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Topological structure of solution sets to asymptotic n-th order vector boundary value problems

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Abstract. The R_δ-structure of solutions is investigated for asymptotic, higher-order, vector boundary value problems. Using the inverse limit technique, the topological structure is also studied, as the first step, on compact intervals. The main theorems are supplied by illustrative examples. One of them is finally applied, on the basis of our recently developed principle, to nontrivial existence problems.

Keywords: asymptotic n-th order vector problems, topological structure, R_δ-set, inverse limit technique, Hukuhara–Kneser–Aronszajn type results.

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1 Introduction

The investigation of a topological structure of sets of solutions to asymptotic boundary value problems is a delicate problem. The majority of the Hukuhara–Kneser–Aronszajn type results about (possibly special) continua of solutions is related to Cauchy initial value problems on compact intervals (see e.g. [5, Chapter III.2], [18–20], [22, Chapters 3 and 4], [26] and the references therein). Less results for Cauchy problems on non-compact and, in particular, infinite intervals were obtained by various techniques, e.g., in [1–4], [5, Chapter III.2], [12, 17, 20, 21, 24, 26, 33, 37, 40]. Rather rare results for boundary vale problems concern again almost exclusively those on compact intervals (see e.g. [11, 15, 38] and [5, Chapter III.3], [22, Chapter 6], where the whole chapters are devoted to this problem). The quite unique related results on non-compact intervals can be found, as far as we know, in [7, 28], [29, Chapter III.13]. The knowledge of a topological structure of solutions for further boundary value problems on non-compact intervals would be therefore highly appreciated.

The reason why such results are so rare consists in counter-examples presented in [3], [5, Example II.2.12], [23], demonstrating the impossibility of asymptotic analogies to the situation on compact intervals. These troubles are due to an "unpleasant" related topology of non-normable Fréchet spaces. For instance, a contractivity of a given operator with respect to a metric need not follow from a contractivity with respect to all seminorms. That is why the

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main theorem in [28] might be empty. On the other hand, although this difficulty can be overcame by means of the inverse limit method, sometimes also called the projective limit technique (see e.g. [3, 4, 23, 27, 34]), the class of appropriate boundary value problems seems to be rather narrow. In fact, besides for the Cauchy initial value problems, simple nontrivial examples for asymptotic boundary value problems via the inverse limit method, were given only by ourselves in [7].

Let us note that, unlike in [29], where (the Hukuhara–Kneser type) continua of solutions were received in a purely analytic way, (the Aronszajn type) R_δ-structure was detected in [7]. The main advantage of these special continua consists in its further possible application to the existence results for nonlinear asymptotic problems (see Section 5 below). The same is true for all special continua (except compact, connected sets themselves) from the proper inclusion scheme (2.1) below.

Hence, our paper is organized as follows. After the auxiliary definitions, lemmas and propositions in Section 2, the topological structure is firstly studied on compact intervals in Section 3. The first obtained result can be regarded in a certain sense as a final theorem. Nevertheless, it will only take there the form of proposition, because it is further employed, via an inverse limit method in Section 4, as a preliminary step for the investigation of structure on non-compact intervals. This knowledge is finally applied, on the basis of our principle developed recently in [6], to the solvability of nontrivial existence asymptotic problems. All the main theorems are supplied by illustrative examples.

2 Preliminaries

At first, we recall some geometric notions of subsets of metric spaces; in particular, of compact absolute retracts, compact contractible sets and R_δ-sets. For more details, see, e.g., [5, 14, 25].

For a subset $A \subset X$ of a metric space $X = (X, d)$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A : d(x, a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X. A subset $A \subset X$ is called a *retract* of X if there exists a retraction $r : X \to A$, i.e. a continuous function satisfying $r(x) = x$, for every $x \in A$.

We say that a metric space X is an *absolute retract* (AR-space) if, for each metric space Y and every closed $A \subset Y$, each continuous mapping $f : A \to X$ is extendable over Y. Let us note that X is an AR-space if and only if it is a retract of some normed space. Moreover, if X is a retract of a convex set in a Fréchet space, then it is an AR-space, too. So, in particular, for an arbitrary interval $J \subset \mathbb{R}$ and $k, n \in \mathbb{N}$, the spaces $C(J, \mathbb{R}^k)$, $C^n(J, \mathbb{R}^k)$, $AC^n_{loc}(J, \mathbb{R}^k)$ are AR-spaces as well as their convex subsets. The foregoing symbols denote, as usually, the spaces of functions $f : J \to \mathbb{R}^k$ which are continuous, have continuous n-th derivatives and locally absolutely continuous n-th derivatives, respectively, endowed with the respective topologies.

We say that a nonempty subset A of a metric space X is *contractible* if there exist a point $x_0 \in A$ and a homotopy $h : A \times [0, 1] \to A$ such that $h(x, 0) = x$ and $h(x, 1) = x_0$, for every $x \in A$. A nonempty set $A \subset X$ is called an *R_δ-set* if there exists a decreasing sequence $\{A_n\}_{n=1}^\infty$ of compact AR-spaces (or, despite of the hierarchy (2.1) below, compact, contractible sets) such that

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Note that any R_δ-set is nonempty, compact and connected. The following hierarchy holds for nonempty compact subsets of a metric space:

$$\begin{aligned}\text{compact + convex} \subset \text{compact } AR\text{-space} \subset \text{compact + contractible} \subset R_\delta\text{-set}\\ \subset \text{compact + acyclic} \subset \text{compact + connected},\end{aligned} \tag{2.1}$$

and all the above inclusions are proper.

We also employ the following definitions and statements from the multivalued analysis in the sequel. Let X and Y be arbitrary metric spaces. We say that F is a *multivalued mapping* from X to Y (written $F : X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is prescribed. We associate with F its *graph* Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

A multivalued mapping $F : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X. Every upper semicontinuous map with closed values has a closed graph.

A multivalued mapping $F : X \multimap X$ with bounded values is called *Lipschitzian* if there exists a constant $L > 0$ such that

$$d_H(F(x), F(y)) \leq Ld(x, y),$$

for every $x, y \in X$, where

$$d_H(A, B) := \inf\{r > 0 \mid A \subset N_r(B) \text{ and } B \subset N_r(A)\}$$

stands for the Hausdorff distance; for its properties, see, e.g., [5,25].

We say that a multivalued mapping $F : X \multimap X$ with bounded values is a *contraction* if it is Lipschitzian with a Lipschitz constant $L \in [0, 1)$.

Let Y be a separable metric space and $(\Omega, \mathcal{U}, \nu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure ν on $\mathcal{U}$. A multivalued mapping $F : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid F(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$.

We say that the mapping $F : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$, is *upper-Carathéodory* if the map $F(\cdot, x) : J \multimap \mathbb{R}^n$ is measurable on every compact subinterval of J, for all $x \in \mathbb{R}^m$, the map $F(t, \cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all (a.a.) $t \in J$, and the set $F(t, x)$ is compact and convex, for all $(t, x) \in J \times \mathbb{R}^m$.

We will employ the following selection statement.

Lemma 2.1 (cf., e.g., [9]). *Let $F : [a, b] \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping satisfying $|y| \leq r(t)(1 + |x|)$, for every $(t, x) \in [a, b] \times \mathbb{R}^m$, and every $y \in F(t, x)$, where $r : [a, b] \to [0, \infty)$ is an integrable function. Then the composition $F(t, q(t))$ admits, for every $q \in C([a, b], \mathbb{R}^m)$, a single-valued measurable selection.*

If $X \cap Y \neq \emptyset$ and $F : X \multimap Y$, then a point $x \in X \cap Y$ is called a *fixed point* of F if $x \in F(x)$. The set of all fixed points of F will be denoted by $\mathrm{Fix}(F)$, i.e.

$$\mathrm{Fix}(F) := \{x \in X \mid x \in F(x)\}.$$

It will be also convenient to recall the following results.

Proposition 2.2 (cf. [35]). *Let X be a closed, convex subset of a Banach space E and let $\phi : X \multimap X$ be a contraction with compact, convex values. Then* $\mathrm{Fix}(\phi)$ *is a nonempty, compact AR-space.*

Lemma 2.3 (cf. [10, Theorem 0.3.4]). *Let $[a,b] \subset \mathbb{R}$ be a compact interval. Assume that the sequence of absolutely continuous functions $x_k : [a,b] \to \mathbb{R}^n$ satisfies the following conditions:*

(*i*) *the set $\{x_k(t) \mid k \in \mathbb{N}\}$ is bounded, for every $t \in [a,b]$,*

(*ii*) *there exists a function $\alpha : [a,b] \to \mathbb{R}$, integrable in the sense of Lebesgue, such that*

$$|\dot{x}_k(t)| \le \alpha(t), \quad \text{for a.a. } t \in [a,b] \text{ and for all } k \in \mathbb{N}.$$

Then there exists a subsequence of $\{x_k\}$ (for the sake of simplicity, denoted in the same way as the sequence) converging to an absolutely continuous function $x : [a,b] \to \mathbb{R}^n$ in the following way:

1. *$\{x_k\}$ converges uniformly to x,*
2. *$\{\dot{x}_k\}$ converges weakly in $L^1([a,b],\mathbb{R}^n)$ to $\dot{x}$.*

The following lemma is a slight modification of the well known result.

Lemma 2.4 (cf. [39, p. 88]). *Let $[a,b] \subset \mathbb{R}$ be a compact interval, E_1, E_2 be Euclidean spaces and $F : [a,b] \times E_1 \multimap E_2$ be an upper-Carathéodory mapping.*

Assume in addition that, for every nonempty, bounded set $\mathcal{B} \subset E_1$, there exists $\nu = \nu(\mathcal{B}) \in L^1([a,b],[0,\infty))$ such that

$$|F(t,x)| \le \nu(t),$$

for a.a. $t \in [a,b]$ and every $x \in \mathcal{B}$.

Let us define the Nemytskiĭ operator $N_F : C([a,b],E_1) \multimap L^1([a,b],E_2)$ in the following way:

$$N_F(x) := \{f \in L^1([a,b],E_2) \mid f(t) \in F(t,x(t)), \text{ a.e. on } [a,b]\},$$

for every $x \in C([a,b],E_1)$. Then, if sequences $\{x_i\} \subset C([a,b],E_1)$ and $\{f_i\} \subset L^1([a,b],E_2)$, $f_i \in N_F(x_i)$, $i \in \mathbb{N}$, are such that $x_i \to x$ in $C([a,b],E_1)$ and $f_i \to f$ weakly in $L^1([a,b],E_2)$, then $f \in N_F(x)$.

3 Topological structure on compact intervals

At first, let us consider the constraint problems for linear systems

$$\left.\begin{array}{c} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) \in C(t), \quad \text{for a.a. } t \in [0,m], \\ x \in S_m, \end{array}\right\} \tag{3.1}$$

$$\left.\begin{array}{c} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) \in C(t), \quad \text{for a.a. } t \in [0,m], \\ (x,\dot{x},\ldots,x^{(n-1)}) \in S'_m, \end{array}\right\} \tag{3.2}$$

where

(*i*) $A_i : [0,m] \to \mathbb{R}^{k\times k}$ are integrable matrix functions such that $|A_i(t)| \le a_i(t)$, for a.a. $t \in [0,m]$ and suitable nonnegative functions $a_i \in L^1([0,m],\mathbb{R})$, for all $i = 1,\ldots,n$,

(*ii*) S_m is a closed, convex subset of $AC^{n-1}([0,m],\mathbb{R}^k)$ (S'_m is a closed, convex subset of $AC^{n-1}([0,m],\mathbb{R}^k) \times AC^{n-2}([0,m],\mathbb{R}^k) \times \cdots \times AC([0,m],\mathbb{R}^k)$),

(*iii*) $C : [0,m] \multimap \mathbb{R}^k$ is an integrable mapping with convex closed values such that $|C(t)| \le c(t)$, for a.a. $t \in [0,m]$ and a suitable nonnegative function $c \in L^1([0,m],\mathbb{R})$, (*iv*) there exist $t_0 \in [0,m]$ and a constant M such that $|x(t_0)| \le M$, $|\dot{x}(t_0)| \le M, \dots, |x^{(n-1)}(t_0)| \le M$, for all solutions of problem (3.1) (all solutions of problem (3.2)).

Proposition 3.1. *Under the above assumptions* (i)–(iv), *the solution set of problem* (3.1) (*the set of solutions and their derivatives up to the* $(n-1)$*-st order of problem* (3.2)) *is convex and compact).*

Proof. Let us prove that the set of solutions and their derivatives of the b.v.p. (3.2) is convex and compact. By the similar reasoning, it is possible to obtain that the solution set of problem (3.1) is convex and compact as well.

