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Positive solutions for a class of semipositone periodic boundary value problems via bifurcation theory

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Abstract. In this paper, we are concerned with the existence of positive solutions of nonlinear periodic boundary value problems like

$$\begin{aligned} &-u'' + q(x)u = \lambda f(x,u), \qquad x \in (0,2\pi), \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{aligned}$$

where $q \in C([0,2\pi],[0,\infty))$ with $q \not\equiv 0$, $f \in C([0,2\pi] \times \mathbb{R}^+, \mathbb{R})$, $\lambda > 0$ is the bifurcation parameter. By using bifurcation theory, we deal with both asymptotically linear, superlinear as well as sublinear problems and show that there exists a global branch of solutions emanating from infinity. Furthermore, we proved that for λ near the bifurcation value, solutions of large norm are indeed positive.

Keywords: semipositone, positive solutions, periodic problems, bifurcation.

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1 Introduction

The purpose of this article is to obtain some existence results for nonlinear periodic boundary value problems (PBVPs) like

$$\begin{aligned} &-u'' + q(x)u = \lambda f(x,u), \qquad x \in (0,2\pi), \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{aligned} \tag{1.1}$$

where $q \in C([0,2\pi],[0,\infty))$ with $q \not\equiv 0$, $\lambda > 0$ is the bifurcation parameter, $f \in C([0,2\pi] \times \mathbb{R}^+, \mathbb{R})$, where $\mathbb{R}^+ := [0,\infty)$. If $f(x,0) \geq 0$ then (1.1) is called a *positone* problem and has been investigated extensively; see [5,11,14,15,19,20] and the references therein.

In the present paper, we deal here with the so called *semipositone* (or *non-positone*) problem, when f is such that

(F1) $f(x,0) < 0, \quad \forall x \in [0,2\pi]$.

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Semipositone problems arise in many different areas of applied mathematics and physics, such as the buckling of mechanical systems, the design of suspension bridges, chemical reactions, and population models with harvesting effort; see [1,10,17].

Existence of positive solutions for nonlinear second order Dirichlet problems in semipositone case was initially studied by Castro and Shivaji in [4]. Henceforth, the existence, multiplicity, and the global behavior of positive solutions of nonlinear second order Dirichlet problems/Robin problems in the semipositone case have been extensively studied by using the method of lower and upper solutions, fixed point theorem in cones as well as the bifurcation theory, see [2,12,16,18] and the references therein.

For nonlinear periodic boundary value problem (1.1), the existence, multiplicity and global behavior of positive solutions have been investigated by several authors via fixed point theorem in cones and the bifurcation theory, one may see J. R. Graef et al. [11], P. J. Torres [19] and Ma et al. [15,20]. In particular, the authors of [15,20] showed that there exists an unbounded continuum $\mathcal{C}$ emanating from $(\mu_1, 0)$, consisting of positive solutions of (1.1) in the positone case, where μ_1 is the first positive eigenvalue of the linear problem corresponding to (1.1). However, in the semipositone case, (1.1) has no positive solutions for λ large. Let us point out that this is in contrast with the positone case.

It is the purpose of this paper to study the global behavior of positive solutions of (1.1) in semipositone case via bifurcation theory. We shall handle the semipositone problems in which nonlinearities are asymptotically linear, superlinear as well as sublinear at infinity.

After some notation and preliminaries listed in Section 2, we deal in Section 3 with asymptotically linear problems and use bifurcation theory to prove an existence result in the frame of semipositone problems. In Section 4 we discuss superlinear problems, we show that (1.1) possesses positive solutions for $0 < \lambda < \lambda_*$. Similar arguments can be used in the sublinear case, discussed in Section 5, to show that (1.1) has positive solutions provided λ is large enough.

2 Notation and preliminaries

We denote the usual norm in $L^r(0, 2\pi)$ by $\|\cdot\|_r$ and the inner product in $L^2(0, 2\pi)$ by $\langle\cdot,\cdot\rangle$. We will work in the Banach space $X = C[0, 2\pi]$ with the norm $\|u\| = \max_{x\in[0,2\pi]} |u(x)|$ or $Y = C^1[0, 2\pi]$ with the norm $\|u\|_1 = \max_{x\in[0,2\pi]} |u(x)| + \max_{x\in[0,2\pi]} |u'(x)|$.

Define the linear operator $L : D(L) \subset X \to X$

$$Lu = -u'' + q(x)u, \qquad u \in D(L)$$

with

$$D(L) = \{u \in C^2[0, 2\pi] | u(0) = u(2\pi),\ u'(0) = u'(2\pi)\}.$$

Then L is a closed operator with compact resolvent, and $0 \in \rho(L)$.

In order to study the semipositone problems (1.1) via bifurcation theory, we must consider the following eigenvalue problem

$$\begin{aligned} &-u'' + q(x)u(x) = \lambda B(x)u, \qquad x \in (0, 2\pi),\\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{aligned} \tag{2.1}$$

where $B(\cdot) \in C([0, 2\pi])$ with $B \not\equiv 0$. From [6], we know that (2.1) has an simply eigenvalue λ_1; ϕ_1 is the corresponding eigenfunction with $\phi_1 > 0$ and $\|\phi_1\| = 1$.

