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Small solutions of the damped half-linear oscillator with step function coefficients

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. In this paper we consider the damped half-linear oscillator

$$x''|x'|^{n-1} + c(t)|x'|^{n-1}x' + a(t)|x|^{n-1}x = 0, \qquad n \in \mathbb{R}^+.$$

We give a sufficient condition guaranteeing the existence of a small solution, that is a non-trivial solution which tends to 0 as t tends to infinity, in the case when both damping and elasticity coefficients are step functions. With our main theorem we not just generalize the corresponding theorem for the linear case $n = 1$, but we even sharpen Hatvani's theorem concerning the undamped half-linear differential equation.

Keywords: small solution, asymptotic stability, half-linear differential equation, step function coefficients, damping, difference equations.

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1 Introduction

Let us consider the second order damped half-linear differential equation

$$x''|x'|^{n-1} + c(t)|x'|^{n-1}x' + a(t)|x|^{n-1}x = 0, \qquad n \in \mathbb{R}^+. \tag{1.1}$$

The term half-linear reflects to the property that the solution set of equation (1.1) is homogeneous but it is not additive. Clearly, equation (1.1) is a generalization of the damped linear oscillator, since for $n = 1$ it takes the form

$$x'' + c(t)x' + a(t)x = 0. \tag{1.2}$$

The qualitative theory for equation (1.1) and for its undamped variant

$$x''|x'|^{n-1} + a(t)|x|^{n-1}x = 0, \qquad n \in \mathbb{R}^+ \tag{1.3}$$

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has an extensive and still growing literature since the foundational works of Bihari [3] and Elbert [11]. This is due to the fact that many theorems concerning the linear case can be generalized to the half-linear case (often with a different machinery), furthermore, half-linear equations have several real life applications as well, see, e.g. monographs [1,8,9] and papers [5,6,10,17,22,31,32,34], and the references therein.

We shall call a non-trivial solution $x_0(t)$ of (1.1) *small* if

$$\lim_{t\to\infty} x_0(t) = 0 \tag{1.4}$$

holds. Note, that this stability property is equivalent to partial asymptotic stability with respect to x. For the linear oscillator

$$x'' + a(t)x = 0 \tag{1.5}$$

Milloux [27] was the first to prove, that if a is differentiable, monotonously increasing and tends to infinity as $t \to \infty$, then it always has at least one small solution. He also gave an example where the coefficient a was a step function and equation (1.5) had a solution which was not small. The first theorem to give a sufficient condition on that all solutions of (1.5) being small, was the celebrated Armellini–Tonelli–Sansone theorem (abbreviated as A–T–S theorem, see, e.g., [26]). This theorem required the "regular" growth of a, which roughly means that the growth of a cannot be located to a set with small measure. Obviously, a step function a is of "irregular" growth type. For the half-linear oscillator (1.3) Atkinson and Elbert [2] generalized Milloux's theorem, while Bihari [4] extended the A–T–S theorem.

Hatvani [19] considered equation (1.3) in the case of increasing step function coefficients:

$$x''|x'|^{n-1} + a_k^{n+1}|x|^{n-1}x = 0, \qquad n \in \mathbb{R}^+, \qquad (t_{k-1} \le t < t_k,\ k = 1,2,\ldots), \tag{1.6}$$

where $\{t_k\}_{k=0}^{\infty}$, $\{a_k\}_{k=1}^{\infty}$ are real sequences, and

$$\begin{aligned} 0 = t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots; \qquad & \lim_{k\to\infty} t_k = \infty, \\ 0 < a_1 < a_2 < \cdots < a_{k-1} < a_k < \cdots; \qquad & \lim_{k\to\infty} a_k = \infty. \end{aligned} \tag{1.7}$$

When we speak about a second order differential equation with step function coefficients we have to define what we mean by a solution of it: a function is a solution if it is continuously differentiable on $[0, \infty)$, furthermore, it is twice continuously differentiable and satisfies the equation on intervals (t_k, t_{k+1}) for all $k = 0, 1, \ldots$ With the aid of a topological method, Hatvani proved the following result for equation (1.6).

Theorem 1.1 (Hatvani [19])**.** *Under assumptions (1.7) equation (1.6) has a small solution.*

We note here that Hatvani and Székely [22] generalized the A–T–S theorem for equation (1.6) in the case $n \ge 1$ under the previous assumptions.