Let us denote by $P(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t)) := C(t) - A_1(t)x^{(n-1)}(t) - \dots - A_n(t)x(t)$. If x_1, x_2 are solutions of problem (3.2), then it follows from the integral representation of a solution that, for a.a. $t \in [0,m]$, we have

$$x_1(t) \in x_1(t_0) + \dot{x}_1(t_0)(t-t_0) + \frac{1}{2}\ddot{x}_1(t_0)(t-t_0)^2 + \dots + \frac{1}{(n-1)!}x_1^{(n-1)}(t_0)(t-t_0)^{n-1} + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}P(s, x_1(s), \dot{x}_1(s), \dots, x_1^{(n-1)}(s))\,ds,$$

and

$$x_2(t) \in x_2(t_0) + \dot{x}_2(t_0)(t-t_0) + \frac{1}{2}\ddot{x}_2(t_0)(t-t_0)^2 + \dots + \frac{1}{(n-1)!}x_2^{(n-1)}(t_0)(t-t_0)^{n-1} + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}P(s, x_2(s), \dot{x}_2(s), \dots, x_2^{(n-1)}(s))\,ds.$$

Let $\theta \in [0,1]$ be arbitrary. Then

$$\begin{aligned}\theta x_1(t) + (1-\theta)x_2(t) &\in \theta x_1(t_0) + (1-\theta)\cdot x_2(t_0) + [\theta\dot{x}_1(t_0) + (1-\theta)\dot{x}_2(t_0)](t-t_0) + \dots \\ &\quad + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}\theta\cdot P(s, x_1(s), \dot{x}_1(s), \dots, x_1^{(n-1)}(s))\,ds \\ &\quad + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}(1-\theta)P(s, x_2(s), \dot{x}_2(s), \dots, x_2^{(n-1)}(s))\,ds \\ &= \theta x_1(t_0) + (1-\theta)x_2(t_0) + [\theta\dot{x}_1(t_0) + (1-\theta)\dot{x}_2(t_0)](t-t_0) + \dots \\ &\quad + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}P(s, \theta x_1(s) + (1-\theta)x_2(s), \dots, \theta x_1^{(n-1)}(s) + (1-\theta)x_2^{(n-1)}(s))ds.\end{aligned}$$

Moreover, for all $k = 1, \dots, n-1$,

$$x_1^{(k)}(t) \in x_1^{(k)}(t_0) + x_1^{(k+1)}(t_0)(t-t_0) + \dots + \frac{1}{(n-1-k)!}x_1^{(n-1-k)}(t_0)(t-t_0)^{n-1-k} + \frac{1}{(n-1-k)!}\int_{t_0}^{t}(t-s)^{n-1-k}P(s, x_1(s), \dot{x}_1(s), \dots, x_1^{(n-1)}(s))\,ds,$$

and

$$x_2^{(k)}(t) \in x_2^{(k)}(t_0) + x_2^{(k+1)}(t_0)(t-t_0) + \dots + \frac{1}{(n-1-k)!}x_2^{(n-1-k)}(t_0)(t-t_0)^{n-1-k} + \frac{1}{(n-1-k)!}\int_{t_0}^{t}(t-s)^{n-1-k}P(s, x_2(s), \dot{x}_2(s), \dots, x_2^{(n-1)}(s))\,ds.$$

By similar arguments as before, we can obtain, for an arbitrary $\theta \in [0,1]$ and all $k = 1, \dots, n-1$, that

$$\theta x_1^{(k)}(t) + (1-\theta)x_2^{(k)}(t) \in \theta x_1^{(k)}(t_0) + (1-\theta)x_2^{(k)}(t_0) + [\theta x_1^{(k+1)}(t_0) + (1-\theta)x_2^{(k)}(t_0)](t-t_0) + \dots + \frac{1}{(n-1-k)!}\int_{t_0}^{t}(t-s)^{n-1-k}P(s, \theta x_1(s) + (1-\theta)x_2(s), \dots, \theta x_1^{(n-1)}(s) + (1-\theta)x_2^{(n-1)}(s))ds.$$

Finally, because of convexity of S'_m, we obtain that

$$\left(\theta x_1 + (1-\theta)x_2,\ \theta\dot{x}_1 + (1-\theta)\dot{x}_2, \ldots, \theta x_1^{(n-1)} + (1-\theta)x_2^{(n-1)}\right) \in S'_m$$

and, therefore, the set of solutions of (3.2) and their derivatives is convex.

Let us also prove that the set of solutions of (3.2) and their derivatives is relatively compact. It follows from the well known Arzelà–Ascoli lemma that the set of solutions is relatively compact in $C^{n-1}([0,m],\mathbb{R}^k)$ if and only if it is bounded and all solutions and their derivatives (up to the $(n-1)$-st order) are equi-continuous.

At first, let us show that the set of solutions of (3.2) is bounded in $C^{n-1}([0,m],\mathbb{R}^k)$. Let x be a solution of (3.2) and let $t \in [0,m]$ be arbitrary.

Since

$$\begin{aligned}
x^{(n-1)}(t) &= x^{(n-1)}(t_0) + \int_{t_0}^{t} x^{(n)}(s)\, ds, \quad \text{for a.a. } t \in [0,m],\\
&\vdots\\
\dot{x}(t) &= \dot{x}(t_0) + \int_{t_0}^{t} \ddot{x}(s)\, ds, \quad \text{for a.a. } t \in [0,m],\\
x(t) &= x(t_0) + \int_{t_0}^{t} \dot{x}(s)\, ds, \quad \text{for a.a. } t \in [0,m],
\end{aligned}$$

it holds, according to conditions (i), (iii) and (iv), that

$$\begin{aligned}
&|x(t)| + |\dot{x}(t)| + \cdots + |x^{(n-1)}(t)|\\
&\le |x(t_0)| + |\dot{x}(t_0)| + \cdots + |x^{(n-1)}(t_0)| + \int_{t_0}^{t} |\dot{x}(s)| + |\ddot{x}(s)| + \cdots + |x^{(n)}(s)|\, ds\\
&\le nM + \int_0^m |\dot{x}(s)| + |\ddot{x}(s)| + \cdots + |x^{(n-1)}(s)| + c(s) + a_1(s)|x^{(n-1)}(s)| + \cdots + a_n(s)|x(s)|\, ds\\
&\le nM + \int_0^m c(s)\, ds + \int_0^m a_n(s)|x(s)| + (1 + a_{n-1}(s))|\dot{x}(s)| + \cdots + (1 + a_1(s))|x^{(n-1)}(s)|\, ds\\
&\le nM + \int_0^m c(s)\, ds + \int_0^m k(s)(|x(s)| + |\dot{x}(s)| + \cdots + |x^{(n-1)}(s)|)\, ds
\end{aligned}$$

where, for all $s \in [0,m]$, $k(s) := \max\{1 + a_1(s), \ldots, 1 + a_{n-1}(s), a_n(s)\}$. Therefore, by Gronwall's lemma (cf. [30]),

$$|x(t)| + |\dot{x}(t)| + \cdots + |x^{(n-1)}(t)| \le \left(nM + \int_0^m c(s)\, ds\right) e^{\int_0^m k(s)\, ds}, \quad \text{for a.a. } t \in [0,m]. \tag{3.3}$$

Therefore, the set of solutions of (3.2) and their derivatives (up to the $(n-1)$-st order) is bounded in $C^{n-1}([0,m],\mathbb{R}^k)$.

Let us now show that all solutions x of (3.2) and their derivatives $\dot{x}, \ldots, x^{(n-1)}$ are also equi-continuous. So, let x be a solution of (3.2) and $t_1, t_2 \in [0,m]$ be arbitrary. Then, we have

$$|x(t_1) - x(t_2)| \le \left|\int_{t_1}^{t_2} |\dot{x}(\tau)|\, d\tau\right| \le \left|\int_{t_1}^{t_2} \left(nM + \int_0^m c(s)\, ds\right) e^{\int_0^m k(s)\, ds}\, d\tau\right|. \tag{3.4}$$

Analogously, we can get, for each $k \in \{1, \ldots, n-2\}$, that

$$|x^{(k)}(t_1) - x^{(k)}(t_2)| \le \left|\int_{t_1}^{t_2} |x^{(k+1)}(\tau)|\, d\tau\right| \le \left|\int_{t_1}^{t_2} \left(nM + \int_0^m c(s)\, ds\right) e^{\int_0^m k(s)\, ds}\, d\tau\right|. \tag{3.5}$$

Moreover,

$$|x^{(n-1)}(t_1) - x^{(n-1)}(t_2)| \le \left| \int_{t_1}^{t_2} c(\tau) + a_1(\tau)|x^{(n-1)}(\tau)| + \cdots + a_n(\tau)|x(\tau)|\, d\tau \right|$$
$$\le \left| \int_{t_1}^{t_2} c(\tau) + l(\tau) \left(nM + \int_0^m c(s)\, ds \right) e^{\int_0^m k(s)\, ds}\, d\tau \right|, \tag{3.6}$$

where, for all $\tau \in [0,m]$, $l(\tau) := \max\{a_1(\tau), \dots, a_n(\tau)\}$.

Taking into account estimates (3.4)–(3.6), $x, \dot{x}, \dots, x^{(n-1)}$ are equi-continuous, because $c(\cdot)$, $k(\cdot)$, $l(\cdot) \in L^1([0,m], \mathbb{R})$. Thus, the set of solutions of (3.2) and their derivatives is relatively compact.

We will still show that the set of solutions of (3.2) and their derivatives (up to the $(n-1)$-st order) is closed. Let $\{x_i\}$ be a sequence of solutions of (3.2) such that $\{(x_i, \dot{x}_i, \dots, x_i^{(n-1)})\} \to (x, \dot{x}, \dots, x^{(n-1)})$. By conditions (i), (iii) and estimate (3.3), the sequences $\{x_i\}$, $\{\dot{x}_i\}, \dots, \{x_i^{(n-1)}\}$ satisfy the assumptions of Lemma 2.3. Thus, there exists a subsequence of $\{x_i\}$, for the sake of simplicity denoted as the sequence, uniformly convergent to x on $[0,m]$, such that $\{\dot{x}_i\}, \dots, \{x_i^{(n-1)}\}$ converges uniformly to $\dot{x}, \dots, x^{(n-1)}$ on $[0,m]$ and that $\{x_i^{(n)}\}$ converges weakly to $x^{(n)}$ in $L^1\left([0,m],\ \mathbb{R}^k\right)$.

If we set $z_i := \left(x_i, \dot{x}_i, \dots, x_i^{(n-1)}\right)$, then $\dot{z}_i \to \left(\dot{x}, \ddot{x}, \dots, x^{(n)}\right)$ weakly in $L^1\left([0,m],\ \mathbb{R}^k\right)$. Let us now consider the following system

$$\dot{z}_i(t) \in G\left(t, z_i(t)\right), \quad \text{for a.a. } t \in [0,m], \tag{3.7}$$

where

$$G\left(t, z_i(t)\right) = \left(\dot{x}_i, \dots, x_i^{(n)}, P(t, z_i(t))\right).$$

Using Lemma 2.4, for $f_i := \dot{z}_i$, $f := (\dot{x}, \ddot{x}, \dots x^{(n)})$, $x_i := (z_i)$, it follows that

$$(\dot{x}(t), \ddot{x}(t), \dots, x^{(n)}(t)) \in G\left(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t)\right),$$

for a.a. $t \in [0,m]$, i.e.

$$x^{(n)}(t) \in P\left(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t)\right), \quad \text{for a.a. } t \in [0,m].$$

Moreover, since the set S'_m is closed, $(x_i, \dots x_i^{(n-1)}) \in S'_m$, for all $i \in \mathbb{N}$, and $(x_i, \dots, x_i^{(n-1)}) \to (x, \dots x^{(n-1)})$, it also holds that $(x, \dot{x}, \dots, x^{(n-1)}) \in S'_m$. After all, the set of solutions of (3.2) and their derivatives is convex and compact, as claimed. □

Remark 3.2. The statement of Proposition 3.1 also holds if we consider multivalued matrix mappings A_i, i.e. if we replace problems (3.1), (3.2) by

$$\left.\begin{array}{r} x^{(n)}(t) \in C(t) - A_1(t)x^{(n-1)}(t) - \cdots - A_n(t)x(t), \quad \text{for a.a. } t \in [0,m], \\ x \in S_m, \qquad\qquad\qquad \end{array}\right\} \tag{3.8}$$

$$\left.\begin{array}{r} x^{(n)}(t) \in C(t) - A_1(t)x^{(n-1)}(t) - \cdots - A_n(t)x(t), \quad \text{for a.a. } t \in [0,m], \\ (x, \dot{x}, \dots, x^{(n-1)}) \in S'_m, \qquad\qquad\qquad \end{array}\right\} \tag{3.9}$$

where $A_i : [0,m] \multimap \mathbb{R}^{k \times k}$ are integrable multivalued matrix mappings such that $|A_i(t)| \le a_i(t)$, for a.a. $t \in [0,m]$ and suitable nonnegative functions $a_i \in L^1([0,m], \mathbb{R})$, for all $i = 1, \dots, n$.

