u_1^*) = k_1(u_1^*)^3$ and $h(u_3^*) = k_2(u_3^*)^3$ both are monotone functions, and only $g(0) = h(0) = 0$. This implies that $u^* = (0,0,0,0,0,0)^T$ is the unique equilibrium point of system (2.1). The proof is completed. $\qquad\square$

For a matrix $D = (d_{ij})_{6\times 6}$, we adopt the matrix norm $\|D\| = \max_{1\le j\le 6} \sum_{i=1}^6 |d_{ij}|$, and the matrix measure $\mu(D) = \max_{1\le j\le 6}(d_{jj} + \sum_{i=1,i\ne j}^6 |d_{ij}|)$.

Lemma 2.5. *Let* $r_{21} = \Omega_1^2 + k_1 u_1^2, r_{43} = \Omega_2^2 + k_2 u_3^2, r_{66} = \varepsilon_3(u_5^2 - 1)$. *Assume that* $0 < \varepsilon_i$ ($i = 1, 2, 3$), $k_j > 0$ ($j = 1, 2$), *if the following condition holds:*

$$\|B\| \le -\mu(R) \quad (2.7)$$

where

$$R = (r_{ij})_{6\times 6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -r_{21} & -\varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -r_{43} & -\varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\Omega_3^2 & -r_{66} \end{pmatrix},$$

then all solutions of system (2.1) are bounded.

Proof. Note that time delay affects the stability of the solutions, it does not affect the boundedness of the solutions. To avoid unnecessary complexity, consider a special case of system (2.1) as $\tau_1 = \tau_3 = \tau_5 = \tau^*$, in the following:

$$\begin{cases} u_1' = u_2, \\ u_2' = -\varepsilon_1 u_2 - (\Omega_1^2 + k_1 u_1^2)u_1 + p_1[u_3(t - \tau^*) - u_1(t - \tau^*)] + q_1[u_5(t - \tau^*) - u_1(t - \tau^*)], \\ u_3' = u_4, \\ u_4' = -\varepsilon_2 u_4 - (\Omega_2^2 + k_2 u_3^2)u_3 + p_2[u_5(t - \tau^*) - u_3(t - \tau^*)] + q_2[u_1(t - \tau^*) - u_3(t - \tau^*)], \\ u_5' = u_6, \\ u_6' = -\varepsilon_3(u_5^2 - 1)u_6 - \Omega_3^2 u_5 + p_3[u_1(t - \tau^*) - u_5(t - \tau^*)] + q_3[u_3(t - \tau^*) - u_5(t - \tau^*)]. \end{cases} \quad (2.8)$$

The matrix form of system (2.8) is the following:

$$U'(t) = RU(t) + BU(t - \tau^*).$$ (2.9)

From $\|B\| \leq -\mu(R)$ we have $\mu(R) < 0$ and $|\mu(R)| > \|B\|$. Define

$$\|U(t)\|_{\tau^*} = \sum_{i=1}^{6} (|u_i(t)| + \int_{t-\tau^*}^{t} |u_i(s)| ds)$$ (2.10)

and

$$\beta(s) = \frac{|\mu(R)| - \|B\|}{1 + \tau^*} s + \frac{|\mu(R)| - \|B\|}{1 + \tau^*} \tau^*.$$ (2.11)

Obviously, $\beta(-\tau^*) = 0, \beta(0) = \frac{\tau^*(|\mu(R)| - \|B\|)}{1+\tau^*} > 0$, and $\beta'(0) = \frac{\beta(0)}{\tau^*} > 0$. Let $U(t)$ $(t \geq -\tau^*)$ be any solution of (2.7). Consider a Lyapunov functional

$$V(t, U(\cdot)) = \sum_{i=1}^{6} (|u_i(t)| + \sum_{j=1}^{6} |b_{ij}| \int_{t-\tau^*}^{t} |u_i(s)| ds + \int_{t-\tau^*}^{t} \beta(s-t) \|U(s)\| ds], \qquad t > \tau^*. $$ (2.12)