We denote by $G(x,s)$ the Green's function associated with the following problem

$$\begin{gathered} -u''+q(x)u=h(x), \qquad x\in(0,2\pi), \\ u(0)=u(2\pi), \qquad u'(0)=u'(2\pi). \end{gathered}$$

From the Theorem 2.5 of [3], we know that $G(x,s)>0$, $\forall x,s\in[0,2\pi]$ and the solution of the above problem is given by

$$u(x)=\int_0^{2\pi} G(x,s)h(s)ds. \tag{2.2}$$

Now, by the positivity of $G(x,s)$ and $h(s)$, we have that $u(x)>0$, $\forall x\in[0,2\pi]$.

Denote

$$m=\min_{0\le x,s\le 2\pi} G(x,s), \qquad M=\max_{0\le x,s\le 2\pi} G(x,s), \qquad \sigma=\frac{m}{M}. \tag{2.3}$$

Obviously, $0<m<M$, and $0<\sigma<1$.

Let $\mathcal{K}: X\to X$ denote the Green operator of L with periodic boundary conditions, i.e. $u=\mathcal{K}v$ if and only if

$$\begin{gathered} -u''+q(x)u=v, \qquad x\in(0,2\pi), \\ u(0)=u(2\pi), \qquad u'(0)=u'(2\pi). \end{gathered}$$

With the above notation, problem (1.1) is equivalent to

$$u-\lambda\mathcal{K}f(u)=0, \qquad u\in X. \tag{2.4}$$

Hereafter we will use the same symbol to denote both the function and the associated Nemitskii operator.

We say that λ_∞ is a *bifurcation from infinity* for (2.4) if there exist $\mu_n\to\lambda_\infty$ and $u_n\in X$, such that $u_n-\mu_n\mathcal{K}f(u_n)=0$ and $\|u_n\|\to\infty$. Extending the preceding definition, we will say that $\lambda_\infty=+\infty$ is a bifurcation from infinity for (2.4) if solutions (μ_n,u_n) of (2.4) exist with $\mu_n\to+\infty$ and $\|u_n\|\to\infty$. This is the case we will meet in Section 5.

In the following, we shall apply the Leray–Schauder degree theory, mainly to the mapping $\Phi_\lambda: X\to X$.

$$\Phi_\lambda(u)=u-\lambda\mathcal{K}f(u).$$

For $R>0$, let $B_R=\{u\in X:\|u\|<R\}$, let $\deg(\Phi_\lambda(u),B_R,0)$ denote the degree of Φ_λ on B_R with respect to 0 and let $i(T,U,X)$ is the fixed point index of T on U with respect to X.

3 Asymptotically linear problems

In this section, we suppose that $f\in C([0,2\pi]\times\mathbb{R}^+,\mathbb{R})$ satisfies (F1) and

(F2) there exists $m>0$ such that

$$\lim_{u\to+\infty}\frac{f(x,u)}{u}=m.$$

Let $\lambda_\infty=\frac{\lambda_1}{m}$ and define

$$a(x)=\liminf_{u\to+\infty}(f(x,u)-mu), \qquad A(x)=\limsup_{u\to+\infty}(f(x,u)-mu).$$

Our main result is the following.

Theorem 3.1. *Suppose that f satisfies (F1) and (F2). Then there exists $\epsilon > 0$ such that (1.1) has positive solutions provided either*

(i) $a > 0$ in $(0, 2\pi)$ and $\lambda \in [\lambda_\infty - \epsilon, \lambda_\infty)$; or

(ii) $A < 0$ in $(0, 2\pi)$ and $\lambda \in (\lambda_\infty, \lambda_\infty + \epsilon]$.

Remark 3.2. Note that in (F2), we can allow that m depends on x.

The proof of Theorem 3.1 will be carried out in several steps. First of all, we extend $f(x, \cdot)$ to all $\mathbb{R}$ by setting

$$F(x, u) = f(x, |u|).$$

Let $X = C[0, 2\pi]$ and set, for $u \in X$,

$$\Psi(\lambda, u) := u - \lambda \mathcal{K} F(u).$$

Obviously, any $u > 0$ such that $\Psi(\lambda, u) = 0$ is a positive solution of (1.1).

Next, we give two lemmas which will be used later.

Lemma 3.3. *For every compact interval $\Lambda \subset \mathbb{R}^+ \backslash \{\lambda_\infty\}$, there exists $r > 0$ such that $\Psi(\lambda, u) \neq 0$, $\forall\, \lambda \in \Lambda$, $\forall\, \|u\| \geq r$.*

Proof. Suppose to the contrary that there exist $\mu_n \in \Lambda$ and $u_n \in X$ with $\|u_n\| \to \infty$ $(n \to \infty)$ be such that

$$u_n = \mu_n \mathcal{K} F(u_n).$$

We may assume $\mu_n \to \mu > 0, \mu \neq \lambda_\infty$. Set $w_n := u_n \|u_n\|^{-1}$, we get

$$w_n = \mu_n \|u_n\|^{-1} \mathcal{K} F(u_n).$$

On the other hand, $\|u_n\|^{-1} F(u_n)$ is bounded in X, $\{w_n\}$ is a relatively compact set in X by the compactness of $\mathcal{K}$. Suppose $w_n \to w$ in X. Then $\|w\| = 1$ and satisfies

$$\begin{aligned} & -w'' + q(x) w = \mu m |w|, \qquad x \in (0, 2\pi), \\ & w(0) = w(2\pi), \qquad w'(0) = w'(2\pi). \end{aligned} \tag{3.1}$$

By (2.2), it is easy to see $w > 0$ in $[0, 2\pi]$. Since $\|w\| = 1$, we infer that $\mu m = \lambda_1$, namely $\mu = \lambda_\infty$. This is a contradiction. □

Lemma 3.4.