Furthermore, let us study the structure of a solution set, on a compact interval, to a semilinear problem.

Hence, let $m \in \mathbb{N}$ and consider the b.v.p.

$$\left.\begin{array}{c} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) \in C(t, x(t), \ldots, x^{(n-1)}(t)), \\ \text{for a.a. } t \in [0, m], \\ l(x, \dot{x}, \ldots, x^{(n-1)}) = 0, \end{array}\right\} (P_m)$$

where

(i) $A_i \in L^1([0, m], \mathbb{R}^{k\times k})$ are such that $|A_i(t)| \le a_i(t)$, for all $t \in [0, m]$ and suitable integrable functions $a_i : [0, m] \to [0, \infty)$, for all $i = 1, \ldots, n$,

(ii) $l : C^{n-1}([0, m], \mathbb{R}^k) \times \cdots \times C([0, m], \mathbb{R}^k) \to \mathbb{R}^{kn}$ is a linear bounded operator,

(iii) the associated homogeneous problem

$$\left.\begin{array}{c} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) = 0, \quad \text{for a.a. } t \in [0, m], \\ l(x, \dot{x}, \ldots, x^{(n-1)}) = 0 \end{array}\right\} (H_m)$$

has only the trivial solution,

(iv) $C : [0, m] \times \mathbb{R}^{kn} \multimap \mathbb{R}^k$ is an upper-Carathéodory mapping,

(v) there exists an integrable function $\alpha : [0, m] \to [0, \infty)$, with $\int_0^m \alpha(t)\, dt$ sufficiently small, such that

$$d_H(C(t, x_1, x_2, \ldots, x_n), C(t, y_1, y_2, \ldots, y_n) \le \alpha(t) \cdot (|x_1 - y_1| + \cdots + |x_n - y_n|),$$

for a.a. $t \in [0, m]$ and all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}^k$,

(vi) there exist a point $(\overline{x}_1, \ldots, \overline{x}_n) \in \mathbb{R}^{kn}$ and a constant $C_0 \ge 0$ such that

$$|C(t, \overline{x}_1, \ldots, \overline{x}_n)| \le C_0 \cdot \alpha(t)$$

holds, for a.a. $t \in [0, m]$

($\overset{(vi)}{\Longrightarrow}$ $|C(t, x_1, \ldots, x_n)| := \sup\{|z| \mid z \in C(t, x_1, \ldots, x_n)\} \le \alpha(t)\big(C_0 + |\overline{x}_1| + \cdots + |\overline{x}_n| + |x_1| + \cdots + |x_n|\big)$ holds, for a.a. $t \in [0, m]$ and all $x_i \in \mathbb{R}^k$, $i = 1, \ldots, n$).

Theorem 3.3. *Under the above assumptions (i)–(vi), the set of solutions of the b.v.p. (P_m) is a nonempty, compact AR-space.*

Proof. Problem (P_m) is equivalent to the first-order problem

$$\left.\begin{array}{l} \dot{\xi}(t) + D(t)\xi(t) \in K(t, \xi(t)), \quad \text{for a.a. } t \in [0, m], \\ l(\xi) = 0, \end{array}\right\} (\tilde{P}_m)$$

where

$$\xi(t)_{kn\times 1} = (x(t), \dot{x}(t), \ldots, x^{(n-1)}(t))^T,$$

$$D(t)_{kn\times kn} = \begin{pmatrix} & 0_{(kn-k)\times k} & -I_{(kn-k)\times(kn-k)} & \\ A_n(t) & A_{n-1}(t) & \cdots & A_1(t) \end{pmatrix}$$

and

$$K(t, \xi)_{kn\times 1} = (0_{(kn-k)\times 1}, C(t, x, \dot{x}, \ldots, x^{(n-1)}(t)))^T.$$

Similarly, the associated homogeneous problem (H_m) is equivalent to the first-order problem

$$\left.\begin{array}{l}\dot{\xi}(t)+D(t)\xi(t)=0, \quad \text{for a.a. } t\in[0,m],\\ l(\xi)=0.\end{array}\right\}\ (\tilde{H}_m)$$

The Fredholm alternative implies (see, e.g., [30]) that there exists the Green function $\tilde{G}$ for the homogeneous problem $(\tilde{H}_m)$ such that each solution $\xi(\cdot)$ of $(\tilde{P}_m)$ can be expressed by the formula $\xi(t) = \int_0^m \tilde{G}(t,s)k(s)\,ds$, where $k(\cdot)$ is a suitable measurable selection of $K(\cdot,\xi(\cdot))$ (cf. Lemma 2.1). If we denote by $\tilde{G}$ the block matrix

$$\tilde{G}_{kn\times kn} = \begin{pmatrix} \tilde{G}^{11}_{k\times k} & \tilde{G}^{12}_{k\times k} & \dots & \tilde{G}^{1n}_{k\times k} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{G}^{n1}_{k\times k} & \tilde{G}^{n2}_{k\times k} & \dots & \tilde{G}^{nn}_{k\times k} \end{pmatrix}, \tag{3.10}$$

then each solution $x(\cdot)$ of (P_m) and its derivatives can be expressed as

$$\begin{aligned} x(t) &= \int_0^m \tilde{G}^{1n}(t,s)c(s)\,ds,\\ \dot{x}(t) &= \int_0^m \tilde{G}^{2n}(t,s)c(s)\,ds,\\ &\vdots\\ x^{(n-1)}(t) &= \int_0^m \tilde{G}^{nn}(t,s)c(s)\,ds, \end{aligned}$$

where $c(\cdot)$ is a suitable measurable selection of $C(\cdot,x(\cdot),\dot{x}(\cdot),\dots,x^{(n-1)}(\cdot))$. Moreover, in view of (v) and (vi),

$$|x(t)|+\dots+|x^{(n-1)}(t))| \le \int_0^m \overline{G}\alpha(s)\left[C_0+|\overline{x}_1|+\dots+|\overline{x}_n|+|x(s)|+\dots+|x^{(n-1)}(s)|\right]\,ds,$$

for a.a. $t\in[0,m]$, where $\overline{G} := \sup_{(t,s)\in[0,m]\times[0,m]}\left\{\left|\tilde{G}^{1n}(t,s)\right|+\left|\tilde{G}^{2n}(t,s)\right|+\dots+\left|\tilde{G}^{nn}(t,s)\right|\right\}$.

Therefore,

$$\max_{t\in[0,m]}\{|x(t)|+|\dot{x}(t)|+\dots+|x^{(n-1)}(t))|\} \le \frac{\overline{G}\cdot(C_0+|\overline{x}_1|+\ldots+|\overline{x}_n|)\cdot\int_0^m\alpha(s)\,ds}{1-\overline{G}\int_0^m\alpha(s)\,ds} =: M,$$

provided

$$\int_0^m \alpha(s)\,ds < \frac{1}{\overline{G}}. \tag{3.11}$$

Therefore, if $\int_0^m \alpha(s)\,ds$ is small enough, namely if the inequality (3.11) holds, then the set of solutions of (P_m) is equal to the set of solutions of the problem

$$\left.\begin{array}{c} x^{(n)}(t)+A_1(t)x^{(n-1)}(t)+\dots+A_n(t)x(t)\in C^*(t,x(t),\dots x^{(n-1)}(t)),\\ \text{for a.a. } t\in[0,m],\\ l(x,\dot{x},\dots x^{(n-1)})=0, \end{array}\right\}\ (R_m)$$

where C^* satisfies conditions (iv)–(v) in Theorem 3.3 with C replaced by C^*, but this time

$$C^*(t,x_1,\dots,x_n) := \begin{cases} C(t,x_1,\dots,x_n), & \text{for} |x_i|\le M,\ i=1,\dots,n,\\ C(t,M_1,\dots,M_n), & \text{otherwise,}\end{cases}$$

where $M_1, \ldots, M_n$ are suitable vectors such that $|M_1| = \cdots = |M_n| = M$. It follows immediately from its definition that C^* satisfies

$$\begin{aligned} |C^*(t, x_1, \ldots x_n)| &:= \sup\{|z| \mid z \in C^*(t, x_1, \ldots, x_n)\} \\ &= \sup\{|z| \mid z \in C(t, x_1, \ldots, x_n), \text{ where } |x_i| \le M,\ i = 1, \ldots, n\} \\ &\le \alpha(t)(C_0^* + |\overline{x}_1^*| + \cdots + |\overline{x}_n^*| + nM) =: \beta(t), \end{aligned} \tag{3.12}$$

where $(\overline{x}_1^*, \ldots, \overline{x}_n^*) \in \mathbb{R}^{kn}$ is such that $|C^*(t, \overline{x}_1^*, \ldots, \overline{x}_n^*)| \le C_0^* \alpha(t)$, for a.a. $t \in [0, m]$.

Let us denote by $G(\cdot, \cdot\cdot) := \tilde{G}^{12}(\cdot, \cdot\cdot)$ the Green function associated to the homogeneous problem (H_m) and define the Nemytskiǐ operator

$$N : C^{n-1}([0, m], \mathbb{R}^k) \multimap C^{n-1}([0, m], \mathbb{R}^k)$$

by the formula

$$\begin{aligned} Nx := \Big\{ h \in C^1([0, m], \mathbb{R}^k) \mid h(\cdot) = \int_0^m G(\cdot, s) f(s)\, ds, \text{ where } f \in L^1([0, m], \mathbb{R}^k), \\ f(t) \in C^*(t, x(t), \dot{x}(t), \ldots, x^{(n-1)}(t)), \text{ for a.a. } t \in [0, m] \Big\}. \end{aligned}$$

Let us note that $Nx \ne \emptyset$, for all $x \in C^{n-1}([0, m], \mathbb{R}^k)$, because, for all $x \in C^{n-1}([0, m], \mathbb{R}^k)$, $C^*(t, x(t), \dot{x}(t), \ldots, x^{(n-1)}(t))$ possesses a measurable selection (again, according to Lemma 2.1).

It is evident that the set of solutions of problem (R_m) is equal to the set of fixed points of the operator N. In order to show that $\mathrm{Fix}(N)$ is, by means of Proposition 2.2, a nonempty, compact AR-space, we will proceed in three steps.

(a) At first, let us show that the operator N has convex values. If $h_1, h_2 \in Nx$, then there exist integrable selections $f_1(\cdot), f_2(\cdot)$ of $C^*(\cdot, x(\cdot), \dot{x}(\cdot), \ldots, x^{(n-1)}(\cdot))$ such that, for a.a. $t \in [0, m]$,

$$h_1(t) = \int_0^m G(t, s) f_1(s)\, ds$$

and

$$h_2(t) = \int_0^m G(t, s) f_2(s)\, ds.$$

Let $\lambda \in [0, 1]$ be arbitrary. Then, for a.a. $t \in [0, m]$,

$$\lambda h_1(t) + (1 - \lambda) h_2(t) = \int_0^m G(t, s) \left[\lambda f_1(s) + (1 - \lambda) f_2(s)\right] ds.$$

Since the mapping C^* has convex values,

$$\lambda f_1(s) + (1 - \lambda) f_2(s) \in C^*(s, x(s), \dot{x}(s), \ldots, x^{(n-1)}(s)),$$

for a.a. $s \in [0, m]$. Therefore, $\lambda h_1 + (1 - \lambda) h_2 \in Nx$, i.e. the operator N has convex values, as claimed.

(b) Secondly, let us show that the operator N has compact values. Let $x \in C^{n-1}([0, m], \mathbb{R}^k)$ be arbitrary and let v be an arbitrary integrable function such that

$$v(t) \in C^*(t, x(t), \dot{x}(t), \ldots, x^{(n-1)}(t)),$$

for a.a. $t \in [0, m]$.

Let us consider the element h of Nx defined, for a.a. $t \in [0, m]$, by

$$h(t) := \int_0^m G(t,s)v(s)\, ds.$$

If $t, \tau \in [0, m]$ are arbitrary, then

$$\begin{aligned} |h(t) - h(\tau)| &= \left| \int_0^m G(t,s)v(s)\, ds - \int_0^m G(\tau,s)v(s)\, ds \right| \\ &\le \int_0^m |G(t,s) - G(\tau,s)| \cdot |v(s)|\, ds \le \int_0^m |G(t,s) - G(\tau,s)| \cdot \beta(s)\, ds. \end{aligned} \tag{3.13}$$

Since $\beta(\cdot)$ is, by the definition, an integrable function, estimate (3.13) implies the equicontinuity of h. Moreover, it immediately follows from condition (3.12) and the properties of the Green function that h is also bounded. Therefore, the well known Arzelà–Ascoli lemma implies that the set Nx is relatively compact.

The relative compactness of values follows also alternatively from the contractivity of N which will be proved in the next step (3). It is namely well known that contractivity implies condensity.