In [18], Hatvani considered the linear oscillator in the case of not necessarily increasing step function coefficients:

$$x'' + a_k^2 x = 0 \qquad (t_{k-1} \le t < t_k,\ k = 1,2,\ldots), \tag{1.8}$$

where $\{t_k\}_{k=0}^{\infty}$, $\{a_k\}_{k=1}^{\infty}$ are real sequences, and

$$\begin{aligned} 0 = t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots; \qquad & \lim_{k\to\infty} t_k = \infty, \\ a_k > 0 \qquad (k = 1,2,\ldots), \qquad & \lim_{k\to\infty} a_k = \infty. \end{aligned} \tag{1.9}$$

He deduced the theorem below.

Theorem 1.2 (Hatvani [18])**.** *If*

$$\sum_{k=1}^{\infty} \max\left\{\frac{a_k}{a_{k+1}} - 1; 0\right\} < \infty, \tag{1.10}$$

then (1.8) *has at least one small solution.*

Condition (1.10) says that the sequence $\{a_k\}_{k=1}^{\infty}$ is "almost" increasing in the sense that the sum of decrements in the not monotone increasing sections is not too large.

It is quite a natural idea that damping helps weaken this condition and even the condition $\lim_{k\to\infty} a_k = \infty$.

Let us turn our attention now to the damped linear oscillator

$$x'' + c_k x' + a_k^2 x = 0 \qquad (t_{k-1} \le t < t_k,\ k = 1,2,\dots), \tag{1.11}$$

where both the elasticity and damping coefficients are step functions. Namely, $\{t_k\}_{k=0}^{\infty}$, $\{a_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ are real sequences with the following properties:

$$\begin{gathered} 0 = t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots; \qquad \lim_{k\to\infty} t_k = \infty, \\ a_k > 0, \quad c_k \ge 0 \quad (k = 1,2,\dots). \end{gathered} \tag{1.12}$$

Hatvani and Székely proved the following.

Theorem 1.3 (Hatvani and Székely [21,23])**.** *Assume that the above conditions on sequences* $\{t_k\}_{k=0}^{\infty}$, $\{c_k\}_{k=1}^{\infty}$ *and* $\{a_k\}_{k=1}^{\infty}$ *are satisfied, and let us introduce the notation*

$$\gamma_k := \frac{c_k}{2a_k + c_k}[(2a_k - c_k)(t_k - t_{k-1}) - 2].$$

Suppose, in addition, that

(i) $a_k > c_k/2$ $(k = 1,2,\dots)$,

(ii) $$\sum_{k=1}^{\infty}\left(-\gamma_k + \ln\frac{a_k}{a_{k+1}}\right) = -\infty,$$

(iii) *there is a number* K *such that for every* $p, q \in \mathbb{N}$, $(1 \le p \le q)$

$$\sum_{k=p}^{q}\left(-\frac{\gamma_k}{2} + \ln\max\left\{\frac{a_k}{a_{k+1}}; 1\right\}\right) < K$$

holds.

Then equation (1.11) *has at least one small solution.*

Remark 1.4. Theorem 1.3 is a generalization and a further developed version of Theorem 3.2 in [21].

Our main purpose is to generalize Theorem 1.3 to the damped half-linear equation (1.1). It will turn out, that as a corollary of our main theorem we even generalize Theorem 1.1 concerning the undamped case. In fact, this corollary is also a generalization of Theorem 1.2.

Finally, we mention that many papers were devoted to examine and extend the above discussed stability problems to several types of equations, we refer the reader to the textbook of Hartman [16] and papers [7,12–15,20,21,24,25,28,29,33].

2 Prerequisites

2.1 Generalized trigonometric functions

During our calculations we will need the well known generalized sine and cosine functions which were introduced for half-linear equations by Elbert [11], and play an essential role in the qualitative analysis of half-linear equations.

Consider the solution $S = S_n(\Phi)$ of the initial value problem

$$\begin{cases} S''|S'|^{n-1} + S|S|^{n-1} = 0 \\ S(0) = 0, \quad S'(0) = 1. \end{cases} \tag{2.1}$$

If we multiply the differential equation by S' and integrate the product over $[0, \Phi]$, we obtain the Pythagorean identity

$$|S'(\Phi)|^{n+1} + |S(\Phi)|^{n+1} = 1 \qquad (-\infty < \Phi < \infty). \tag{2.2}$$