(i) Assume $a > 0$. Then the assertion of Lemma 3.3 holds with $\Lambda = [\lambda_\infty, \lambda]$, $\forall\, \lambda > \lambda_\infty$.

(ii) Assume $A < 0$. Then we can take $\Lambda = [0, \lambda_\infty]$ in Lemma 3.3.

Proof. We prove statement (i); (ii) follows similarly. By Lemma 3.3, the assertion holds for any interval $\Lambda_\epsilon = [\lambda_\infty + \epsilon, \beta]$, $\epsilon > 0$. Suppose now there exist sequences $\{u_n\}$ in X and $\{\lambda_n\}$ in $\mathbb{R}^+$ with $\|u_n\| \to \infty, \lambda_n \downarrow \lambda_\infty$, such that $\Psi(\lambda_n, u_n) = 0$ $\forall\, n$. Setting $w_n = \|u_n\|^{-1} u_n$, as in the proof of Lemma 3.3, we conclude that $w_n \to w$ in X with $w > 0$. Thus, there exists $\beta > 0$ such that $w = \beta\phi_1$. Then one has $u_n = \|u_n\| w_n \to +\infty$ for all $x \in [0, 2\pi]$ and $F(x, u_n) = f(x, u_n)$ for n large enough.

From $\Psi(\lambda_n, u_n) = 0$ it follows that

$$\lambda_1 \langle u_n, \phi_1 \rangle = \lambda_n \langle f(x, u_n) - m u_n, \phi_1 \rangle + \lambda_n m \langle u_n, \phi_1 \rangle.$$

Since $\lambda_n > \lambda_\infty$ and $\langle u_n, \phi_1 \rangle > 0$ for n large enough, we infer that $\langle f(x, u_n) - m u_n, \phi_1 \rangle < 0$ for n large enough and the Fatou lemma yields

$$0 \geq \liminf \langle f(x, u_n) - m u_n, \phi_1 \rangle \geq \langle a, \phi_1 \rangle,$$

a contradiction if $a > 0$. □

Lemma 3.5. *Let $k \in L^1(0, 2\pi)$ with $k \geq 0$, and let $u \in X$ satisfy*

$$\begin{aligned} &-u'' + q(x)u \geq -k(x), \quad \text{a.e. in } (0, 2\pi), \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \end{aligned}$$

Then

$$u(x) \geq \sigma \left(\|u\| - \left(\frac{1}{\sigma} + 1 \right) M \|k\|_1 \right), \qquad x \in [0, 2\pi],$$

where σ and M are from (2.3).

Proof. Let w_0 be the unique solution of the problem

$$\begin{aligned} &-w'' + q(x)w = -k(x), \quad \text{a.e. in } (0, 2\pi), \\ &w(0) = w(2\pi), \qquad w'(0) = w'(2\pi). \end{aligned}$$

Then

$$w_0(x) = -\int_0^{2\pi} G(x, s) k(s) ds.$$

Set $y = u - w_0$. Then

$$\begin{aligned} &-y'' + q(x)y \geq 0, \quad \text{a.e. in } (0, 2\pi), \\ &y(0) = y(2\pi), \qquad y'(0) = y'(2\pi), \end{aligned}$$

and accordingly

$$y(x) \geq \sigma \|y\|, \; x \in [0, 2\pi].$$

Since $w_0(x) = -\int_0^{2\pi} G(x, s) k(s) ds \geq -M||k||_1$. Thus

$$\begin{aligned} u(x) &= y(x) + w_0(x) \\ &\geq \sigma \|y\| - M\|k\|_1 \\ &= \sigma \|u - w_0\| - M\|k\|_1 \\ &\geq \sigma(\|u\| - M\|k\|_1) - M\|k\|_1 \\ &= \sigma \left(\|u\| - \left(\frac{1}{\sigma} + 1 \right) M\|k\|_1 \right). \end{aligned}$$

□

Lemma 3.6. *If $\lambda > \lambda_\infty$ there exists $r > 0$ such that*

$$\Psi(\lambda, u) \neq t\phi_1, \qquad \forall t \geq 0,\ \|u\| \geq r.$$

Proof. Let us assume that for some sequence $\{u_n\}$ in X with $\|u_n\| \to \infty$ and numbers $\tau_n \geq 0$, such that $\Psi(\lambda, u_n) = \tau_n\phi_1$. Then

$$Lu_n = \lambda F(x, u_n) + \tau_n\lambda_1\phi_1,$$

and since $F(x, u) \approx m|u|$ as $|u| \to \infty$, and $\tau_n\lambda_1\phi_1 \geq 0$ in $[0, 2\pi]$, by (2.2), we know that $u_n > 0$ in $[0, 2\pi]$.