The closedness of values follows from the fact that, according to [31], N can be expressed as the closed graph composition of operators $\phi \circ S_{C^*}$, where $S_{C^*} : C^{n-1}([0, m], \mathbb{R}^k) \multimap L^1([0, m], \mathbb{R}^k)$ and $\phi : L^1([0, m], \mathbb{R}^k) \to C^{n-1}([0, m], \mathbb{R}^k)$ are defined by

$$S_{C^*}(x) := \left\{ f \in L^1([0, m], \mathbb{R}^k) \mid f(t) \in C^*(t, x(t), \dots, x^{(n-1)}(t)), \text{ for a.a. } t \in [0, m] \right\}$$

and

$$\phi(f) := \left\{ h \in C^{n-1}([0, m], \mathbb{R}^k) \mid h(t) = \int_0^m G(t,s)f(s)\, ds, \text{ for a.a. } t \in [0, m] \right\}.$$

(c) In order to show that the operator N is a contraction, let us consider the Banach space $C^{n-1}([0, m], \mathbb{R}^k)$ endowed with the norm

$$|x|_{C^{n-1}} := \sup_{t \in [0,m]} \left\{ |x(t)| + |\dot{x}(t)| + \ldots + |x^{(n-1)}(t)| \right\},$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^k$. If $x, y \in C^{n-1}([0, m], \mathbb{R}^k)$ are arbitrary, then there exist $h_x \in Nx$, $h_y \in Ny$ and integrable selections (cf. Lemma 2.1) $f_x(\cdot)$ of $C^*(\cdot, x(\cdot), \dot{x}(\cdot), \dots, x^{(n-1)}(\cdot))$ and $f_y(\cdot)$ of $C^*(\cdot, y(\cdot), \dot{y}(\cdot), \dots, y^{(n-1)}(\cdot))$ such that

$$\begin{aligned} d_H(Nx, Ny) &= |h_x - h_y|_{C^{n-1}} = \left| \int_0^m G(t,s)f_x(s)\, ds - \int_0^m G(t,s)f_y(s)\, ds \right|_{C^{n-1}} \\ &= \sup_{t \in [0,m]} \left\{ \left| \int_0^m G(t,s)[f_x(s) - f_y(s)]\, ds \right| + \dots + \left| \int_0^m \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s)[f_x(s) - f_y(s)]\, ds \right| \right\} \\ &\le \sup_{t \in [0,m]} \int_0^m \left\{ |G(t,s)| + \left| \frac{\partial}{\partial t} G(t,s) \right| + \dots + \left| \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s) \right| \right\} \cdot |f_x(s) - f_y(s)|\, ds \\ &\le \sup_{t \in [0,m]} \left\{ |x(t) - y(t)| + \dots + |x^{n-1}(t) - y^{n-1}(t)| \right\} \end{aligned}$$

$$\times \sup_{(t,s)\in[0,m]\times[0,m]} \left\{ |G(t,s)| + \cdots + \left| \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s) \right| \right\} \cdot \int_0^m \alpha(t)\, dt$$
$$= \sup_{(t,s)\in[0,m]\times[0,m]} \left\{ |G(t,s)| + \left| \frac{\partial}{\partial t} G(t,s) \right| + \cdots + \left| \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s) \right| \right\}$$
$$\times \int_0^m \alpha(t)\, dt \cdot |x-y|_{C^{n-1}}.$$

If the integral $\int_0^m \alpha(t)\, dt$ is small enough, namely if

$$\mathfrak{L} := \sup_{(t,s)\in[0,m]\times[0,m]} \left\{ |G(t,s)| + \cdots + \left| \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s) \right| \right\} \cdot \int_0^m \alpha(t)\, dt < 1, \tag{3.14}$$

then the operator N is a desired contraction with a Lipschitz constant $\mathfrak{L} \in [0,1)$.

Finally, since N is a contraction with compact and convex values, the set $\mathrm{Fix}(N)$ is, according to Proposition 2.2, a nonempty, compact AR-space which completes the proof. □

Remark 3.4. It follows from the proof of Theorem 3.3 that the smallness of the integral $\int_0^m \alpha(t)\, dt$ in conditions (v) and (vi) is given by the identical inequalities (3.11) and (3.14), namely

$$\int_0^m \alpha(t)\, dt < \frac{1}{\sup_{(t,s)\in[0,m]\times[0,m]} \left\{ |G(t,s)| + \left| \frac{\partial}{\partial t} G(t,s) \right| + \cdots + \left| \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s) \right| \right\}}. \tag{3.15}$$

Remark 3.5. If the mapping $C(t,\cdot)$ is Lipschitzian with a sufficiently small constant L, i.e. if condition (v) takes the form

(v') there exists a sufficiently small constant $L \geq 0$ such that

$$d_H(C(t,x_1,\ldots,x_{n-1}), C(t,y_1,\ldots,y_{n-1})) \leq L \cdot (|x_1-y_1| + \cdots + |x_{n-1}-y_{n-1}|),$$

for a.a. $t \in [0,m]$ and all $x_i, y_i \in \mathbb{R}^k$, $i = 1,\ldots,n-1$, then the same conclusion holds, provided

$$L < \frac{1}{\sup_{t\in[0,m]} \int_0^m |G(t,s)| + \left| \frac{\partial}{\partial t} G(t,s) \right| + \cdots + \left| \frac{\partial^{n-1}}{\partial t^{n-1}} G(t,s) \right| \, ds}. \tag{3.16}$$

Remark 3.6. Theorem 3.3 reduces, for $n = 1$, to [11, Theorem 4] and, for $n = 2$, to [6, Lemma 3.2]. Moreover, unlike in [11], the smallness of $\int_0^m \alpha(t)\, dt$, resp. L, is expressed here explicitly in (3.15), (3.16) (see also (3.20), (3.21), (3.23) and (3.25) below).

Remark 3.7. For scalar ($k = 1$) problem (P_m), the topological structure of the set of solutions was studied in [38]. Our related conditions (i)–(vi) are, in this particular case, more explicit and our conclusion is more precise, because in [38] an R_δ-set was obtained.

Example 3.8. Consider the n-point vector interpolation b.v.p. in $\mathbb{R}^k$:

$$\left.\begin{array}{c} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) \in C(t,x(t),\ldots,x^{(n-1)}(t)), \\ \text{for a.a. } t \in [0,m], \\ x(t_1) = \cdots = x(t_n) = 0, \end{array}\right\} \tag{3.17}$$

where $0 = t_1 < \cdots < t_n = n$, $A_j \in C([0,m], \mathbb{R}^{k\times k})$, and the associated homogenous problem for $C \equiv 0$, i.e.

$$\left.\begin{aligned} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) = 0, \\ \text{for a.a. } t \in [0,m], \\ x(t_1) = \cdots = x(t_n) = 0. \end{aligned}\right\} \tag{3.18}$$

Setting $\alpha_{jrs} := \|a_{jrs}\| = \max_{t\in[0,m]} |a_{jrs}|$, where $A_j(\cdot) = [a_{jrs}(\cdot)]$, i.e. a_{jrs} are the entries of A_j, for $j = 1,\dots,n$, let us assume that

$$1 > \max_{r\in\{1,\dots,k\}} \left\{ \frac{m^n}{n!} \left[\frac{(n-1)^{n-1}}{n^n}\alpha_{nrr} + \sum_{s\neq r} \alpha_{nrs} \right] + \sum_{j=1}^{n-1} \frac{m^j}{j!} \sum_{l=1}^{k} \alpha_{jrl} \right\}. \tag{3.19}$$

Setting still $b_{rs} := \sum_{j=1}^{n} \alpha_{jrs} m^j$, the inequality (3.19) means that the spectral radius $\rho(B) < 1$, where $B = [b_{rs}]$.

Thus, under (3.19), homogenous problem (3.18) has according to [8, Theorem 1] only the trivial solution, i.e. condition (iii) is satisfied. Observe that for $A_j(t) \equiv 0$, $j = 1,\dots,n$, the inequality (3.19) holds trivially.

If $C : [0,m] \times \mathbb{R}^{kn} \multimap \mathbb{R}^k$ is an upper-Carathéodory as in (iv), then, after all, the set of solutions of (3.17) is, in view of Theorem 3.3, a nonempty, compact AR-space, provided (v) and (vi) hold with (3.15) or (3.16), provided $C(t,\cdot)$ is Lipschitzian.

For $A_j(t) \equiv 0$, $j = 1,\dots,n$, the associated Green function $G(t,s)$ in inequalities (3.15) and (3.16) takes the following scalar form:

$$G(t,s) = \frac{1}{(n-1)!} \left(\chi_{[0,t]}(s)(t-s)^{n-1} - \sum_{j=1}^{n} \chi_{[0,tj]}(s) \frac{\prod_{i=1,i\neq j}^{n}(t-t_i)}{\prod_{i=1,i\neq j}^{n}(tj-t_i)} (t_j - s)^{n-1} \right),$$

where $(t,s) \in [0,m]^2$ and $\chi_{[0,t]}$ stands for the characteristic function, i.e.

$$\chi_{[0,t]}(s) = \begin{cases} 1, & \text{for } s \in [0,t] \\ 0, & \text{for } s \notin [0,t]. \end{cases}$$

Thus,

$$\frac{\partial^j}{\partial t^j G(t,s)} \leq \frac{m^{n-j-1}}{(n-j-1)!} \left(1 + \frac{m^{n-1}n}{\triangle^{n-1}} \right),$$

for $j = 0,1,\dots,n-1$, where $(t,s) \in [0,m]^2$ and $\triangle = \min_{i\in\{1,\dots,n-1\}} |t_{i+1} - t_i|$.

Therefore, condition (3.15) can be still specified as follows:

$$\int_0^m \alpha(t)\,dt < \frac{1}{\left(1 + \frac{m^{n-1}n}{\triangle^{n-1}}\right) \sum_{j=0}^{n-1} \frac{m^{n-j-1}}{(n-j-1)!}}. \tag{3.20}$$

In particular, for the equidistant points $t_j = j = 1,\dots,n$, we obtain that

$$\int_0^m \alpha(t)\,dt < \frac{1+n^n}{\sum_{j=0}^{n-j} \frac{n^{n-j-1}}{(n-j-1)!}}. \tag{3.21}$$

For the properties of Green's functions of some further scalar boundary value problems, see e.g. [13,36] and the references therein. For vector problems like (3.17), where $A_j \neq 0$, $j = 1,\dots,n$, the calculations would be technically rather cumbersome, but still possible.

Remark 3.9. The topological structure of the set of solutions to another n-point ($n > 3$) b.v.p. was considered for the second-order differential inclusion in a separable Banach space in [15]. Let us note that such results in abstract spaces are quite rare.

Example 3.10. Let us consider the third-order three point b.v.p.

$$\left.\begin{aligned} x^{(3)}(t) \in C(t, x(t), \dot{x}(t), \ddot{x}(t)), &\quad \text{for a.a. } t \in [0,1], \\ \dot{x}(0) = x(1) = 0, &\quad \ddot{x}(\eta) + \beta x(0) = 0, \end{aligned}\right\} \tag{3.22}$$

where $\beta \in [0,2)$, $\eta \in \left[\frac{\sqrt{121+24\beta}-5}{3(4+\beta)}, 1\right)$, $C : [0,1] \times \mathbb{R}^{3n} \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping such that, for a.a. $t \in [0,1]$, and all $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}^n$,

$$d_H(C(t, x_1, x_2, x_3), C(t, y_1, y_2, y_3)) \le \alpha(t) \cdot (|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|)\,,$$

with $\alpha \in L^1([0,1], [0,\infty))$ satisfying

$$\int_0^1 \alpha(t)\, dt < \frac{4 - 2\beta}{22 - \beta}. \tag{3.23}$$

Moreover, let there exist $C_0 > 0$ such that

$$|C(t, 0, 0, 0)| \le C_0 \cdot \alpha(t), \quad \text{for a.a. } t \in [0,1]. \tag{3.24}$$

We will show that, under the above assumptions, the set of solutions of (3.22) is a non-empty, compact AR-space. The homogeneous problem associated to (3.22), i.e.

$$\left.\begin{aligned} x^{(3)}(t) = 0, &\quad \text{for a.a. } t \in [0,1], \\ \dot{x}(0) = x(1) = 0, &\quad \ddot{x}(\eta) + \beta x(0) = 0, \end{aligned}\right\}$$

has only the trivial solution and the related Green function G takes the form (cf. [32])

$$G(t,s,\eta) := \begin{cases} -\frac{(2-\beta t^2)(1-s)^2}{2(2-\beta)}, & 0 \le t \le s \le 1,\ s \ge \eta \\ -\frac{(2-\beta t^2)(1-s)^2}{2(2-\beta)} + \frac{1-t^2}{2-\beta}, & 0 \le t \le s \le 1,\ s < \eta \\ -\frac{(2-\beta t^2)(1-s)^2}{2(2-\beta)} + \frac{(t-s)^2}{2}, & 0 \le s \le t \le 1,\ s \ge \eta \\ -\frac{(2-\beta t^2)(1-s)^2}{2(2-\beta)} + \frac{(t-s)^2}{2} + \frac{1-t^2}{2-\beta}, & 0 \le s \le t \le 1,\ s < \eta. \end{cases}$$