Calculating the upper right derivative D^+V of V along the solution of (2.9), we derive that

$$
\begin{aligned}
D^+V(t, U(\cdot))|_{(2.9)} &\leq \sum_{i=1}^{6} (|u_i'(t)| + \sum_{j=1}^{6} |b_{ij}|(|u_j(t)| - |u_j(t-\tau^*)|)) + \beta(0)\|U(t)\| \\
&\quad - \beta(-\tau^*)\|U(t-\tau^*)\| - \int_{t-\tau^*}^{t} \beta'(s-t)\|U(s)\| ds \\
&\leq -(|\mu(R)| - \|B\| - \beta(0))\|U(t)\| - \frac{\beta(0)}{\tau^*} \int_{t-\tau^*}^{t} \|U(s)\| ds \\
&< -\left(\frac{|\mu(R)| - \|B\|}{1 + \tau^*}\right) \|U(t)\|_{\tau}^* \\
&< -\frac{(|\mu(R)| - \|B\|)V(t, U(\cdot))}{(1 + \tau^*)[(1 + \|B\|) + \beta(0)]} \\
&< 0.
\end{aligned}
$$ (2.13)

From the definition of V and $D^+V < 0$, this implies the boundedness of the solutions of system (2.9) [7]. □

3 Periodic and partial periodic oscillations

Note that k_1, k_2 and ε_3 are constants, u_1^3, u_3^3 and u_5^2 are high order infinitesimal as u_1, u_3 and u_5 tend toward to zero respectively. So, the unique equilibrium point which is exactly the zero point of system (2.1) and system (2.3), have the same stability or instability. The oscillatory behavior of the solution of system (2.3) implied that the solution of system (2.1) is also oscillatory. Assume that $\varepsilon_i > 0$ $(i = 1, 2, 3)$ and all solutions of system (2.1) are bounded. We first point out that the component u_6 of the trivial solution of system (2.3) is unstable. Consider the subsystem constructing by the fifth and sixth equations of system (2.3) as follows $(u_1 = u_3 = 0)$:

$$
\begin{cases}
u_5' = u_6 \\
u_6' = -\varepsilon_3(u_5^2 - 1)u_6 - \Omega_3^2 u_5 - p_3 u_5(t - \tau_5) - q_3 u_5(t - \tau_5).
\end{cases}
$$ (3.1)

Suppose that t_1 and t_2 are suitably large zero points such that $u_6(t_1) = u_6(t_2) = 0$. To show that the component $u_6(t)$ of the trivial solution is unstable we shall prove that there exists $t(t_1 < t < t_2)$ such that $|u_6(t)| > 0$. If such t does not exist, then $u_6(t) = 0$ for arbitrary $t \in [t_1, t_2]$. Thus, from the first equation of (3.1), $u_5(t)(t_1 + \tau_5 \leq t \leq t_2 - \tau_5)$ is a constant which equals $u_5(t_1)$. If $u_5(t_1) = 0$, then from the second equation of (3.1) we get

$$u_6' = \varepsilon_3 u_6. \tag{3.2}$$

Note that $u_6(t_1) = 0$, integrating both sides of (3.2) from t_1 to t we get

$$u_6(t) = e^{\varepsilon_3(t-t_1)}. \tag{3.3}$$

(3.3) implies that $u_6(t) \neq 0$ for arbitrary t $(t_1 < t < t_2)$. This contradicts $u_6(t) = 0$ for arbitrary $t \in [t_1, t_2]$. If $u_5(t_1) \neq 0$, then

$$u_6(t) = \frac{-\Omega_3^2 u_5(t_1) - p_3 u_5(t_1) - q_3 u_5(t_1)}{\varepsilon_3(u_5^2(t_1) - 1))}$$
$$- \frac{(-\Omega_3^2 u_5(t_1) - p_3 u_5(t_1) - q_3 u_5(t_1)) \exp(\varepsilon_3(u_5^2(t_1) - 1)t_1)}{(\varepsilon_3(u_5^2(t_1) - 1))) \exp(\varepsilon_3(u_5^2(t_1) - 1)t)}. \tag{3.4}$$

Obviously, $u_6(t) \neq 0$ $(t_1 < t < t_2)$. This means that the component $u_6(t)$ of the trivial solution of system (2.3) is unstable. It is easy to see that $u_5(t)$ is also unstable since $u_5'(t) = u_6(t)$. Therefore, for unbalanced damped Duffing oscillators model (2.3), if the components u_1, u_2, u_3, and u_4 of the trivial solution are globally asymptotically stable, then the system generates a partial periodic oscillation. In order to discuss the asymptotic stability of components u_1, u_2, u_3, and u_4, we investigate the subsystem constructed by the first four equations of system (2.3) $(u_5(t) = 0)$:

$$\begin{cases} u_1' = u_2, \\ u_2' = -\varepsilon_1 u_2 - \Omega_1^2 u_1 + p_1[u_3(t - \tau_3) - u_1(t - \tau_1)] - q_1 u_1(t - \tau_1), \\ u_3' = u_4, \\ u_4' = -\varepsilon_2 u_4 - \Omega_2^2 u_3 - p_2 u_3(t - \tau_3) + q_2[u_1(t - \tau_1) - u_3(t - \tau_3)]. \end{cases} \tag{3.5}$$