Note that $u_n \in D(L)$ has a unique decomposition

$$u_n = v_n + s_n\phi_1, \tag{3.2}$$

where $s_n \in \mathbb{R}$ and $\langle v_n, \phi_1\rangle = 0$. Since $u_n > 0$, $\phi_1 > 0$ on $[0, 2\pi]$, we have from (3.2) that $s_n = \langle u_n, \phi_1\rangle\langle\phi_1, \phi_1\rangle^{-1} > 0$, $\forall n \in \mathbb{N}$.

Choose $\kappa > 0$ such that

$$\kappa < 1 - \frac{\lambda_\infty}{\lambda}.$$

By (F2), there exists $M_0 > 0$, such that

$$f(x, u) \geq (1 - \kappa)mu, \qquad \forall u > M_0,\ x \in [0, 2\pi].$$

From $\|u_n\| \to \infty$ and Lemma 3.5, we know that there exits $N^* > 0$, such that

$$u_n > M_0, \qquad \forall n \geq N^*,$$

and consequently

$$f(x, u_n) \geq (1 - \kappa)mu_n. \tag{3.3}$$

Applying (3.3), it follows that

$$\begin{aligned}
s_n\lambda_1\langle\phi_1, \phi_1\rangle &= \langle u_n, L\phi_1\rangle \\
&= \langle Lu_n, \phi_1\rangle \\
&= \lambda\langle F(x, u_n), \phi_1\rangle + \tau_n\lambda_1\langle\phi_1, \phi_1\rangle \\
&\geq \lambda\langle F(x, u_n), \phi_1\rangle \\
&\geq \lambda\langle(1 - \kappa)mu_n, \phi_1\rangle \\
&= \lambda(1 - \kappa)m\langle\phi_1, u_n\rangle \\
&= \lambda(1 - \kappa)ms_n\langle\phi_1, \phi_1\rangle.
\end{aligned}$$

Thus

$$\lambda_\infty \geq \lambda(1 - \kappa).$$

This is a contradiction. □

In order to investigate the bifurcation from infinity, we follow the standard pattern and perform the change of variable $z = u\|u\|^{-2}$ $(u \neq 0)$.

Letting

$$\Phi(\lambda, z) = \|u\|^{-2}\Psi(\lambda, u) = z - \lambda\|z\|^2\mathcal{K}F\left(\frac{z}{\|z\|^2}\right),$$

one has that λ_∞ is a bifurcation from infinity for (2.4) if and only if it is a bifurcation from the trivial solution $z = 0$ for $\Phi = 0$. From Lemmas 3.3 and 3.4 it follows by homotopy that

$$\begin{aligned} \deg(\Phi(\lambda,\cdot), B_{1/r}, 0) &= \deg(\Phi(0,\cdot), B_{1/r}, 0) \\ &= \deg(I, B_{1/r}, 0) = 1, \qquad \forall \lambda < \lambda_\infty. \end{aligned} \tag{3.4}$$

Similarly, by Lemma 3.6 one infers, for all $\tau \in [0,1]$ and for all $\lambda > \lambda_\infty$,

$$\begin{aligned} \deg(\Phi(\lambda,\cdot), B_{1/r}, 0) &= \deg(\Phi(0,\cdot) - \tau\phi, B_{1/r}, 0) \\ &= \deg(\Phi(0,\cdot) - \phi, B_{1/r}, 0) = 0. \end{aligned} \tag{3.5}$$

Let us set

$$\Sigma = \{(\lambda, u) \in \mathbb{R}^+ \times X : u \neq 0,\ \Psi(\lambda, u) = 0\}.$$

From (3.4) and (3.5) and the preceding discussion we deduce

Lemma 3.7. *λ_∞ is a bifurcation from infinity for* (2.4)*. More precisely there exists an unbounded closed connected set $\Sigma_\infty \subset \Sigma$ that bifurcates from infinity. Moreover, Σ_∞ bifurcates to the left (to the right) provided $a > 0$ (respectively $A < 0$).*

Proof of Theorem 3.1. By the above lemmas, it suffices to show that if $\mu_n \to \lambda_\infty$ and $\|u_n\| \to \infty$ then $u_n > 0$ in $[0, 2\pi]$ for n large enough. Setting $w_n = u_n\|u_n\|^{-1}$ and using the preceding arguments, we find that, up to subsequence, $w_n \to w$ in X, and $w = \beta\phi_1$, $\beta > 0$. Then, it follows that $u_n > 0$ in $[0, 2\pi]$, for n large enough. □

Remark 3.8. The proof of Theorem 3.1 actually shows that there exists $k > 0$ such that for all $(\lambda, u) \in \Sigma_\infty$ with $\|u\| \geq k$ one has that $u > 0$ in $[0, 2\pi]$. Thus such (λ, u) are solutions of (1.1).

Example 3.9. Let us consider the second-order periodic boundary value problem

$$\begin{aligned} &-u''(x) + q(x)u = \lambda f(x, u), \qquad x \in (0, 2\pi), \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{aligned} \tag{3.6}$$

where $q \in C([0, 2\pi], [0, \infty))$ with $q \not\equiv 0$, $f(x, u) = 10u + x\ln(1 + u) - x$, $\lambda > 0$ is a parameter.