By direct computation, we obtain that

$$\frac{\partial G(t,s,\eta)}{\partial t} = \begin{cases} \frac{\beta t(1-s)^2}{2-\beta}, & 0 \le t \le s \le 1,\ s \ge \eta \\ \frac{\beta t(1-s)^2}{2-\beta} + \frac{-2t}{2-\beta}, & 0 \le t \le s \le 1,\ s < \eta \\ \frac{\beta t(1-s)^2}{2-\beta} + (t-s), & 0 \le s \le t \le 1,\ s \ge \eta \\ \frac{\beta t(1-s)^2}{2-\beta} + (t-s) + \frac{-2t}{2-\beta}, & 0 \le s \le t \le 1,\ s < \eta, \end{cases}$$

and

$$\frac{\partial^2 G(t,s,\eta)}{\partial t^2} = \begin{cases} \frac{\beta(1-s)^2}{2-\beta}, & 0 \le t \le s \le 1,\ s \ge \eta \\ \frac{\beta(1-s)^2}{2-\beta} + \frac{-2}{2-\beta}, & 0 \le t \le s \le 1,\ s < \eta \\ \frac{\beta(1-s)^2}{2-\beta} + 1, & 0 \le s \le t \le 1,\ s \ge \eta \\ \frac{\beta(1-s)^2}{2-\beta} + 1 + \frac{-2}{2-\beta}, & 0 \le s \le t \le 1,\ s < \eta. \end{cases}$$

Since, for all $\eta \in \left[\frac{\sqrt{121+24\beta}-5}{3(4+\beta)}, 1\right)$,

$$\sup_{(t,s)\in[0,1]\times[0,1]} \left\{ |G(t,s,\eta)| + \left|\frac{\partial}{\partial t}G(t,s,\eta)\right| + \left|\frac{\partial^2}{\partial t^2}G(t,s,\eta)\right| \right\} \le \frac{22-\beta}{4-2\beta}, \tag{3.25}$$

condition (3.23) ensures that, the problem (3.22) is, according to Theorem 3.3 (cf. condition (3.15)), solvable with a compact AR-space of solutions.

4 Topological structure on non-compact intervals

One of the efficient methods which can be used for studying b.v.p.s on non-compact intervals is an *inverse limit method*. Let us recall that by the *inverse system*, we mean a family $\mathcal{S} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$, where Σ is a set directed by the relation $\le$, X_α is, for all $\alpha \in \Sigma$, a metric space and $\pi_\alpha^\beta : X_\beta \to X_\alpha$ is a continuous function, for all $\alpha, \beta \in \Sigma$ such that $\alpha \le \beta$. Moreover, $\pi_\alpha^\alpha = id_{X_\alpha}$ and $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$, for all $\alpha \le \beta \le \gamma$. The *limit of inverse system* $\mathcal{S}$ is denoted by $\varprojlim \mathcal{S}$ and it is defined by

$$\varprojlim \mathcal{S} := \left\{ (x_\alpha) \in \Pi_{\alpha\in\Sigma} X_\alpha \mid \pi_\alpha^\beta(x_\beta) = x_\alpha, \text{ for all } \alpha \le \beta \right\}.$$

If we denote by $\pi_\alpha : \varprojlim \mathcal{S} \to X_\alpha$ the restriction of the projection $p_\alpha : \Pi_{\alpha\in\Sigma} X_\alpha \to X_\alpha$ onto α-th axis, then it holds $\pi_\alpha = \pi_\alpha^\beta \pi_\beta$, for all $\alpha \le \beta$.

Let us now consider two inverse systems $\mathcal{S} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ and $\mathcal{S}' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}$. By a *multivalued mapping of the system $\mathcal{S}$ into the system $\mathcal{S}'$*, we mean a family $\{\sigma, \varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma : \Sigma' \to \Sigma$ and multivalued mappings $\varphi_{\sigma(\alpha')} : X_{\sigma(\alpha')} \multimap Y_{\alpha'}$ such that, for all $\alpha' \le \beta'$,

$$\pi_{\alpha'}^{\beta'} \varphi_{\sigma(\beta')} = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')}^{\sigma(\beta')}.$$

Mapping $\{\sigma, \varphi_{\sigma(\alpha')}\}$ induces a *limit mapping* $\varphi : \varprojlim \mathcal{S} \multimap \varprojlim \mathcal{S}'$ satisfying, for all $\alpha' \in \Sigma'$,

$$\pi_{\alpha'} \varphi = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')}.$$

We will make use of the following result. For more details about the inverse limit method, see, e.g., [3–5,23,27,34].

Proposition 4.1 (cf. [3,4,23]). *Let $\mathcal{S} = \{X_m, \pi_m^p, \mathbb{N}\}$ and $\mathcal{S}' = \{Y_m, \pi_m^p, \mathbb{N}\}$ be two inverse systems such that $X_m \subset Y_m$. If $\varphi : \varprojlim \mathcal{S} \multimap \varprojlim \mathcal{S}'$ is a limit map induced by a mapping $\{id, \varphi_m\}$, where $\varphi_m : X_m \multimap Y_m$, and if $Fix(\varphi_m)$ are, for all $m \in \mathbb{N}$, R_δ-sets, then the fixed point set $Fix(\varphi)$ of φ is an R_δ-set, too.*

The following corollary is a direct consequence of Proposition 4.1.

Corollary 4.2. *Let us consider the sequence of b.v.p.s $\{(K_m)\}_{m=1}^\infty$, where*

$$\left.\begin{array}{c} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) \in C(t, x(t), \dots, x^{(n-1)}(t)), \\ \text{for a.a. } t \in [t_0, t_0+m], \\ x \in S_m \ (\text{resp. } (x, \dot{x}, \dots, x^{(n-1)}) \in S'_m), \end{array}\right\} \tag{K_m}$$

and let us assume that each problem (K_m), $m \in \mathbb{N}$, has an R_δ-set of solutions which correspond to fixed points of the associated integral operator. Moreover, let the boundary condition be such that, for all $m \in \mathbb{N}$, the following holds:

If $x : [t_0, t_0 + m] \to \mathbb{R}^k$ is a solution of problem (K_m) *then* $x|_{[t_0,t_0+m-1]} : [t_0, t_0 + m - 1] \to \mathbb{R}^k$ *is a solution of problem* (K_{m-1}).

Then the set of solutions of the problem

$$\left.\begin{array}{c} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) \in C(t, x(t), \ldots, x^{(n-1)}(t)), \\ \text{for a.a. } t \in [t_0, \infty), \\ x \in S \ (\text{resp. } (x, \dot{x}, \ldots, x^{(n-1)}) \in S'), \end{array}\right\} (K_\infty)$$

where S is such that if $x \in S$, then $x \in S_m$, for all $m \in \mathbb{N}$, (resp. if $(x, \dot{x}, \ldots, x^{(n-1)}) \in S'$, then $(x, \dot{x}, \ldots, x^{(n-1)}) \in S'_m$, for all $m \in \mathbb{N}$), is an R_δ-set.

Remark 4.3. The class of b.v.p. (K_∞) for which Corollary 4.2 applies seems to be rather narrow. On the other hand, one can easily interface problems (K_m) on compact intervals with suitable Cauchy initial value problems on infinite intervals $[m, \infty)$.

For each solution $x(\cdot)$ of (K_m) with $(\dot{x}(m), \ldots, x^{(n-1)}(m)) = (x_1, \ldots, x_{n-1})$, the initial value problems

$$\left.\begin{array}{c} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) \in C(t, x(t), \ldots, x^{(n-1)}(t)), \\ \text{for a.a. } t \in [m, m + \triangle], \ 0 < \triangle < \infty, \\ x(m) = 0, \ \dot{x}(m) = x_1, \ldots, x^{(n-1)}(m) = x_{n-1} \end{array}\right\}$$

have, according to [19], R_δ-sets of solutions. According to Corollary 4.2, the same problem has therefore an R_δ-set of solutions on $[m, \infty)$. Together, on $[0, \infty)$, we can have in this way a union of R_δ-sets of solutions of the matched problems (K_m) and the Cauchy problem on $[m, \infty)$.

As an illustration, we can give the following simple example.

Example 4.4. Consider the problem

$$\left.\begin{array}{c} x^{(3)}(t) \in C(t, x(t), \dot{x}(t), \ddot{x}(t)), \quad \text{for a.a. } t \in [0, \infty), \\ \dot{x}(0) = x(1) = 0, \quad \ddot{x}(\eta) + \beta x(0) = 0, \end{array}\right\} \tag{4.1}$$

where $\beta \in [0, 2)$, $\eta \in \left[\frac{\sqrt{121+24\beta}-5}{3(4+\beta)}, 1\right)$, $C : [0, \infty) \times \mathbb{R}^{3n} \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping such that, for a.a. $t \in [0, 1]$, and all $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}^n$,

$$d_H(C(t, x_1, x_2, x_3), C(t, y_1, y_2, y_3)) \leq \alpha(t) \cdot (|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|),$$

with $\alpha \in L^1([0, 1], [0, \infty))$ satisfying

$$\int_0^1 \alpha(t)\, dt < \frac{4 - 2\beta}{22 - \beta}.$$

Moreover, let there exist $C_0 > 0$ such that

$$|C(t, 0, 0, 0)| \leq C_0 \cdot \alpha(t), \quad \text{for a.a. } t \in [0, 1].$$

We will show that, under the above assumptions, the set of solutions of (4.1) can be expressed as a special union of R_δ-sets.

In order to solve (4.1), we will consider separately the b.v.p.

$$\left.\begin{array}{c} x^{(3)}(t) \in C(t, x(t), \dot{x}(t), \ddot{x}(t)), \quad \text{for a.a. } t \in [0, 1], \\ \dot{x}(0) = x(1) = 0, \quad \ddot{x}(\eta) + \beta x(0) = 0, \end{array}\right\} \tag{4.2}$$

and the Cauchy (initial value) problem

$$\left.\begin{array}{c} x^{(3)}(t) \in C(t, x(t), \dot{x}(t), \ddot{x}(t)), \quad \text{for a.a. } t \in [1, \infty), \\ x(1) = 0, \quad \dot{x}(1) = x_1, \quad \ddot{x}(1) = x_2. \end{array}\right\} \tag{4.3}$$

By means of Theorem 3.3, the b.v.p. (4.2) is solvable with an R_δ-set of solutions (cf. Example 3.10). In fact, the set of solutions of (4.2) is, according to Theorem 3.3, a nonempty, compact AR-space.

Let $x(\cdot)$ be a solution of the b.v.p. (4.2) and let us put $x_1 := \dot{x}(1)$ and $x_2 := \dot{x}(2)$. Now, let us consider, for these interface values of the derivatives, the problem (4.3). The Cauchy problem, considered on an arbitrary compact interval $[1, m]$, $m \in \mathbb{N}$, has an R_δ-set of solutions (cf. [19]). Using the inverse limit method, we can conclude that, for the fixed $x_1 = \dot{x}(1)$ and $x_2 := \dot{x}(2)$, the Cauchy problem (4.3) has, according to Corollary 4.2, an R_δ-set of solutions on $[1, \infty)$ which, in particular, implies that the related solution set is nonempty. If we denote by $x_D : [0, 1] \to \mathbb{R}^n$ the solution of the b.v.p. (4.2) satisfying $\dot{x}_D(1) = x_1$, $\ddot{x}_D(1) = x_2$ and by $x_H : [1, \infty] \to \mathbb{R}^n$ the solution of the Cauchy problem (4.3), then

$$x(t) := \begin{cases} x_D(t), & \text{for all } t \in [0, 1], \\ x_H(t), & \text{for all } t \in [1, \infty), \end{cases}$$

is the solution of the original problem (4.1).

Although the solution set of each separate problem was proved to be an R_δ-set, the solution set of the whole problem can be more complex. Nevertheless, if, for instance, the problem (4.2) is uniquely solvable, then the solution set of the whole problem is an R_δ-set, too.

Remark 4.5. Quite analogously, we can match problem (3.17) in Example 3.8 with an appropriate Cauchy initial value problem on $[m, \infty)$, to get a union of R_δ-sets of solutions of the interface problem on $[0, \infty)$.

Combining Corollary 4.2 with Proposition 3.1, we obtain immediately the following result which we state here already in the form of theorem.