S and S' are periodic functions with period $2\hat{\pi}$, where $\hat{\pi}$ is defined as

$$\hat{\pi} = \frac{2\frac{\pi}{n+1}}{\sin\frac{\pi}{n+1}}.$$

Similarly to the "ordinary" tangent function, the generalized tangent function is

$$T(\Phi) = \frac{S(\Phi)}{S'(\Phi)}.$$

We recall the following identity (see e.g. formulas (2.10) and (2.12) in [2]):

$$\frac{d}{d\Phi}\left(\frac{1}{n+1}|S'(\Phi)|^{n-1}S'(\Phi)S(\Phi)\right) = |S'(\Phi)|^{n+1} - \frac{n}{n+1}. \tag{2.3}$$

2.2 Small solutions of systems of non-linear difference equations

The proof of Theorem 1.3 relies on a theorem (see Theorem 2.2 in [21]) which guarantees the existence of small solutions to systems of linear first order non-autonomous difference equations. In order to generalize Theorem 1.3 to the half-linear case, we will need a similar theorem for systems of non-linear first order non-autonomous difference equations as well, therefore we recall some concepts and Theorem 9 from [23]. We have to note that this theorem is a consequence of the very general theorem of Karsai, Graef and Li (see Theorem 1 in [25]), which is based on a Lyapunov function. Instead of that we will use the following theorem, which relies only on the right hand side of the equation.

Let us consider the system of non-linear difference equations

$$\mathbf{x}_{k+1} = \mathbf{f}(k, \mathbf{x}_k), \qquad k = 0, 1, 2, \dots, \tag{2.4}$$

where $\mathbf{x}_k \in \mathbb{R}^m$ $(m \in \mathbb{N})$ is a column vector, and the functions $\mathbf{f}(k, \cdot)$ satisfy the following conditions for all $k \in \mathbb{N}_0$:

$$\mathbf{f}(k,\cdot)\colon \mathrm{D}_k \subset \mathbb{R}^m \to \mathbb{R}^m, \qquad \operatorname{ran}\mathbf{f}(k,\cdot) \subset \mathrm{D}_{k+1},$$

$$\mathbf{f}(k, 0) = 0, \qquad \mathbf{f}(k,\cdot) \in C^1(\mathrm{D}_k),$$

where D_k is a convex domain $(k = 0, 1, \ldots)$. Let $q \geq p$ $(p, q \in \mathbb{N}_0)$, then introduce the notation

$$\mathbf{F}(q, p\,; \cdot) = \mathbf{f}(q, \cdot) \circ \cdots \circ \mathbf{f}(p, \cdot),$$

furthermore, let $F^j(q, p\,; \cdot)\colon \mathrm{D}_p \to \mathbb{R}$ $(j = 1, \ldots, m)$ be the jth component function of $\mathbf{F}(q, p\,; \cdot)$, that is

$$\mathbf{F}(q, p\,; \mathbf{x}) = \begin{pmatrix} F^1(q, p\,; \mathbf{x}) \\ \vdots \\ F^m(q, p\,; \mathbf{x}) \end{pmatrix}.$$

Note that $\mathbf{F}(k, 0\,; \cdot)$ is the flow of equation (2.4). For a function $g\colon \mathbb{R}^m \to \mathbb{R}$, $\operatorname{grad} g(\mathbf{x})$ denotes the gradient of g, i.e.

$$\operatorname{grad} g(\mathbf{x}) = \left(\frac{\partial g(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial g(\mathbf{x})}{\partial x_m} \right)^{\mathrm{T}}.$$

Clearly, the Jacobian of a function $\mathbf{G}\colon \mathbb{R}^m \to \mathbb{R}^m$ is the $m \times m$ matrix

$$\mathbf{G}'(\mathbf{x}) = \begin{pmatrix} \operatorname{grad} G^1(\mathbf{x}) \\ \vdots \\ \operatorname{grad} G^m(\mathbf{x}) \end{pmatrix},$$

where $G^1, \ldots, G^m$ are the component functions of $\mathbf{G}$. Let $H_0 \subset \mathrm{D}_0$ be the closure of a bounded and connected open set, then define H_k as the kth image of H_0 among the flow of (2.4), that is $H_k = \mathbf{F}(k, 0; H_0)$. The phase volume of H_k can be calculated as

$$\mu(H_k) = \int_{H_0} \left| \det \mathbf{F}'(k, 0; \mathbf{x}) \right| \, \mathrm{d}\mathbf{x}, \tag{2.5}$$

where μ is the Lebesgue measure.