Let λ_1 be the first positive eigenvalue corresponding to the linear problem

$$\begin{aligned} &-u''(x) + q(x)u = \lambda h(x)u, \qquad x \in (0, 2\pi), \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi), \end{aligned}$$

where $h(\cdot) \in C([0, 2\pi])$ with $h \not\equiv 0$. Let ϕ be the positive eigenfunction corresponding to λ_1. Next, we will check that all of conditions in Theorem 3.1 are fulfilled.

In fact,

$$f(x,0) = -x < 0, \qquad x \in (0,2\pi);$$
$$m = \lim_{u\to+\infty} \frac{f(x,u)}{u} = 10;$$
$$a(x) = \liminf_{u\to+\infty}(f(x,u) - mu) = \liminf_{u\to+\infty}(x\ln(1+u) - x) > 0, \qquad x \in (0,2\pi).$$

Notice that $\lambda_\infty = \frac{\lambda_1}{10}$. Thus, from Theorem 3.1, there exists $\epsilon > 0$, such that (3.6) has positive solutions provided $\lambda \in (\lambda_\infty - \epsilon, \lambda_\infty)$. Moreover, Lemma 3.7 guarantees that there exists an unbounded closed connected set $\Sigma_\infty \subset \Sigma$ that bifurcates from infinity. Moreover, Σ_∞ bifurcates to the left.

4 Superlinear problems

We will study the existence of positive solutions for problems (1.1) when $f(x,\cdot)$ is superlinear. Precisely, we suppose that $f \in C([0,2\pi] \times \mathbb{R}^+, \mathbb{R})$ satisfies (F1) and

(F3) there exists $b \in C[0,2\pi]$, $b > 0$, such that $\lim_{u\to\infty} u^{-p} f(x,u) = b$, uniformly in $x \in [0,2\pi]$ with $1 < p < \infty$.

Our main result is the following theorem.

Theorem 4.1. *Let $f \in C([0,2\pi] \times \mathbb{R}^+, \mathbb{R})$ satisfy (F1) and (F3). Then there exists $\lambda_* > 0$ such that* (1.1) *has positive solutions for all $0 < \lambda \le \lambda_*$. More precisely, there exists a connected set of positive solutions of* (1.1) *bifurcating from infinity at $\lambda_\infty = 0$.*

The following well-known result of the fixed point index is crucial in our arguments.

Lemma 4.2 ([8])**.** *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{v \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \ne x$ for $x \in \partial K_r = \{v \in K : \|x\| = r\}$.*

(i) If $\|Tx\| \ge \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K)$=0.

(ii) If $\|Tx\| \le \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K)$=1.

Proof of Theorem 4.1. As before we set

$$F(x,u) = f(x,|u|)$$

and let

$$\hat{G}(x,u) = F(x,u) - b|u|^p.$$

For the remainder of the proof, we will omit the dependence with respect to $x \in [0,2\pi]$.

In order to prove that $\lambda_\infty = 0$ is a bifurcation from infinity for

$$u - \lambda\mathcal{K}F(u) = 0, \tag{4.1}$$

we use the rescaling $w = \gamma u$, $\lambda = \gamma^{p-1}$, $\gamma > 0$. A direct calculation shows that (λ,u), $\lambda > 0$, is a solution of (4.1) if and only if

$$w - \mathcal{K}\tilde{F}(\gamma,w) = 0, \tag{4.2}$$

where

$$\tilde{F}(\gamma, w) := b|w|^p + \gamma^p \hat{G}(\gamma^{-1} w).$$

We can extend $\tilde{F}$ to $\gamma = 0$ by setting

$$\tilde{F}(0, w) = b|w|^p$$

and, by (F3), such an extension is continuous. We set

$$S(\gamma, w) = w - \mathcal{K}\tilde{F}(\gamma, w), \qquad \gamma \in \mathbb{R}^+.$$

Let us point out explicitly that $S(\gamma, \cdot) = I - \mathcal{K}\tilde{F}(\gamma, \cdot)$, with $\mathcal{K}\tilde{F}(\gamma, \cdot)$ is compact. For $\gamma = 0$, solutions of $S_0(w) := S(0, w) = 0$ are nothing but solutions of

$$\begin{aligned} &-w'' + q(x)w = b|w|^p, \qquad x \in (0, 2\pi), \\ &w(0) = w(2\pi), \qquad w'(0) = w'(2\pi). \end{aligned} \tag{4.3}$$

Now, we claim that there exist two constants $r_1,\ R_1$ with $0 < r_1 < R_1$, such that

$$S_0(w) \neq 0, \qquad \forall \, \|w\| \geq R_1 \tag{4.4}$$

$$S_0(w) \neq 0, \qquad \forall \, \|w\| \leq r_1 \tag{4.5}$$

and

$$\deg \left(S_0, K_R \backslash \bar{K}_r, 0\right) = -1, \qquad \forall \, 0 < r \leq r_1,\ R \geq R_1. \tag{4.6}$$

In order to prove (4.4), (4.5) and (4.6), we divide the proof into two steps.