Theorem 4.6. *Let us consider the linear problems on compact intervals* (3.1) *and* (3.2) *together with the asymptotic problems*

$$\left.\begin{array}{c} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) \in C(t), \text{ for a.a. } t \in [0, \infty), \\ x \in S, \end{array}\right\} \tag{4.4}$$

$$\left.\begin{array}{c} x^{(n)}(t) + A_1(t)x^{(n-1)}(t) + \cdots + A_n(t)x(t) \in C(t), \text{ for a.a. } t \in [0, \infty), \\ (x, \dot{x}, \ldots, x^{(n-1)}) \in S', \end{array}\right\} \tag{4.5}$$

where

(*i*) *$A_i : [0, \infty) \to \mathbb{R}^{k \times k}, i = 1, \ldots, n$, are locally integrable matrix functions such that $|A_i(t)| \leq a_i(t)$, for all $i = 1, \ldots, n$, a.a. $t \in [0, \infty)$ and suitable nonnegative functions $a_i \in L^1_{\text{loc}}([0, \infty), \mathbb{R})$,*

(*ii*) *S is a closed, convex subset of $AC^{n-1}_{loc}([0, \infty), \mathbb{R}^k)$, S_m are, for all $m \in \mathbb{N}$, closed, convex subsets of $AC^{n-1}([0, m], \mathbb{R}^k)$*
(S' is a closed, convex subset of $AC^{n-1}_{\text{loc}}([0, \infty), \mathbb{R}^k) \times \cdots \times AC_{loc}([0, \infty), \mathbb{R}^k)$ and S'_m are, for all $m \in \mathbb{N}$, closed, convex subset of $AC^{n-1}([0, m], \mathbb{R}^k) \times \cdots \times AC([0, m], \mathbb{R}^k)$) ,

(*iii*) $C : [0,\infty) \multimap \mathbb{R}^k$ *is a locally integrable mapping with convex, closed values such that* $|C(t)| \le c(t)$, *for a.a.* $t \in [0,\infty)$ *and a suitable nonnegative function* $c \in L^1_{loc}([0,\infty),\mathbb{R})$,

(*iv*) *there exists* $t_0 \in [0,\infty)$ *such that, for all* $m \in \mathbb{N}$, *we are able to find a constant* M_m *such that* $|x(t_0)| \le M_m$, $|\dot{x}(t_0)| \le M_m, \dots, |x^{(n-1)}(t_0)| \le M_m$, *for all solutions* $x(\cdot)$ *of problem* (3.1) (*all solutions* $x(\cdot)$ *of problem* (3.2)),

(*v*) S *is such that if* $x \in S$, *then* $x \in S_m$, *for all* $m \in \mathbb{N}$, *(resp. if* $(x,\dot{x},\dots,x^{(n-1)}) \in S'$, *then* $(x,\dot{x},\dots,x^{(n-1)}) \in S'_m$, *for all* $m \in \mathbb{N}$*).*

Moreover, let, for all $m \in \mathbb{N}$, *the set of solutions of* (3.1) (*the set of solutions of* (3.2) *and their derivatives up to the* $(n-1)$*-st order) is nonempty and corresponds to fixed points of the associated integral operator. Furthermore, let the boundary condition be such that, for all* $m \in \mathbb{N}$, *the following holds:*

If $x : [0,m] \to \mathbb{R}^k$ *belongs to* S_m, *then* $x|_{[0,m-1]} : [0,m-1] \to \mathbb{R}^k$ *belongs to* S_{m-1}.

(*If*

$$(x,\dot{x},\dots,x^{(n-1)}) : [0,m] \times [0,m] \times \dots \times [0,m] \to \mathbb{R}^{kn}$$

belongs to S'_m, *then*

$$(x|_{[0,m-1]}, \dot{x}|_{[0,m-1]}, \dots, x^{(n-1)}|_{[0,m-1]}) : [0,m-1] \times [0,m-1] \times \dots \times [0,m-1] \to \mathbb{R}^{kn}$$

belongs to S'_{m-1}.)

Then the set of solutions of the problem (4.4) *(the set of solutions of the problem* (4.5) *and their derivatives up to the* $(n-1)$*-st order) is an* R_δ*-set.*

Remark 4.7. The statement of Theorem 4.6 holds if we consider multivalued matrix mappings A_i. More concretely, let us consider the linear problems on compact intervals (3.8) and (3.9), together with the asymptotic problems

$$\left.\begin{aligned} x^{(n)}(t) \in C(t) - A_1(t)x^{(n-1)}(t) - \dots - A_n(t)x(t), \quad & \text{for a.a. } t \in [0,\infty), \\ x \in S_m, \quad & \end{aligned}\right\} \tag{4.6}$$

$$\left.\begin{aligned} x^{(n)}(t) \in C(t) - A_1(t)x^{(n-1)}(t) - \dots - A_n(t)x(t), \quad & \text{for a.a. } t \in [0,\infty), \\ (x,\dot{x},\dots,x^{(n-1)}) \in S'_m, \quad & \end{aligned}\right\} \tag{4.7}$$

where $A_i : [0,\infty) \multimap \mathbb{R}^{k\times k}$ are locally integrable multivalued matrix mappings such that $|A_i(t)| \le a_i(t)$, for a.a. $t \in [0,\infty)$ and suitable nonnegative functions $a_i \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R})$, for all $i = 1,\dots,n$.

Moreover, let all the conditions from Theorem 4.6, but with (i) replaced by the above assumption, hold. Then the set of solutions of problem (4.6) (the set of solutions of problem (4.7) and their derivatives up to the $(n-1)$-st order) is an R_δ-set.

As an application of Theorem 4.6 and Remark 4.7, let us study the n-th order asymptotic (Kneser-type) b.v.p.

Example 4.8. Let us consider the asymptotic (Kneser-type) b.v.p.

$$\left.\begin{aligned} x^{(n)}(t) \in -A_1(t)x^{(n-1)}(t) - \dots - A_n(t)x(t), \quad & \text{for a.a. } t \in [a,\infty), \\ (-1)^i x^{(i)}(t) \ge 0, \quad \text{for all } t \in [a,\infty),\ i = 0,\dots n-1, & \end{aligned}\right\} \tag{4.8}$$

where

(i) $a \in (0, \infty)$,

(ii) $A_i : [a, \infty) \multimap \mathbb{R}$ are locally integrable multivalued mappings such that $|A_i(t)| \le \alpha_i(t)$, for a.a. $t \in [a, \infty)$ and suitable nonnegative functions $\alpha_i \in L^1_{\mathrm{loc}}([a, \infty), \mathbb{R})$, for all $i = 1, \ldots, n$,

(iii) there exists $r \in (0, \infty)$ such that

$$(-1)^n(-a_1(t)x_n - \cdots - a_n(t)x_1) \ge 0,$$

for all $t \ge a$, all measurable selections a_i of A_i, $i = 1, \ldots, n$, and all x_i satisfying

$$0 \le (-1)^{i-1}x_i \le rt^{1-i}, \qquad i = 1, \ldots, n,$$

(iv) $0 \notin A_n(t)$, for all t in a right neighbourhood of a.

Moreover, let us define

$$f^*(t) := \max\{|-A_1(t)x_n - \cdots - A_n(t)x_1| \;:\; 0 \le (-1)^{i-1}x_i \le rt^{1-i},\ i = 1, \ldots, n\} \tag{4.9}$$

and choose $\delta \in \left(0, \frac{1}{a+1}\right)$ so small that

$$2(a+1)^{n-1}\int_a^{a+\delta} f^*(\tau)\, d\tau \le r.$$

Together with (4.8), let us consider the initial condition

$$x(a) = c_0, \tag{4.10}$$

where

$$c_0 \in \left(0, \left(\frac{\delta}{a+\delta}\right)^{n-1}\frac{r}{2n!}\right). \tag{4.11}$$

Let us show that, under the above assumptions, the set of solutions of (4.8), (4.10) and their derivatives up to the $(n-1)$-st order) is an R_δ-set.

For this goal, together with the b.v.p. (4.8), (4.10), let us consider the family of associated problems on compact intervals

$$\left.\begin{array}{c} x^{(n)}(t) \in -A_1(t)x^{(n-1)}(t) - \cdots - A_n(t)x(t), \quad \text{for a.a. } t \in [a, m], \\ x(a) = c_0 \\ (-1)^i x^{(i)}(t) \ge 0, \quad \text{for all } t \in [a, m],\ i = 0, \ldots n-1, \end{array}\right\} \tag{P_m}$$

where $m \in \mathbb{N}$, $m > a$.

Let v_i be a measurable selection of A_i, $i = 1, \ldots, n$. It was shown in [16] (see Lemma 2.1 in [16] and the remarks below) that, under the above assumptions imposed on A_i, the following two norms in $AC^{n-1}([a, m], \mathbb{R})$, where $m > a$ is arbitrary, are equivalent:

$$\|x\| := \sup_{t\in[a,m]}|x(t)| + \sup_{t\in[a,m]}|\dot{x}(t)| + \cdots + \sup_{t\in[a,m]}|x^{(n-1)}(t)| + \int_a^m |x^{(n)}(t)|\, dt,$$

$$\|x\|_* := \sup_{t\in[a,m]}|x(t)| + \int_a^m \left|x^{(n)}(t) + v_n(t)x(t) + v_{n-1}(t)\dot{x}(t) + \cdots + v_1(t)x^{(n-1)}(t)\right|\, dt.$$

If $x(\cdot)$ is a solution of the b.v.p. (P_m), for some $m \in \mathbb{N}$, $m > a$, then

$$\|x\|_* = c_0.$$

Condition (iv) from Theorem 4.6 is therefore satisfied, because $\sup_{t\in[a,m]} |x^{(i)}(t)| \le \|x\|$, $i = 1, \dots, n-1$, and since the norms $\|x\|_*$ and $\|x\|$ are equivalent.

Moreover, since the sets

$$\begin{aligned} S'_m &:= \{(x, \dot{x}, \ddot{x}, \dots x^{(n-1)}) \in AC^{n-1}([a,m], \mathbb{R}) \times \dots \times AC([a,m], \mathbb{R}),\ x(0) = c_0, \\ &\quad (-1)^i x^{(i)}(t) \ge 0, \quad \text{for all } t \in [a,m],\ i = 0, \dots, n-1\}, \\ S' &:= \{(x, \dot{x}, \ddot{x}, \dots, x^{(n-1)}) \in AC^{(n-1)}_{\mathrm{loc}}([a,\infty), \mathbb{R}) \times \dots \times AC_{\mathrm{loc}}([a,\infty), \mathbb{R}), \\ &\quad x(0) = c_0,\ (-1)^i x^{(i)}(t) \ge 0, \quad \text{for all } t \in [a,\infty),\ i = 0, \dots, n-1\} \end{aligned}$$

are closed and convex, the b.v.p.s (4.8), (4.10), (P_m) satisfy the assumptions (i)–(v) of Theorem 4.6.

The non-emptiness of the set of solutions of (P_m) follows from Theorem 13.1 in [29] and the fact that A_i, $i = 1, \dots, n$, admit (according to Lemma 2.1) single-valued measurable selections v_i, $i = 1, \dots, n$.

If we denote by $P(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t)) := -A_1(t)x^{(n-1)}(t) - \dots - A_n(t)x(t)$, then $x(\cdot)$ is a solution of (P_m) if and only if, for a.a. $t \in [a,m]$,

$$x(t) \in x(u) - |x(u)| + c_0 + \dot{x}(a) \cdot t + \dots + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} P(s, x(s), \dots, x^{(n-1)}(s))\, ds, \tag{4.12}$$

$$\dot{x}(t) \in \dot{x}(u) + |\dot{x}(u)| + \dot{x}(a) + \dots + \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} P(s, x(s), \dots, x^{(n-1)}(s))\, ds, \tag{4.13}$$

$$\vdots$$

$$x^{(n-1)}(t) \in x^{(n-1)}(u) \pm |x^{(n-1)}(u)| + x^{(n-1)}(a) + \int_a^t P(s, x(s), \dots, x^{(n-1)}(s))\, ds, \tag{4.14}$$

for each $u \in [a,m]$, provided

$$0 \notin A_n(t), \tag{4.15}$$

on a subset of $[a,m]$ with a nonzero measure.