Theorem 2.1 (Hatvani and Székely [23]). *Suppose that there exists a closed ball H_0 around the origin and a number $K > 0$, such that for all $p, q \in \mathbb{N}_0$ $(0 \leq p \leq q)$, $j = 1, \ldots, m$ and $\mathbf{x} \in H_0$*

$$\left\| \operatorname{grad} F^j(q, p\,; \mathbf{x}) \right\| \leq K \tag{2.6}$$

holds, furthermore

$$\lim_{k \to \infty} \int_{H_0} \left| \det \mathbf{F}'(k, 0; \mathbf{x}) \right| \, \mathrm{d}\mathbf{x} = 0. \tag{2.7}$$

Then equation (2.4) *has at least one small solution.*

Remark 2.2. If

$$\lim_{k \to \infty} \det \mathbf{F}'(k, 0; \mathbf{x}) = 0 \qquad (\mathbf{x} \in H_0) \tag{2.8}$$

is satisfied, then (2.7) holds.

Remark 2.3. Observe that according to the chain rule,

$$\left| \det \mathbf{F}'(k, 0; \mathbf{x_0}) \right| = \prod_{i=0}^{k} \left| \det \mathbf{f}'(i, \mathbf{x_i}) \right| \qquad (\mathbf{x_0} \in H_0). \tag{2.9}$$

3 Main result

Let us now consider the damped half-linear differential equation with step function coefficients

$$x''|x'|^{n-1} + c_k|x'|^{n-1}x' + a_k^{n+1}|x|^{n-1}x = 0, \qquad n \in \mathbb{R}^+, \qquad (t_{k-1} \le t < t_k,\ k = 1,2,\dots), \tag{3.1}$$

where $\{t_k\}_{k=0}^{\infty}$, $\{a_k\}_{k=1}^{\infty}$ and $\{c_k\}_{k=1}^{\infty}$ are real sequences, and

$$\begin{gathered} 0 = t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots; \qquad \lim_{k\to\infty} t_k = \infty, \\ a_k > 0, \quad c_k \ge 0 \quad (k = 1,2,\dots) \end{gathered} \tag{3.2}$$

are satisfied. Note that these conditions are identical to (1.12).

Before we can state our main result, we recall Hadamard's lemma, which we will need in the proof of the main theorem.

Lemma 3.1 (Hadamard, see e.g. [30, p. 176])**.** *Let N be an $m \times m$ matrix having rows v_i ($v_i \in \mathbb{R}^m$, $1 \le i \le m$), then*

$$|\det N| \le \prod_{i=1}^{m} \|v_i\|\,.$$

Now we are ready to state our main theorem.

Theorem 3.2. *Assume that conditions (3.2) on sequences $\{t_k\}_{k=0}^{\infty}$, $\{c_k\}_{k=1}^{\infty}$ and $\{a_k\}_{k=1}^{\infty}$ are satisfied, and let us introduce the notation*

$$\gamma_k := \frac{c_k}{a_k + c_k}[n(a_k - c_k)(t_k - t_{k-1}) - 2]. \tag{3.3}$$

Suppose, in addition, that

(i) $a_k > c_k$ $(k = 1,2,\dots)$,

(ii) $$\sum_{i=1}^{\infty}\left(-\frac{2\gamma_i}{n+1} + \ln\frac{a_i}{a_{i+1}}\right) = -\infty,$$

(iii) *there is a number K such that for every $p, q \in \mathbb{N}$, $(1 \le p \le q)$*

$$\sum_{i=p}^{q}\left(-\frac{\gamma_i}{n+1} + \ln\max\left\{\frac{a_i}{a_{i+1}};1\right\}\right) < K \tag{3.4}$$

holds.

Then equation (3.1) has at least one small solution.

Proof. First, let us introduce the notation $a(t) := a_k^{n+1}$ $(t_{k-1} \le t < t_k,\ k = 1,2\dots)$ and then we define a new time variable in the following way

$$\tau = \varphi(t) = \int_0^t a^{\frac{1}{n+1}}(s)\,\mathrm{d}s, \qquad \tau_k := \varphi(t_k). \tag{3.5}$$