Step 1: We show that there exists $R > 0$ such that $S_0(w) \neq 0$, $\forall \, \|w\| \geq R$.

Assume to the contrary that there exists a sequence $\{w_n\}$ of solutions of (4.3) satisfying

$$\lim_{n \to \infty} \|w_n\| = \infty,$$

that is,

$$\begin{aligned} &-w_n'' + q(x)w_n = (b|w_n|^{p-1})w_n, \qquad x \in (0, 2\pi), \\ &w_n(0) = w_n(2\pi), \qquad w_n'(0) = w_n'(2\pi). \end{aligned}$$

Notice that

$$\lim_{n \to \infty} b|w_n|^{p-1} = \infty, \qquad \forall x \in [0, 2\pi].$$

From the Sturm comparison theorem [13, Theorem 2.6] or the special case of [7, Lemma 5.1] when $p = 2$, we have w_n must change its sign in $[0, 2\pi]$. This contradicts the fact that $w_n > 0$ on $[0, 2\pi]$.

Step 2: We show that there exists $r_1 > 0$ such that $S_0(w) \neq 0$ for all $0 < \|w\| \leq r_1$.

Assume on the contrary that (4.5) is not true. Then there exists a sequence w_n of solutions of (4.3) satisfying

$$\|w_n\| \to 0, \qquad n \to \infty.$$

Let $v_n = w_n / \|w_n\|$. From (4.3), we have

$$\begin{aligned} & -v_n'' + q(x)v_n = b(x)\frac{|w_n|^p}{\|w_n\|}, \qquad x \in (0, 2\pi), \\ & v_n(0) = v_n(2\pi), \qquad v_n'(0) = v_n'(2\pi), \end{aligned} \tag{4.7}$$

that is,

$$v_n(x) = \int_0^{2\pi} G(x,s)b(s)\frac{|w_n|^p}{\|w_n\|}ds.$$

Since $b \in C[0, 2\pi]$ and $p > 1$ we have that

$$\lim_{n\to\infty}\left(b\frac{|w_n|^p}{\|w_n\|}\right) \le \lim_{n\to\infty}\left(b\frac{\|w_n\|^p}{\|w_n\|}\right) = 0, \quad \text{uniformly in } x \in [0, 2\pi].$$

So $\lim_{n\to\infty} v_n = 0$ uniformly but this is a contradiction since $\|v_n\| = 1$ for all $n \in \mathbb{N}$.

To show (4.6) is valid. Define a cone K in X by

$$K := \left\{ u \in X : u(x) \ge 0 \text{ on } [0, 2\pi] \text{ and } \min_{0\le x\le 2\pi} u(x) \ge \sigma\|u\| \right\},$$

where σ is from (2.3). A standard argument can be used to show that $\mathcal{K}\tilde{F}(0, \cdot) : K \to K$.

Denote

$$K_r := \{u \in K : \|u\| < r\}.$$

Now, from (4.4) and (4.5), we deduce

$$S_0(w) \ne 0, \quad \forall w \in \partial K_R, \qquad S_0(w) \ne 0, \quad \forall w \in \partial K_r.$$

This implies

$$S_0(w) \ne 0, \qquad \forall w \in \partial(K_R \backslash \bar{K}_r).$$

Thus the degree $\deg(S_0, K_R\backslash\bar{K}_r, 0)$ is well defined.

Next, we show that $\deg(S_0, K_R\backslash\bar{K}_r, 0) = -1$.

The remaining arguments are the same as that of Theorem 3 of [9] and we will only give a short sketch.

Denote

$$f_1(w) := |w|^p, \qquad \forall x \in [0, 2\pi].$$

It is easy to verify the following conditions

(A1) $f_0 := \lim_{w\to 0^+} \frac{f_1(w)}{w} = 0$;

(A2) $f_\infty := \lim_{w\to +\infty} \frac{f_1(w)}{w} = \infty$.

Choose $M_1 > 0$ such that

$$\sigma m M_1 \int_0^{2\pi} b(s)ds > 1,$$

where σ, m are from (2.3).

By (A2), there is $R_2 > 0$ such that $f(w) \geq M_1 w$ for all $w \geq R_2$. Choose $R > \max\{R_1, R_2\}$, we claim that $\|\mathcal{K}\tilde{F}(0,w)\| > \|w\|$ for $w \in \partial K_R$. In fact, for $w \in \partial K_R$

$$
\begin{aligned}
(\mathcal{K}\tilde{F}(0,w))(x) &= \int_0^{2\pi} G(x,s)b(s)f_1(w)ds \\
&\geq \sigma m M_1 \|w\| \int_0^{2\pi} b(s)ds \\
&> \|w\|.
\end{aligned}
$$

Hence, Lemma 4.2 implies

$$
i(\mathcal{K}\tilde{F}(0,\cdot), K_R, K) = 0. \tag{4.8}
$$

On the other hand, by (A1) there is a $\delta > 0$ such that $0 \leq w \leq \delta$ implies

$$
f_1(w) \leq \eta w,
$$

where $\eta > 0$ satisfying

$$
M\eta \int_0^{2\pi} b(s)ds \leq 1.
$$

Choose $0 < r < \min\left\{\delta, \frac{r_1}{2}\right\}$, for $w \in \partial K_r$,

$$
\begin{aligned}
\|\mathcal{K}\tilde{F}(0,w)\| &= \max_{x\in[0,2\pi]} \int_0^{2\pi} G(x,s)b(s)f_1(w)ds \\
&\leq M\eta\|w\| \int_0^{2\pi} b(s)ds \\
&\leq \|w\|.
\end{aligned}
$$