More concretely, since the constraint in (P_m) can be equivalently expressed as

$$\left.\begin{array}{c} x(0) = c_0, \\ x(u) - |x(u)| = 0,\ \dot{x}(u) + |\dot{x}(u)| = 0, \dots, x^{(n-1)}(u) \pm |x^{(n-1)}(u)| = 0, \\ \text{for all } u \in [a,m], \end{array}\right\} \tag{4.16}$$

every solution $x(\cdot)$ of (P_m) and its derivatives $\dot{x}(\cdot), \dots, x^{(n-1)}(\cdot)$ obviously satisfy (4.12)–(4.14). Reversely, derivating (4.14), we obtain

$$x^{(n)}(t) \in P(t, x(t), \dot{x}(t), \dots, x^{(n-1)}(t)).$$

Moreover, $x(a) \in x(u) - |x(u)| + c_0$, $\dot{x}(a) \in \dot{x}(u) + |\dot{x}(u)| + \dot{x}(a)$, $\ddot{x}(a) \in \ddot{x}(u) - |\ddot{x}(u)| + \ddot{x}(a), \dots, x^{(n-1)}(a) \in x^{(n-1)}(u) \pm |x^{(n-1)}(u)| + x^{(n-1)}(a)$, for each $u \in [a,m]$, i.e. $\dot{x}(u) + |\dot{x}(u)| =$

0, $\ddot{x}(u) - |\ddot{x}(u)| = 0, \dots, x^{(n-1)}(u) \pm |x^{(n-1)}(u)| = 0$ and, in particular, for $u = a$, $|x(a)| = c_0$. Thus, for $x(a) = c_0$, we also have $x(u) - |x(u)| = 0$, by which (4.16) (i.e. the constraint in (P_m)) is satisfied. On the other hand, if $x(a) = -c_0$, we arrive at $x(u) - |x(u)| = -2c_0$, i.e. $x(u) = -c_0$, for all $u \in [a, m]$, and subsequently $0 \in A_n(t)$, for a.a. $t \in [a, m]$, which is a contradiction with (4.15).

The set of solutions of (P_m) and their derivatives is a fixed point set of the map $\varphi_m : C^{n-1}([a,m], \mathbb{R}) \times \cdots \times C^1([a,m], \mathbb{R}) \times C([a,m], \mathbb{R}) \multimap C^{n-1}([a,m], \mathbb{R}) \times \cdots \times C^1([a,m], \mathbb{R}) \times C([a,m], \mathbb{R})$, where, for all $t \in [a, m]$,

$$
\begin{aligned}
&\varphi_m(x, \dots, x^{(n-1)})(t) \\
&\quad := \Bigg\{ \Bigg(\bigcup_{u \in [a,m]} x(u) - |x(u)| + c_0 + \ldots + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\, ds, \\
&\qquad \bigcup_{u \in [a,m]} \dot{x}(u) + |\dot{x}(u)| + \dot{x}(a) + \ddot{x}(a) \cdot t + \ldots + \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} f(s)\, ds, \dots, \\
&\qquad \bigcup_{u \in [a,m]} x^{(n-1)}(u) \pm |x^{(n-1)}(u)| + x^{(n-1)}(a) + \int_a^t f(s)\, ds \Bigg) \mid f \in L^1([a,m], \mathbb{R}) \text{ and} \\
&\qquad f(s) \in P(t, x(s), \dot{x}(s), \dots, x^{(n-1)}(s)), \text{ for a.a. } s \in [a, m] \Bigg\}.
\end{aligned}
$$

It can be easily seen that $\{\varphi_m\}_{m=1}^{\infty}$ is a map of the inverse system

$$
\{C^{n-1}([a,m], \mathbb{R}) \times \ldots C^1([a,m], \mathbb{R}) \times C([a,m], \mathbb{R}), \pi_m^p, \mathbb{N}\}
$$

into itself, where, for all $p \geq m$, $x \in C^{n-1}([a,p], \mathbb{R}) \times \cdots \times C^1([a,p], \mathbb{R}) \times C([a,p], \mathbb{R})$, $\pi_m^p(x, \dot{x}, \dots, x^{(n-1)}) = (x|_{[a,m]}, \dot{x}|_{[a,m]}, \dots, x^{(n-1)}|_{[a,m]})$.

Mappings $\{\varphi_m\}_{m=1}^{\infty}$ induce the limit mapping $\varphi : C^{n-1}([a,\infty), \mathbb{R}) \times \cdots \times C^1([a,\infty), \mathbb{R}) \times C([a,\infty), \mathbb{R}) \multimap C^{n-1}([a,\infty), \mathbb{R}) \times \cdots \times C^1([a,\infty), \mathbb{R}) \times C([a,\infty), \mathbb{R})$, where, for all $t \geq a$,

$$
\begin{aligned}
&\varphi(x, \dots, x^{(n-1)})(t) \\
&\quad := \Bigg\{ \Bigg(\bigcup_{u \in [a,\infty)} x(u) - |x(u)| + c_0 + \cdots + \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\, ds, \\
&\qquad \bigcup_{u \in [a,\infty)} \dot{x}(u) + |\dot{x}(u)| + \dot{x}(a) + \ddot{x}(a) \cdot t + \cdots + \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} f(s)\, ds, \dots, \\
&\qquad \bigcup_{u \in [a,\infty)} x^{(n-1)}(u) \pm |x^{(n-1)}(u)| + x^{(n-1)}(a) + \int_a^t f(s)\, ds \Bigg) \mid f \in L^1_{\mathrm{loc}}([a,\infty), \mathbb{R}) \text{ and} \\
&\qquad f(s) \in P(t, x(s), \dot{x}(s), \dots, x^{(n-1)}(s)), \text{ for a.a. } s \in [0, \infty) \Bigg\}.
\end{aligned}
$$

The fixed point set of the mapping φ is the set of solutions and their derivatives of the problem (4.8), (4.10). Applying Theorem 4.6 (cf. Remark 4.7), the set of solutions and their derivatives of the original problem (4.8), (4.10) is therefore an R_δ-set, as claimed.

The second illustrative example of the application of Theorem 4.6 is as follows.

Example 4.9. Consider the asymptotic problem

$$\left.\begin{array}{c} x^{(n)}(t) \in C(t), \quad \text{for a.a. } t \in [0,\infty), \\ x^{(i)}(0) = A_i, \quad i = 0,1,\dots,n-3, \\ |x_j^{(n-2)}(0)| = a x_j^{(n-1)}(0), \quad j = 1,\dots,k, \\ x_j^{(n-1)}(t) \le b_j, \quad j = 1,\dots,k, \text{ for all } t \in [0,\infty), \end{array}\right\} \tag{4.17}$$

where $x = (x_1,\dots x_k)$, $a > 0, A_i \in \mathbb{R}^k, i = 1,\dots,n-3$, $B = (b_1,\dots,b_k) \in [0,\infty)^k$, and $C = (c_1,\dots,c_k) : [0,\infty) \multimap [0,\infty)^k$ is an (Aumann-like) integrable mapping with convex, closed values such that $c_j(t) \le \gamma_j(t)$, for a.a. $t \in [0,\infty)$ and suitable (non-negative) functions $\gamma_j \in L^1([0,\infty),\mathbb{R})$, $j = 1,\dots,k$, such that

$$\int_0^\infty \gamma_j(t)\, dt \le b_j, \qquad j = 1,\dots,k. \tag{4.18}$$

Observe that every solution $x(\cdot)$ of (4.17) as well as $(4.17)_m$, where $(4.17)_m$ denotes the restriction of (4.17) to the interval $[0,m]$, $m \in \mathbb{N}$, must satisfy

$$x_j^{(n-1)}(t) = x_j^{(n-1)}(0) + \int_0^t c_j^{\text{sel}}(s)\, ds, \qquad j = 1,\dots,k, \tag{4.19}$$

where $C^{\text{sel}} = (c_1^{\text{sel}},\dots,c_k^{\text{sel}}) \subset C : [0,\infty) \to [0,\infty)^k$ is a measurable selection of C which exists, according to the well known Kuratowski–Ryll–Nardzewski selection theorem (see e.g. [5, Theorem I.3.49]).

Thus, in view of (4.18) and (4.19), both problems (4.17) as well as $(4.17)_m$ are solvable, at least with $x^{(n-1)}(0) = 0$ ($\Rightarrow x^{(n-2)}(0) = 0$). In particular, for $C(t) \equiv 0$, we get $x^{(n-1)}(t) \equiv x^{(n-1)}(0) = 0$.

Moreover, every solution $x(\cdot)$ of (4.17), resp. $(4.17)_m$, takes the form

$$x_j(t) = x_j^{\text{hom}}(t) + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} c_j^{\text{sel}}(t_{n-1})\, dt_{n-1} \dots dt_1\, dt, \qquad j = 1,\dots,k, \tag{4.20}$$

where $x_j^{\text{hom}}(t) = \sum_{j=0}^{n-1} \frac{A_j}{j!} t^j \pm \frac{aP}{(n-2)!} t^{n-2} + \frac{P}{(n-1)!} t^{n-1}$, $P = (p_1,\dots p_k) \in \mathbb{R}^k$ is a parameter such that $x^{(n-1)}(0) = P$ ($\Rightarrow 0 \le p_j \le b_j, j = 1,\dots,k$), and subsequently, in view of (4.19), $x_j^{(n-1)}(t) \ge 0$, $j = 1,\dots,k$, for all $t \in [0,\infty)$.

By the last two conditions in (4.17) (cf. (4.20)), $x(\cdot)$ also satisfies $x^{(n-2)}(0) = aP$, and so $x_j^{(n-2)}(0) \in \big[-\frac{b_j}{a}, \frac{b_j}{a}\big], j = 1,\dots,k$.

Hence, in order to apply Theorem 4.6, problems (4.17), and $(4.17)_m$ can be, under our assumptions, rewritten into the equivalent forms:

$$\left.\begin{array}{c} x^{(n)}(t) \in C(t), \quad \text{for a.a. } t \in [0,\infty), \\ x \in S := \big\{x \in AC_{\text{loc}}^{n-1}([0,\infty),\mathbb{R}^k) \mid x^{(i)}(0) = A_i, \quad i = 0,1,\dots,n-3, \\ x^{(n-2)}(0) = \pm a x^{(n-1)}(0),\ x^{(n-1)}(0) = P, \quad \text{where} \\ 0 \le p_j \le b_j - \int_0^\infty c_j^{\text{sel}}(s)\, ds, \quad j = 1,\dots,k\big\}, \end{array}\right\} \tag{4.21}$$

$$\left.\begin{array}{c} x^{(n)}(t) \in C(t), \quad \text{for a.a. } t \in [0,m], \\ x \in S_m := \big\{x \in AC^{n-1}([0,m],\mathbb{R}^k) \mid x^{(i)}(0) = A_i, \quad i = 0,1,\dots,n-3, \\ x^{(n-2)}(0) = \pm a x^{(n-1)}(0),\ x^{(n-1)}(0) = P, \quad \text{where} \\ 0 \le p_j \le b_j - \int_0^m c_j^{\text{sel}}(s)\, ds, \quad j = 1,\dots,k\big\}. \end{array}\right\} \tag*{$(4.21)_m$}$$

In this way, one can easily check that S is a closed, convex subset of $AC^{n-1}_{\mathrm{loc}}([0,\infty),\mathbb{R}^k)$ and S_m are, for all $m \in \mathbb{N}$, closed, convex subsets of $AC^{n-1}([0,m],\mathbb{R}^k)$, i.e. condition (ii) in Theorem 4.6 is satisfied.

Because of $x^{(i)}(0) = A_i$, $i = 0,1,\dots,n-3$, and $|x^{(n-2)}(0)| = ax^{(n-1)}(0) \leq b_j, j = 1,\dots,k$, condition (iv) also trivially holds.

As we have already pointed out, the set of solutions of (4.17), and subsequently under our assumptions of (4.21), is nonempty and corresponds to selections of the associated (Aumann-like) integral operator in (4.20).

Finally, one can readily check that if $x : [0,m] \to \mathbb{R}^k$ belongs to S_m, then $x|_{[0,m-1]} : [0,m-1] \to \mathbb{R}^k$ belongs to S_{m-1}, for all $m \in \mathbb{N}$.

Since the remaining conditions (i), (iii), (v) are trivially satisfied, Theorem 4.6 applies, and the set of solutions of the original problem (4.17) is an R_δ-set.

5 Application to existence results

Finally, we will show how the topological structure of (Schauder-like) parametrized systems can be used to the solvability of nontrivial asymptotic problems involving "unpleasant" nonlinearities, or so. For this goal, we will employ the following special case of our principle, developed recently in [6, Theorem 3.1 and Corollary 4.2].

Proposition 5.1. *Let us consider the b.v.p.*

$$\left.\begin{array}{c} x^{(n)}(t) \in C\left(t, x(t), \dots, x^{(n-1)}(t)\right), \text{ for a.a. } t \in J, \\ x \in S, \end{array}\right\} \tag{5.1}$$

where J is a given (possibly noncompact) interval, $C : J \times \mathbb{R}^{kn} \multimap \mathbb{R}^k$ is an upper-Carathéodory mapping and $S \subset AC^{n-1}_{loc}(J,\mathbb{R}^k)$.