For convenience, we make the change of variables as $y_1(t) = u_1\left(t - \frac{\tau_1 - \tau_3}{2}\right)$, $y_2(t) = u_2\left(t - \frac{\tau_1 - \tau_3}{2}\right)$, $y_3(t) = u_3(t)$, $y_4(t) = u_4(t)$ if $\tau_1 > \tau_3$, or $y_1(t) = u_1\left(t - \frac{\tau_3 - \tau_1}{2}\right)$, $y_2(t) = u_2\left(t - \frac{\tau_3 - \tau_1}{2}\right)$, $y_3(t) = u_3(t)$, $y_4(t) = u_4(t)$ if $\tau_1 < \tau_3$ [32]. We can then rewrite system (3.5) as the following equivalent system

$$\begin{cases} y_1' = y_2, \\ y_2' = -\varepsilon_1 y_2 - \Omega_1^2 y_1 + p_1[y_3(t - \bar{\tau}) - y_1(t - \bar{\tau})] - q_1 y_1(t - \bar{\tau}), \\ y_3' = y_4, \\ y_4' = -\varepsilon_2 y_4 - \Omega_2^2 y_3 - p_2 y_3(t - \bar{\tau}) + q_2[y_1(t - \bar{\tau}) - y_3(t - \bar{\tau})], \end{cases} \tag{3.6}$$

where $\bar{\tau} = \frac{\tau_1 + \tau_3}{2}$. The matrix form of (3.6) is as follows:

$$Y'(t) = A_1 Y(t) + B_1 Y(t - \bar{\tau}), \tag{3.7}$$

where $Y(t) = (y_1(t), \ldots, y_4(t))^T$, $Y(t - \bar{\tau}) = (y_1(t - \bar{\tau}), 0, y_3(t - \bar{\tau}), 0)^T$,

$$A_1 = (a_{ij})_{4 \times 4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\Omega_1^2 & -\varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\Omega_2^2 & -\varepsilon_2 \end{pmatrix},$$

$$B_1 = (b_{ij})_{4\times4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -p_1 - q_1 & 0 & p_1 & 0 \\ 0 & 0 & 0 & 0 \\ q_2 & 0 & -p_2 - q_2 & 0 \end{pmatrix}.$$

Theorem 3.1. *Suppose that there exists a unique equilibrium point and all solutions of system* (2.1) *are bounded. Let $\varepsilon_i > 0$ $(i = 1, 2)$, $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be the eigenvalues of matrix $A_1 + B_1$, which has a negative real part, namely, $\mathrm{Re}\,\gamma_i < 0$ $(i = 1, 2, 3, 4)$. We set $\gamma = \min\{|\mathrm{Re}\,\gamma_1|, |\mathrm{Re}\,\gamma_2|, |\mathrm{Re}\,\gamma_3|, |\mathrm{Re}\,\gamma_4|\}$. Assume that*

$$\frac{\bar{\tau}(\|A_1\| + \|B_1\|)\|B_1\|}{\gamma} < 1. \tag{3.8}$$

Then system (2.1) *has a partial periodic oscillation.*

Proof. Since $\varepsilon_i > 0$ $(i = 1, 2)$, and all solutions of system (2.1) are bounded, according to the above analysis, the component u_6 and u_5 are unstable. Therefore, we only need to show that the components u_i or y_i $(i = 1, \dots, 4)$ are stable. Then system (2.1) generates a partial periodic oscillation. Consider system (3.7) for $t \geq \bar{\tau}$ we have

$$Y'(t) = (A_1 + B_1)Y(t)) - B_1 \int_{t-\bar{\tau}}^{t} Y'(s)ds$$

$$= (A_1 + B_1)Y(t)) - B_1 \int_{t-\bar{\tau}}^{t} [A_1 Y(s) + B_1 Y(s - \bar{\tau})]ds \tag{3.9}$$