It is obvious that $\mathcal{K}\tilde{F}(0,w) \neq w$ for $w \in \partial K_r$. An application of Lemma 4.2 again shows that

$$
i(\mathcal{K}\tilde{F}(0,\cdot), K_r, K) = 1. \tag{4.9}
$$

Now, the additivity of the fixed point index and (4.8), (4.9) together implies

$$
i(\mathcal{K}\tilde{F}(0,\cdot), K_R \backslash \bar{K}_r, K) = -1.
$$

Combining this together with the fact $S_0 : X \to K_R \backslash \bar{K}_r$, it deduces that

$$
\deg\left(S_0, K_R \backslash \bar{K}_r, 0\right) = -1.
$$

Therefore, the claim is proved. □

Next we show the following result.

Lemma 4.3. *There exists $\gamma_0 > 0$ such that*

(i) $\deg\left(S(\gamma,\cdot), K_R \backslash \bar{K}_r, 0\right) = -1, \ \ \forall\, 0 \leq \gamma \leq \gamma_0$;

(ii) if $S(\gamma, w) = 0,\ \gamma \in [0, \gamma_0],\ r \leq \|w\| \leq R$, then $w > 0$ in $[0, 2\pi]$.

Proof. Clearly (i) follows if we show that $S(\gamma, w) \neq 0$ for all $\|w\| \in \{r, R\}$ and all $0 \leq \gamma \leq \gamma_0$. Otherwise, there exists a sequence (γ_n, w_n) with $\gamma_n \to 0$, $\|w_n\| \in \{r, R\}$ and $w_n = \mathcal{K}\tilde{F}(\gamma_n, w_n)$. Since $\mathcal{K}$ is compact then, up to a subsequence, $w_n \to w$ and $S_0(w) = 0$, $\|w\| \in \{r, R\}$, a contradiction with (4.4) and (4.5).

To prove (ii), we argue again by contradiction. As in the preceding argument, we find a sequence $w_n \in X$, with $\{x \in [0, 2\pi] : w_n(x) \leq 0\} \neq \emptyset$, such that $w_n \to w$, $\|w\| \in [r, R]$ and $S_0(w) = 0$; namely, w solves (4.3). From the positivity of Green's function $G(x, s)$ and $b|w|^p$, we have $w > 0$. Therefore $w_n > 0$ on $[0, 2\pi]$ for n large enough, a contradiction. □

Proof of Theorem 4.1 completed. By Lemma 4.3, we know that problem (4.2) has a positive solution w_γ, for all $0 \leq \gamma \leq \gamma_0$. Recalling, for $\gamma > 0$, the rescaling $\lambda = \gamma^{p-1}$, $u = w/\gamma$, gives a solution (λ, u_λ) of (4.1) for all $0 < \lambda < \lambda := \gamma_0^{p-1}$. Since $w_\gamma > 0$, (λ, u_λ) is a positive solution of (1.1). Finally $\|w_\gamma\| \geq r$ for all $\gamma \in [0, \gamma_0]$ implies that $\|u_\lambda\| = \|w_\gamma\|/\gamma \to \infty$ as $\gamma \to 0$. This completes the proof. □

5 Sublinear problems

In this section, we deal with sublinear f, namely $f \in C([0, 2\pi] \times \mathbb{R}^+, \mathbb{R})$ that satisfy (F1) and

(F4) there exists $b \in C[0, 2\pi]$ with $b > 0$ in $[0, 2\pi]$ such that $\lim_{u \to \infty} u^{-q} f(x, u) = b$, uniformly in $x \in [0, 2\pi]$ with $0 \leq q < 1$.

We will show that in this case positive solutions of (1.1) branch off from ∞ for $\lambda_\infty = +\infty$. First, some preliminaries are in order. It is convenient to work on $Y = C^1[0, 2\pi]$. Following the same procedure as for the superlinear case, we employ the rescaling $w = \gamma u$, $\lambda = \gamma^{q-1}$ and use the same notation, with q instead of p and Y instead of X. As before, (λ, u) solves (4.1) if (γ, w) satisfies (4.2). Note that now, since $0 \leq q < 1$, one has that

$$\lambda \to +\infty \Leftrightarrow \gamma \to 0. \tag{5.1}$$

Furthermore, it follows from the special case of Dai et al. [7, Theorem 6.1] when $p = 2$, we get that

$$\begin{aligned} &-u''(x) + q(x)u(t) = bu^q, \qquad x \in (0, 2\pi), \\ &u(0) = u(2\pi), \qquad u'(0) = u'(2\pi) \end{aligned} \tag{5.2}$$

has a unique positive solution w_0 with $w_0(t) > 0$ in $[0, 2\pi]$.