Moreover, let $H : J \times \mathbb{R}^{2kn} \multimap \mathbb{R}^k$ be an upper-Carathéodory map such that

$$H(t, c_1, \dots, c_n, c_1, \dots, c_n) \subset C(t, c_1, \dots, c_n), \text{ for all } (t, c_1, \dots, c_n) \in J \times \mathbb{R}^{kn}. \tag{5.2}$$

Assume that

(i) *there exists a retract Q of $C^{n-1}(J,\mathbb{R}^k)$ such that the associated problem*

$$\left.\begin{array}{c} x^{(n)}(t) \in H\left(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t)\right), \text{ for a.a. } t \in J, \\ x \in S \cap Q \end{array}\right\} \tag{5.3}$$

is solvable with an R_δ-set of solutions, for each $q \in Q$,

(ii) *there exists a non-negative, locally integrable function $\alpha : J \to \mathbb{R}$ such that*

$$|H(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t))| \leq \alpha(t)\left(1 + |x(t)| + \dots + |x^{(n-1)}(t)|\right),$$

a.e. in J, for any $(q,x) \in \Gamma_{\mathfrak{T}}$, where $\mathfrak{T}$ denotes the multivalued map which assigns to any $q \in Q$ the set of solutions of (5.3),

(iii) *$\mathfrak{T}(Q) \subset Q$,*

(iv) *$\mathfrak{T}(Q)$ is bounded in $C(J,\mathbb{R}^k)$.*

Then problem (5.1) *admits a solution in* $S \cap Q$.

Finally, let us illustrate the application of Proposition 5.1, on the basis of the knowledge of the structure of the solution sets from the foregoing Section 4, in two examples.

Example 5.2. Let us consider the n-th order nonlinear (Kneser-type) asymptotic b.v.p.

$$\left.\begin{array}{c} x^{(n)}(t) \in -A_1(t,x(t),\dots,x^{(n-1)}(t))x^{(n-1)}(t) - \cdots - A_n(t,x(t),\dots,x^{(n-1)}(t))x(t), \\ \text{for a.a. } t \in [a,\infty), \\ x(a) = c_0, \\ (-1)^i x^{(i)}(t) \geq 0, \text{ for all } i = 0,\dots,n-1, \text{ and } t \in [a,\infty), \end{array}\right\} \tag{5.4}$$

where

- $a \in (0,\infty)$,
- $A_i : [a,\infty) \times \mathbb{R}^n \to \mathbb{R}$, $i = 1,\dots,n$, are upper-Carathéodory mappings with
$$|A_i(t,x_1,x_2,\dots,x_n)| \leq \beta(t)(1+|x_1|),$$
for all $(x_1,x_2,\dots,x_n) \in \mathbb{R}^n$ and $t \in [a,\infty)$, where $\beta \in L^1_{\text{loc}}([a,\infty),\mathbb{R})$,
- c_0 satisfies (4.11) with $f^*(\cdot)$ from (4.9) defined by
$$f^*(t) := \max\Big\{ | - A_1(t,x_1,\dots,x_n)x_n - \cdots - A_n(t,x_1,\dots,x_n)x_1 \mid : \\ 0 \leq (-1)^{i-1}x_i \leq rt^{1-i},\ i = 1,\dots,n \Big\}$$
- $0 \notin A_n(t,x_1,x_2,\dots,x_n)$, for all $(x_1,x_2,\dots,x_n) \in \mathbb{R}^n$ and for t in a right neighbourhood of a.

In order to apply Proposition 5.1, let us define the set of candidate solutions as follows

$$Q := \Big\{ x \in C^{n-1}([0,\infty),\mathbb{R}) \,|\, x(a) = c_0,\ (-1)^i x^{(i)}(t) \geq 0,\ \text{for all } i = 0,\dots n-1, \text{ and } t \in [a,\infty) \Big\}.$$

Let us still consider the associated problems

$$\left.\begin{array}{c} x^{(n)}(t) \in -A_1(t,q(t),\dots,q^{(n-1)}(t))x^{(n-1)}(t) - \cdots - A_n(t,q(t),\dots,q^{(n-1)}(t))x(t), \\ \text{for a.a. } t \in [a,\infty), \\ x(a) = c_0, \\ (-1)^i x^{(i)}(t) \geq 0, \quad \text{for all } i = 0,\dots n-1, \text{ and } t \in [a,\infty). \end{array}\right\} \tag{P_q}$$

Let us check that if there exists $r \in (0,\infty)$ such that, for all $q \in Q$,

$$(-1)^n(-a_1(t,q(t),\dots,q^{(n-1)}(t))x_n - \cdots - a_n(t,q(t),\dots,q^{(n-1)}(t))x_1) \geq 0,$$

for all $t \geq a$, all measurable selections a_i of A_i, $i = 1,\dots,n$, and all x_i satisfying

$$0 \leq (-1)^{i-1}x_i \leq rt^{1-i}, \qquad i = 1,\dots,n,$$

then the b.v.p. (5.4) has a solution.

More concretely, let us verify, that the b.v.p. (P_q) satisfies, for all $q \in Q$, all assumptions of Proposition 5.1.

ad (i) It can be proved exactly in the same way as in Example 4.8 that the b.v.p. (P_q) has, for each $q \in Q$, an R_δ-set of solutions.

ad (ii) Assumption (ii) follows immediately from the properties of mappings A_i, $i=1,\ldots,n$, and the definition of (P_q).

ad (iii) Since the set $S := Q$ is closed and each solution of the b.v.p. (P_q) belongs to Q, it holds that $\overline{\mathfrak{T}(Q)} \subset S$, where the map $\mathfrak{T}$ is the solution mapping that assigns to each $q \in Q$ the set of solutions of (P_q).

ad (iv) It follows directly from the boundary conditions that $\mathfrak{T}(Q)$ is bounded in $C([a,\infty),\mathbb{R})$.

Since all the assumptions of Proposition 5.1 are satisfied, the b.v.p. (5.4) admits a solution $x(\cdot)$ such that $0 \le x(t) \le c_0$, for all $t \in [a,\infty)$.

Remark 5.3. For single-valued maps $A_i, i = 1,\ldots,n$, the set of solutions to problem (5.4) was proved in [29, Theorem III.13.1] to be a continuum.

Now, the result in Example 4.9 will be applied, by means of Proposition 5.1, to the following existence problem

$$\left.\begin{aligned} x^{(n)}(t) \in C(t,x(t),\ldots,x^{(n-1)}(t)), &\quad \text{for a.a. } t \in [0,\infty),\\ x^{(i)}(0) = A_i, &\quad i = 0,1,\ldots,n-3,\\ |x_j^{(n-2)}(0)| = a x_j^{(n-1)}(0), &\quad j = 1,\ldots,k,\\ x_j^{(n-1)}(t) \le b_j, &\quad j = 1,\ldots,k, \text{ for all } t \in [0,\infty), \end{aligned}\right\} \tag{5.5}$$

where $C : [0,\infty) \times \mathbb{R}^{kn} \multimap \mathbb{R}^k$ is an upper-Carathéodory mapping and the other symbols in (5.5) have the same meaning as those in Example 4.9.

Example 5.4. Consider (5.5) and assume, additionally, that $C = (c_1,\ldots,c_k)$ satisfies

$$c_j(t, X_0, X_1 \ldots, X_n - 1) \le \gamma_j(t), \tag{5.6}$$

for a.a. $t \in [0,\infty)$, all $(X_0, X_1, \ldots, X_n - 1) \in \mathbb{R}^{kn}$, and suitable (non-negative) functions $\gamma_j \in L^1([0,\infty),\mathbb{R})$, $j = 1,\ldots,k$, such that

$$\int_0^\infty \gamma_j(t)\, dt \le b_j, \qquad j = 1,\ldots,k. \tag{5.7}$$

Then problem (5.5) admits a solution.

Taking $J = [0,\infty)$, $Q = C^{n-1}([0,\infty),\mathbb{R}^k)$ and S to be the same as in Example 4.9, we have obviously $S \cap Q = S$. Since $C(t,q(t),\ldots,q^{(n-1)}(t))$ is, under (5.6) and (5.7), (Aumann-like) integrable (see e.g. [9]) with convex, closed values, for every $q \in Q$, problem (5.3) with $H\big(t,x(t),\ldots,x^{(n-1)}(t),q(t),\ldots,q^{(n-1)}(t)\big) = C(t,q(t),\ldots,q^{(n-1)}(t))$ in Proposition 5.1 is, in view of the conclusions in Example 4.9, solvable with an R_δ-set of solutions, for each $q \in Q$.

The inequalities (5.6), (5.7) immediately imply the existence of a suitable $\alpha \in L^1_{\mathrm{loc}}([0,\infty),\mathbb{R})$ such that $\gamma_j(t) \le \alpha(t)$, $j = 1,\ldots,k$, a.e. in $[0,\infty)$ and, because of $Q = C^{n-1}([0,\infty),\mathbb{R}^k)$, (iii) must be also fulfilled (cf. the solution form (4.20) in Example 4.9).

Since condition (iv) trivially holds for the initial values $x_j(0)$, $j = 1,\ldots,k$, Proposition 5.1 applies, and subsequently (5.5) is solvable, as claimed.

Remark 5.5. One can easily check that the sole existence of a solution $x(\cdot)$ of problem (5.5) with initial conditions $x^{(i)}(0) = A_i$, $i = 0, 1, \ldots, n-3$, and $x^{(n-2)}(0) = x^{(n-1)}(0) = 0$ follows already from the analysis in Example 4.9.

On the other hand, problem (5.5) has in fact a one-parameter family of R_δ-sets of solutions which is a subset of a larger R_δ-set described below, because by means of the parametric transformation $x = X_0 \in \mathbb{R}^k, X_l = \dot{X}_l - A_{l-1},\ l = 1, \ldots, n-3,\ X_{n-2} = \dot{X}_{n-3} \mp aP,\ X_{n-1} = \dot{X}_{n-2} - P$, where $P \in \prod_{j=1}^k [0, D_j]$, it can be equivalently rewritten into the zero initial-value problem for the first-order system of inclusions

$$\left.\begin{array}{c} \dot{X}_0 = X_1 + A_1, \dot{X}_1 = X_2 + A_2, \ldots, \dot{X}_{n-4} = X_{n-3} + A_{n-3}, \\ \dot{X}_{n-3} = X_{n-2} \pm aP, \dot{X}_{n-2} = X_{n-1} + P, \dot{X}_{n-1} \in C_P(t, X_0, \ldots X_{n-1}), \\ \text{with } (X_0(0), \ldots, X_{n-1}(0)) = 0 \in \mathbb{R}^{kn}, \end{array}\right\} \tag{5.8}$$

where $C_P(t, X_0, \ldots X_{n-1}) = C(t, X_0 + A_0, \ldots, X_{n-3} + A_{n-3}, X_{n-2} \pm aP, X_{n-1} + P)$, $P \in \prod_{j=1}^k [0, D_j]$, and $D_j \geq 0$, $j = 1, \ldots, k$, are suitable constants.

It is well known that, according to [19], (5.8) has for each $P \in \prod_{j=1}^k [0, D_j]$ an R_δ-set of solutions on every compact subinterval of $[0, \infty)$ and, according to [5, Theorem III.2.12], even on the whole $[0, \infty)$. Moreover, the one-parameter family of right-hand sides of (5.8), i.e. $(X_1 + A_1, \ldots, X_{n-3} + A_{n-3}, X_{n-2} \pm aP, X_{n-1} + P, C_P)$, $P \in \prod_{j=1}^k [0, D_j]$ is obviously (as a whole) a multivalued selection of the right-hand side of the multivalued problem

$$\left.\begin{array}{c} \dot{X}_0 = X_1 + A_1, \dot{X}_1 = X_2 + A_2, \ldots, \dot{X}_{n-4} = X_{n-3} + A_{n-3}, \\ \dot{X}_{n-3} \in X_{n-2} + a\prod_{j=1}^k [-D_j, D_j], \dot{X}_{n-2} \in X_{n-1} + \prod_{j=1}^k [0, D_j], \\ \dot{X}_{n-1} \in C(t, X_0 + A_0, \ldots X_{n-3} + A_{n-3}, X_{n-2} + a\prod_{j=1}^k [-D_j, D_j], X_{n-1} + \prod_{j=1}^k [0, D_j]) \\ \text{with } (X_0(0), \ldots, X_{n-1}(0)) = 0 \in \mathbb{R}^{kn}. \end{array}\right\} \tag{5.9}$$

Problem (5.9) has, by the same standard reference sources, an R_δ-set of solutions on $[0, \infty)$, as well as on each compact subset of $[0, \infty)$.

It is therefore a question, whether the union of one-parameter family of R_δ-sets of solutions to (5.5) forms an R_δ-set itself.

Remark 5.6. Although all illustrative examples in this paper could take the form of theorems, we decided to reserve this form exclusively for those having the character of the general methods (see Theorem 3.3 and Theorem 4.6). For asymptotic b.v.p.s, Theorem 4.6 still allows us to fulfil the crucial condition (i) in Proposition 5.1, in order to solve problem (5.1).

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