leading to

$$Y(t) = e^{(A_1+B_1)(t-\bar{\tau})}Y(\bar{\tau})) - \int_{\bar{\tau}}^{t} ds \int_{s-\bar{\tau}}^{s} e^{(A_1+B_1)(t-s)} B_1[A_1 Y(\sigma) + B_1 Y(\sigma - \bar{\tau})]d\sigma \tag{3.10}$$

and hence

$$\|Y(t)\| \leq \|Y\|_{\bar{\tau}} e^{-\gamma(t-\bar{\tau})} + \|B_1\| \int^{t} ds \int_{s-\bar{\tau}}^{s} e^{-\gamma(t-s)}(\|A_1\|\|Y(\sigma)\| + \|B_1\|\|Y(\sigma-\bar{\tau})\|)d\sigma, \tag{3.11}$$

where $\|Y\|_{\bar{\tau}} = \sup_{t\in[-\bar{\tau},\bar{\tau}]} \|Y(t)\|$. From (3.8), there exists a positive constant α $(\alpha < \gamma)$ such that for arbitrary $t > \bar{\tau}$ the following inequality holds:

$$\left(1 - \frac{\bar{\tau}(\|A_1\| + \|B_1\|)\|B_1\|}{\gamma}\right) e^{-\alpha(t-\bar{\tau})} \geq e^{-\gamma(t-\bar{\tau})}. \tag{3.12}$$

From (3.11) and (3.12), we get

$$\|Y(t)\| \leq \|Y\|_{\bar{\tau}} e^{-\alpha(t-\bar{\tau})}, \qquad t > \bar{\tau}. \tag{3.13}$$

Inequality (3.13) implies the global asymptotic stability of the equilibrium point of system (3.8). This suggests that the equilibrium point of system (3.6) is globally asymptotically stable. So system (2.1) has a partial periodic oscillation. □

In the following let A^T and A^{-1} be the transpose and the inverse of a square matrix A respectively. $A > 0$ (< 0) will be denoted a positive (negative) definite matrix A.

Theorem 3.2. *Suppose that there is a unique equilibrium point and all solutions of system* (2.1) *are bounded. In addition, there is a positive definite four by four matrix P and a positive diagonal four by four matrix Q such that*

$$\begin{bmatrix} A_1^T P + PA_1 + Q & PB_1 \\ B_1^T P & -Q \end{bmatrix} < 0, \tag{3.14}$$

then system (2.1) *has a partial periodic oscillation.*

Proof. Similar to Theorem 3.1, we only need to prove that the components y_i $(i = 1, \ldots, 4)$ of the trivial solution of system (3.7) are stable. For system (3.7), we will employ the following positive definite Lyapunov functional

$$W(t) = Y^T PY(t) + \int_{t-\bar{\tau}}^{t} Y^T(s) QY(s) ds. \tag{3.15}$$

The upper right derivative of $W(t)$ along the trajectories of the system (3.7) is obtained as follows:

$$
\begin{aligned}
D^+ W(t)|_{(3.7)} &= (Y^T(t) A_1^T + Y^T(t-\bar{\tau}) B_1^T) PY(t) + Y^T(t) P(A_1 Y(t) + B_1 Y(t-\bar{\tau})) \\
&\quad + Y^T(t) QY(t) - Y^T(t-\bar{\tau}) QY(t-\bar{\tau}) \\
&= Y^T(t)(A_1^T P + PA_1 + Q)Y(t) + Y^T(t) PB_1 Y(t-\bar{\tau}) \\
&\quad + Y^T(t-\bar{\tau}) B_1^T PY(t) - Y^T(t-\bar{\tau}) QY(t-\bar{\tau}) \\
&= (Y^T(t) \quad Y^T(t-\bar{\tau})) \begin{bmatrix} A_1^T P + PA_1 + Q & PB_1 \\ B_1^T P & -Q \end{bmatrix} \begin{pmatrix} Y(t) \\ Y(t-\bar{\tau}) \end{pmatrix} \\
&< 0. \tag{3.16}
\end{aligned}
$$

This means that the trivial solution of system (3.7) is asymptotically stable, and implies that system (2.1) for arbitrary τ_1 and τ_3 has a partial periodic oscillation. $\square$

Theorem 3.3. *Suppose that system* (2.1) *has a unique equilibrium point and all solutions of system* (2.1) *are bounded. If $A_1 + B_1 > 0$, then system* (2.1) *generates a periodic oscillation.*