Let $\lambda_1[bw_0^{q-1}]$ denote the first eigenvalue of the linearized problem

$$\begin{aligned} &-v''(x) + q(x)v(x) = \lambda b w_0^{q-1} v, \qquad x \in (0, 2\pi), \\ &v(0) = v(2\pi), \qquad v'(0) = v'(2\pi). \end{aligned} \tag{5.3}$$

(5.2) implies that $v = w_0$ is an eigenfunction corresponding to

$$\lambda_1 \left[b w_0^{q-1} \right] = 1. \tag{5.4}$$

We set $D_\delta = \{w \in Y : \|w - w_0\|_1 \leq \delta\}$ and extend $\tilde{F}$ to $\gamma = 0$ by

$$\tilde{F}_0(w) = \tilde{F}(0, w) := b|w|^q.$$

Lemma 5.1. *There exists $\delta > 0$ such that $\mathcal{K}\tilde{F} : [0,\infty) \times D_\delta \to Y$ is compact and continuous.*

Proof. First of all, we proved that $\mathcal{K}\tilde{F} : [0,\infty) \times D_\delta \to Y$ is continuous. If $0 < q < 1$ the same arguments used for $p > 1$ show that $\mathcal{K}\tilde{F}$ is continuous. Now we consider a situation where $q = 0$. Let $\delta > 0$ be such that $w > 0$ for all $w \in D_\delta$. Obviously, it suffices to show that $\mathcal{K}F(\gamma_n, w_n) \to \mathcal{K}\tilde{F}_0(w)$ whenever $\gamma_n \to 0$ and $w_n \to w$ in Y. Since $w > 0$ then $\gamma_n^{-1} w_n \to +\infty$, pointwise in $[0, 2\pi]$. Notice $q = 0$ implies that $\lim_{u\to\infty} f(x,u) = b$, and accordingly,

$$G(\gamma_n^{-1} w_n) \to 0 \quad \text{in } L^r(0,2\pi), \quad \forall r \geq 1.$$

Then

$$\mathcal{K}\tilde{F}(\gamma_n, w_n) = \mathcal{K}\tilde{F}_0(w_n) + \mathcal{K}G(\gamma_n^{-1} w_n) \to \mathcal{K}\tilde{F}_0(w),$$

in the Sobolev space $H^{2,r}$, $\forall r \geq 1$. A standard argument can be used to show that $\mathcal{K}\tilde{F} : [0,\infty) \times D_\delta \to Y$ is compact. $\square$

Theorem 5.2. *Let $f \in C([0,2\pi] \times \mathbb{R}^+, \mathbb{R})$ satisfy (F1) and (F4). Then there exists $\lambda^* > 0$ such that (1.1) has positive solutions for all $\lambda \geq \lambda^*$. More precisely, there exists a connected set of positive solutions of (1.1) bifurcating from infinity for $\lambda_\infty = +\infty$.*

Proof. By Lemma 5.1, degree theoretic arguments apply to $S(\gamma, w) = w - \mathcal{K}\tilde{F}(\gamma, w)$. Moreover, note that $S_0(w) = S(0,w) = w - \mathcal{K}\tilde{F}_0(w)$ is C^1 on D_δ and its Fréchet derivative $S_0'(w_0)$ is given by

$$S_0'(w_0)v = \begin{cases} v - \mathcal{K}[qbw_0^{q-1}v], & 0 < q < 1, \\ v, & q = 0. \end{cases}$$

In particular, for $0 < q < 1$, (5.4) implies that all the characteristic values of $I - S_0'(w_0)$ are greater than 1.

Since w_0 is the unique positive solution of (5.2). By [8, Theorem 8.10], we have

$$\deg(S_0, D_\delta, 0) = \deg(I - \mathcal{K}\tilde{F}_0, D_\delta, 0) = \deg(S_0'(w_0), D_\delta, 0) = (-1)^{m(\lambda)}, \qquad \forall q \in [0,1),$$

where $m(\lambda)$ is the sum of algebraic multiplicity of the eigenvalues μ of problem (5.3) satisfying $\lambda^{-1}\mu < 1$. If $\lambda \in [0, \lambda_1)$, we know that there is no such a μ at all, then

$$\deg(S_0, D_\delta, 0) = (-1)^{m(\lambda)} = (-1)^0 = 1, \qquad \forall q \in [0,1).$$

By continuation, we deduce that there exists a connected subset Γ of solutions of $S(\gamma, w) = 0 (\gamma > 0)$, such that $(0, w_0) \in \bar{\Gamma}$. Moreover, by an argument similar to that of Lemma 4.3, we get that there exists $\gamma_0 > 0$ such that these solutions are positive provided $0 < \gamma \leq \gamma_0$. By the rescaling $\lambda = \gamma^{q-1}$, $u = w/\gamma$, Γ is transformed into a connected subset Σ_∞ of solutions of (1.1). These solutions are indeed positive for all $\lambda > \lambda^* := \gamma_0^{q-1}$ and, according to (5.1), Σ_∞ bifurcates from infinity for $\lambda_\infty = +\infty$. $\square$

Remark 5.3. In general, solutions on Σ_∞ can change sign and the behavior of Σ_∞ depends on the definition of f for $u < 0$. Let us point out that this is in contrast with the positone case; see, for example, the article [7,15,20].

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