Proof. According to Lemma 2.3 we only need to show that the equilibrium point of subsystem (3.7) is unstable since the components u_5 and u_6 of the equilibrium point of system (2.3) are unstable. The characteristic equation associated with system (3.7) is given by:

$$\lambda = A_1 + B_1 e^{-\lambda \bar{\tau}}. \tag{3.17}$$

Note that (3.17) is a transcendental equation and λ may be a complex number. We prove that there exists a positive eigenvalue of (3.17) under the condition $A_1 + B_1 > 0$. If we set $f(\lambda) = \lambda - A_1 - B_1 e^{-\lambda \bar{\tau}}$, then $f(\lambda)$ is a continuous function of λ. Since $A_1 + B_1 > 0$, then $f(0) = -A_1 - B_1 = -(A_1 + B_1) < 0$. When λ is sufficiently large, say $\lambda = \lambda^* > 0$, $e^{-\lambda^* \bar{\tau}}$ is sufficiently small, and $f(\lambda^*) = \lambda^* - A_1 - B_1 e^{-\lambda^* \bar{\tau}} > 0$, then there exists $\lambda = \tilde{\lambda}$, $\tilde{\lambda} \in (0, \lambda^*)$ such that $f(\tilde{\lambda}) = 0$. This means that there is a positive eigenvalue of the characteristic equation (3.17) for any time delay $\bar{\tau}$, implying that the equilibrium point of system (2.1) is unstable for arbitrary time delays τ_1 and τ_3, and system (2.1) generates a periodic oscillation. $\square$

Theorem 3.4. *Suppose that system* (2.1) *has a unique equilibrium point and all solutions are bounded. Let $\alpha_1, \alpha_2, \ldots, \alpha_6$ and $\beta_1, \beta_2, \ldots, \beta_6$ denote the eigenvalues of matrices A and B respectively. $\alpha_i = \alpha_{i1} + i\alpha_{i2}$ (α_{i2} may be zero), and for some j ($j \in \{1, 2, \ldots, 6\}$, $\alpha_{j1} > 0$), then the trivial solution of system* (2.3) *is unstable and system* (2.1) *generates a periodic oscillation.*

Proof. In this case, let $\tau_* = \min\{\tau_1, \tau_3, \tau_5\}$. Corresponding system (2.3) we consider the following special system

$$U'(t) = AU(t) + BU(t - \tau_*), \tag{3.18}$$

where $\tau_1 = \tau_3 = \tau_5 = \tau_*$. The characteristic equation of (3.18) is the following:

$$\det[\lambda E - A - B e^{-\lambda \tau_*}] = 0, \tag{3.19}$$

where E is the six by six unit matrix. Immediately we have that

$$\prod_{k=1}^{6}[\lambda - \alpha_k - \beta_k e^{-\lambda \tau_*}] = 0. \tag{3.20}$$

If we let $\lambda = \sigma + i\omega$ be an eigenvalue of system (3.18), then for some $\alpha_{j1} > 0$ we get

$$\begin{cases} \sigma - \alpha_{j1} - \beta_{j1} e^{-\sigma \tau_*} \cos(\omega \tau_*) = 0, \\ \omega - \alpha_{j2} - \beta_{j2} e^{-\sigma \tau_*} \sin(\omega \tau_*) = 0. \end{cases} \tag{3.21}$$

We shall show that $\sigma > 0$ and there is an eigenvalue which has a positive real part for system (3.18). Indeed, let $f(\sigma) = \sigma - \alpha_{j1} - \beta_{j1} e^{-\sigma \tau_*} \cos(\omega \tau_*)$, then $f(\sigma)$ is a continuous function of σ. Since $\alpha_{j1} > 0$, one can select a suitable delay τ_* such that $\beta_{j1} \cos(\omega \tau_*) > -\alpha_{j1}$. Therefore, $f(0) = -\alpha_{j1} - \beta_{j1} \cos(\omega \tau_*) < 0$. Noting that $e^{-\sigma \tau_*} \to 0$ as $\sigma \tau_* \to +\infty$, obviously, there exists a suitably large $\bar{\sigma}(>0)$ such that $f(\bar{\sigma}) = \bar{\sigma} - \alpha_{j1} - \beta_{j1} e^{-\bar{\sigma} \tau_*} \cos(\omega \tau_*) > 0$. By the continuity of $f(\sigma)$, there exists a positive $\sigma^* \in (0, \bar{\sigma})$ such that $f(\sigma^*) = 0$. Thus, there is an eigenvalue of the characteristic equation associated with system (3.20) which has a positive real part. This means that the trivial solution of system (3.20) is unstable, implying that the trivial solution of system (2.3) for $\tau_1 = \tau_3 = \tau_5 = \tau_*$ is unstable. For a time delay system, if the trivial solution is unstable, then the instability of the trivial solution will be maintained as time delay increases. So for any delays the trivial solution of system (2.3) is also unstable. This implies that system (2.1) generates a periodic oscillation. We select a suitable delay such that the system has an oscillatory solution. This oscillation is said to induce by time delay. The proof is completed. □

4   Computer simulation result

In system (1.4), we select $\varepsilon_1 = 0.035, \varepsilon_2 = 0.025, \varepsilon_3 = 0.015; \Omega_1^2 = 0.016, \Omega_2^2 = 0.025, \Omega_3^2 = 0.12; \kappa_1 = 25, \kappa_2 = 15, p_1 = 0.015, p_2 = 0.025, p_3 = 0.085, q_1 = 0.0075, q_2 = 0.0065, q_3 = 0.0085$. Thus $\|A_1\| = 1.035, \|B_1\| = 0.029$. The eigenvalues of matrix $A_1 + B_1$ are $-0.0166 \pm 0.1842i$, $-0.0134 \pm 0.2474i$, and $\gamma = 0.0134$. It is easy to check that the conditions of Lemma 2.4 and Lemma 2.5 hold. When time delays are selected as $\tau_1 = 0.012, \tau_2 = 0.015, \tau_3 = 0.02$, and $\tau_1 = 0.12, \tau_2 = 0.15, \tau_3 = 0.2$, respectively, we see that $\frac{\bar{\tau}(\|A_1\| + \|B_1\|)\|B_1\|}{\gamma} < \frac{0.2 \cdot (1.035 + 0.029) \cdot 0.029}{0.0134} = 0.4605 < 1$. From Theorem 3.1, system (2.1) generates a partial periodic oscillation (see Figures 4.1a and 4.1b). When delays are increased, the convergent rate is slightly changed (see Figures 4.2a and 4.2b).

When we select $\varepsilon_1 = 0.35, \varepsilon_2 = 0.25, \varepsilon_3 = 0.15; \Omega_1^2 = 0.96, \Omega_2^2 = 1.25, \Omega_3^2 = 2.15; \kappa_1 = 0.2, \kappa_2 = 0.5, p_1 = 1.15, p_2 = 1.25, p_3 = 1.85, q_1 = 0.75, q_2 = 0.65, q_3 = 0.85$, and delays are $\tau_1 = 0.2, \tau_2 = 0.4, \tau_3 = 0.5$, and $\tau_1 = 1, \tau_2 = 2, \tau_3 = 2.5$, respectively, then the eigenvalues of matrix A are $0.1750 \pm 0.9460i, -0.1250 \pm 1.0651i, -1.3932$, and 1.5432. Note that $1.5432 > 0$, and the conditions of Theorem 3.4 are satisfied. Thus, system (2.1) generates a periodic oscillation (see Figures 4.3 and 4.4).
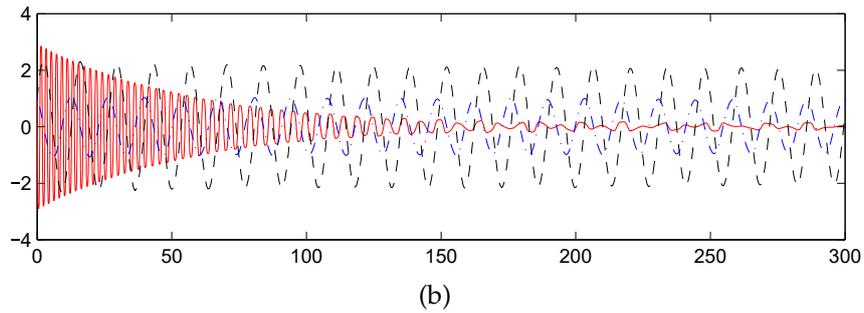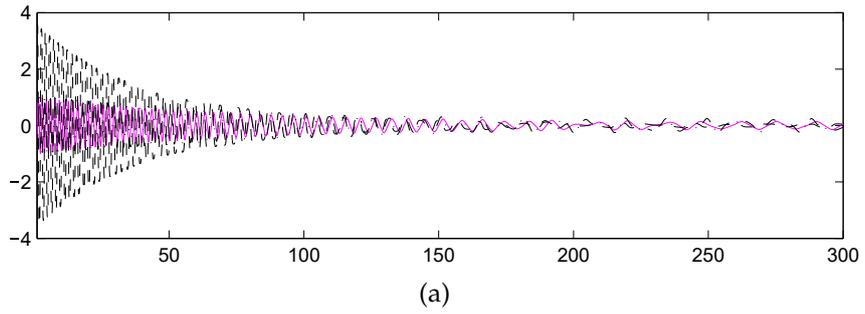
Figure 4.1: *(a)* $u_1(t), u_2(t)$, and $u_3(t)$ are convergent; delays: 0.012, 0.015, 0.020. Solid line: $u_1(t)$, dashed line: $u_2(t)$, dashdotted line: $u_3(t)$. *(b)* $u_4(t)$ is convergent, both $u_5(t)$ and $u_6(t)$ are oscillatory; delays: 0.012, 0.015, 0.020. Solid line: $u_4(t)$, dashed line: $u_5(t)$, dashdotted line: $u_6(t)$.
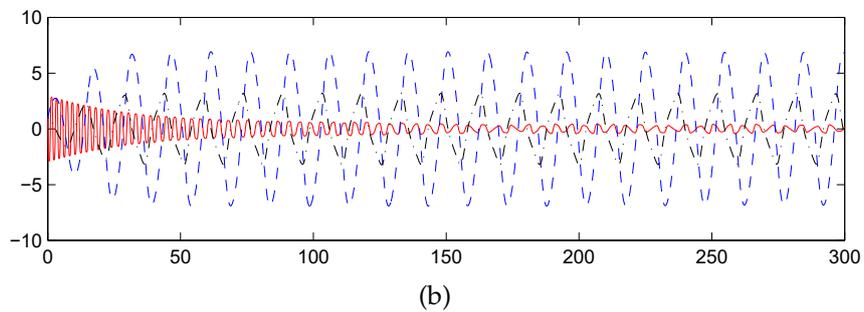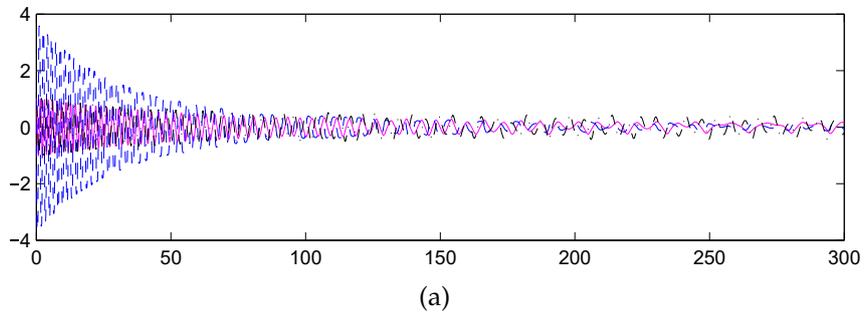

Figure 4.2: *(a)* $u_1(t), u_2(t)$, and $u_3(t)$ are convergent; delays: 1.2, 1.5, 2. Solid line: $u_1(t)$, dashed line: $u_2(t)$, dashdotted line: $u_3(t)$. *(b)* $u_4(t)$ is convergent, both $u_5(t)$ and $u_6(t)$ are oscillatory; delays: 1.2, 1.5, 2. Solid line: $u_4(t)$, dashed line: $u_5(t)$, dashdotted line: $u_6(t)$.
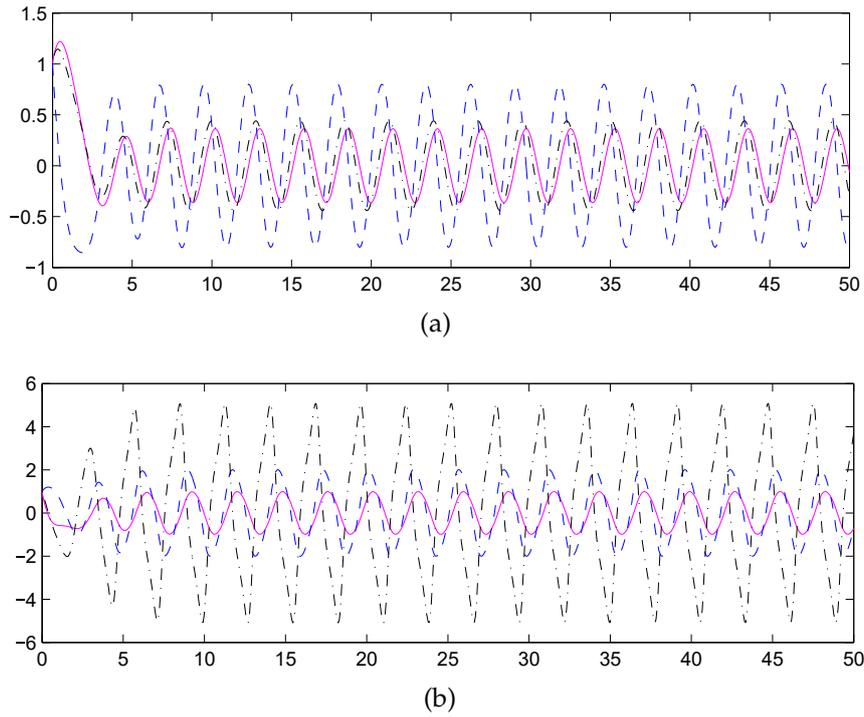
Figure 4.3: Oscillation of the solution; delays: 0.02, 0.04, 0.05. *(a)* Solid line: $u_1(t)$, dashed line: $u_2(t)$, dashdotted line: $u_3(t)$. *(b)* $u_4(t)$, dashed line: $u_5(t)$, dashdotted line: $u_6(t)$.
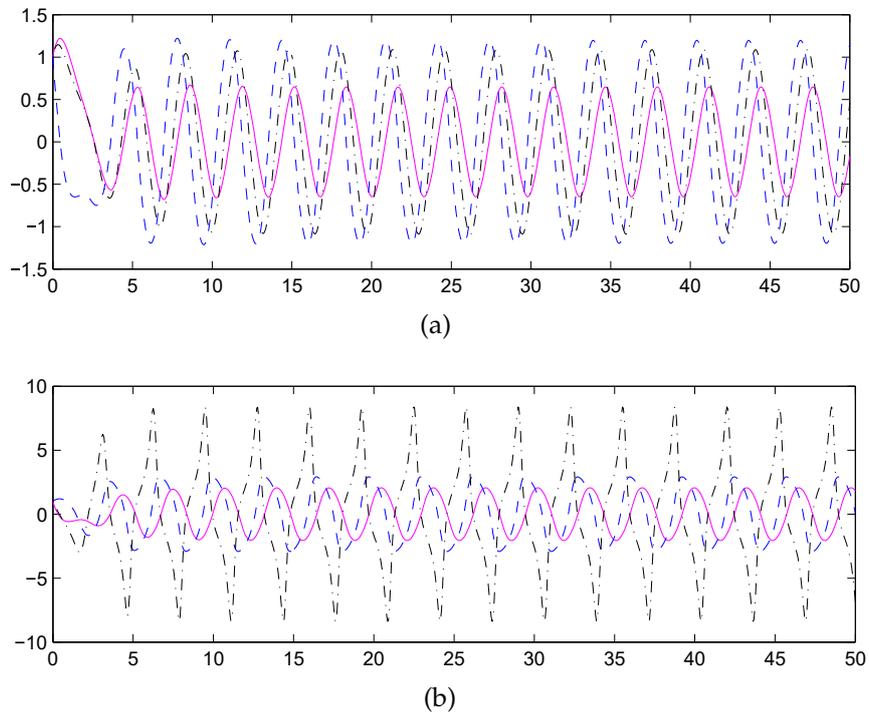

Figure 4.4: Oscillation of the solution; delays: 0.2, 0.4, 0.5. *(a)* Solid line: $u_1(t)$, dashed line: $u_2(t)$, dashdotted line: $u_3(t)$. *(b)* $u_4(t)$, dashed line: $u_5(t)$, dashdotted line: $u_6(t)$.

5 Conclusion

This paper discussed a system of two coupled damped Duffing oscillators driven by a van der Pol oscillator with delays. Some sufficient conditions to ensure the periodic and partial periodic oscillations for the system are established. Interestingly, this partial periodic oscillation is induced by unbalanced damped oscillators. When periodic and partial periodic oscillations occur, delays only affect the oscillation frequency. The study of micro-electro-mechanical phenomena and nano-electro-mechanical phenomena often requires experimental methods that can accurately control and manipulate the interaction between micro- and nano-objects. Our results are helpful for developing of novel MEMS or NEMS devices, which can precisely control of these nanoscale interactions, provide an ideal platform for interacting with the micro- and nano-world.

Acknowledgements

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