Note that

$$\tau_{k+1} - \tau_k = a_{k+1}(t_{k+1} - t_k). \tag{3.6}$$

Let $x(t) = x(\varphi^{-1}(\tau)) =: y(\tau)$, where φ^{-1} denotes the inverse function of φ. This way, for the first and second derivative of x we have

$$x'(t) = \dot{y}(\tau) a^{\frac{1}{n+1}}(t), \quad x''(t) = \ddot{y}(\tau) a^{\frac{2}{n+1}}(t) \qquad (t \neq t_k,\ k = 0,1,2,\ldots),$$

where $(\cdot)^{\cdot} = \mathrm{d}(\cdot)/\mathrm{d}\tau$. Thus, equation (3.1) is transformed into the form

$$\ddot{y}(\tau)|\dot{y}(\tau)|^{n-1} + h(\tau)|\dot{y}(\tau)|^{n-1}\dot{y}(\tau) + |y(\tau)|^{n-1}y(\tau) = 0, \qquad (\tau_{k-1} < \tau < \tau_k,\ k = 1,2,\ldots), \tag{3.7}$$

where

$$h(\tau) = \frac{c_k}{a_k} \qquad (\tau_{k-1} < \tau < \tau_k,\ k = 1,2,\ldots). \tag{3.8}$$

Let us use the notations $f(t-0)$ and $f(t+0)$ for the left-hand side and the right-hand side limit of a function f at t, respectively. Since any solution x of equation (3.1) has to be continuously differentiable on $(0,\infty)$, $x'(t_k - 0) = x'(t_k + 0) = x'(t_k)$ must hold for every $k \in \mathbb{N}$, i.e.,

$$\dot{y}(\tau_k) = \dot{y}(\tau_k + 0) = \frac{a_k}{a_{k+1}}\dot{y}(\tau_k - 0).$$

Then (3.1) is equivalent to the following differential equation with impulses:

$$\begin{cases} \ddot{y}(\tau)|\dot{y}(\tau)|^{n-1} + h(\tau)|\dot{y}(\tau)|^{n-1}\dot{y}(\tau) + |y(\tau)|^{n-1}y(\tau) = 0, & \tau \neq \tau_k \\ \dot{y}(\tau_k) = \frac{a_k}{a_{k+1}}\dot{y}(\tau_k - 0), \qquad k = 1,2,\ldots \end{cases} \tag{3.9}$$

Let us apply now the so-called generalized Prüfer transformation, that is, let us introduce the generalized polar coordinates $\dot{y} = \rho S'(\Phi)$, $y = \rho S(\Phi)$, where

$$\rho = (|\dot{y}|^{n+1} + |y|^{n+1})^{\frac{1}{n+1}}, \qquad T(\Phi) = \frac{y}{\dot{y}}, \quad -\infty < \Phi < \infty.$$

With the aid of these variables we can rewrite equation (3.7) into the system of equations

$$\begin{aligned} \dot{\rho} &= -h(\tau)\rho|\dot{S}(\Phi)|^{n+1}, \\ \dot{\Phi} &= 1 + h(\tau)|\dot{S}(\Phi)|^{n-1}\dot{S}(\Phi)S(\Phi), \qquad (\tau_{k-1} \leq \tau < \tau_k,\ k = 1,2,\ldots). \end{aligned} \tag{3.10}$$

Observe that $\dot{\rho}(\tau) \leq 0$, furthermore, assumption (i) and (3.8) together imply that $\dot{\Phi}(\tau) > 0$ holds for all $\tau \in [\tau_{k-1}, \tau_k)$ $(k = 1,2,\ldots)$. With the aid of equations (3.8), (2.3), and the Newton–Leibniz theorem we obtain the following estimation:

$$\begin{aligned} \int_{\tau_{k-1}}^{\tau_k} \frac{\dot{\rho}(\tau)}{\rho(\tau)}\mathrm{d}\tau = \ln\frac{\rho(\tau_k - 0)}{\rho(\tau_{k-1})} &= -\frac{c_k}{a_k}\int_{\tau_{k-1}}^{\tau_k} |\dot{S}(\Phi(\tau))|^{n+1}\,\mathrm{d}\tau \\ &= -\frac{c_k}{a_k}\int_{\tau_{k-1}}^{\tau_k} \frac{|\dot{S}(\Phi(\tau))|^{n+1}\dot{\Phi}(\tau)}{\dot{\Phi}(\tau)}\,\mathrm{d}\tau \\ &= -\frac{c_k}{a_k}\int_{\tau_{k-1}}^{\tau_k} \frac{|\dot{S}(\Phi(\tau))|^{n+1}\dot{\Phi}(\tau)}{1 + \frac{c_k}{q_k}|\dot{S}(\Phi(\tau))|^{n-1}\dot{S}(\Phi(\tau))S(\Phi(\tau))}\,\mathrm{d}\tau \\ &= -\frac{c_k}{a_k}\int_{\Phi(\tau_{k-1})}^{\Phi(\tau_k)} \frac{|\dot{S}(u)|^{n+1}}{1 + \frac{c_k}{a_k}|\dot{S}(u)|^{n-1}\dot{S}(u)S(u)}\,\mathrm{d}u \\ &\leq -\frac{c_k}{a_k}\cdot\frac{1}{1 + \frac{c_k}{a_k}}\int_{\Phi(\tau_{k-1})}^{\Phi(\tau_k)} |\dot{S}(u)|^{n+1}\,\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
&= -\frac{c_k}{a_k+c_k}\int_{\Phi(\tau_{k-1})}^{\Phi(\tau_k)}\left(\frac{n}{n+1}+|\dot S(u)|^{n+1}-\frac{n}{n+1}\right)\mathrm{d}u\\
&= -\frac{c_k}{a_k+c_k}\left\{\left[\frac{n}{n+1}u+\frac{1}{n+1}|S'(u)|^{n-1}S'(u)S(u)\right]_{\Phi(\tau_{k-1})}^{\Phi(\tau_k)}\right\}\\
&= -\frac{c_k}{(n+1)(a_k+c_k)}\Big[n(\Phi(\tau_k-0)-\Phi(\tau_{k-1}))\\
&\qquad +\Big(|S'(\Phi(\tau_k-0))|^{n-1}S'(\Phi(\tau_k-0))S(\Phi(\tau_k-0))\\
&\qquad -|S'(\Phi(\tau_{k-1}))|^{n-1}S'(\Phi(\tau_{k-1}))S(\Phi(\tau_{k-1}))\Big)\Big].
\end{aligned}
\tag{3.11}
$$

From the second equation of system (3.10) we get the estimation from below for $\dot\Phi$

$$
\dot\Phi = 1+\frac{c_k}{a_k}|\dot S(\Phi)|^{n-1}\dot S(\Phi)S(\Phi)\geq 1-\frac{c_k}{a_k}\qquad(\tau_{k-1}\leq\tau<\tau_k,\ k=1,2,\dots).
$$

By integration over $[\tau_{k-1},\tau_k]$ and using (3.6) yields

$$
\Phi(\tau_k-0)-\Phi(\tau_{k-1})=\int_{\tau_{k-1}}^{\tau_k}\dot\Phi(\tau)\,\mathrm{d}\tau\geq\left(1-\frac{c_k}{a_k}\right)(\tau_k-\tau_{k-1})=(a_k-c_k)(t_k-t_{k-1}). \tag{3.12}
$$

Now, we are able to continue estimation (3.11)

$$
\ln\frac{\rho(\tau_k-0)}{\rho(\tau_{k-1})}\leq-\frac{c_k}{(n+1)(a_k+c_k)}[n(a_k-c_k)(t_k-t_{k-1})-2]=-\frac{\gamma_k}{n+1}. \tag{3.13}
$$

Next, we are moving towards to apply Theorem 2.1. In order to do this, let $\hat{\mathbf f}(k-1,\cdot)$ be the non-linear mapping corresponding to system (3.10), that is

$$
\begin{pmatrix}\dot y(t_k-0)\\ y(t_k-0)\end{pmatrix}=\hat{\mathbf f}(k-1,(\dot y(t_{k-1}),y(t_{k-1}))),\qquad k=1,2,\dots
$$

According to system (3.9) the vector $(\dot y(t_k),y(t_k))^{\mathrm T}$ is

$$
\begin{pmatrix}\dot y(t_k)\\ y(t_k)\end{pmatrix}=\begin{pmatrix}\frac{a_k}{a_{k+1}} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\dot y(t_k-0)\\ y(t_k-0)\end{pmatrix}=\mathbf f(k-1,(\dot y(t_{k-1}),y(t_{k-1}))),
$$

where

$$
\mathbf f(k-1,(\dot y(t_{k-1}),y(t_{k-1})))=\begin{pmatrix}\frac{a_k}{a_{k+1}} & 0\\ 0 & 1\end{pmatrix}\hat{\mathbf f}(k-1,(\dot y(t_{k-1}),y(t_{k-1}))).
$$

Clearly, the stability properties of system (3.9) are equivalent to the stability properties of the following system of non-linear difference equations

$$
\begin{pmatrix}x_k\\ y_k\end{pmatrix}=\mathbf f(k-1,(x_{k-1},y_{k-1}))\qquad(k=1,2,\dots). \tag{3.14}
$$

Since ρ is a monotonically decreasing function, it follows that

$$
\|(x_k,y_k)\|\leq\|(x_{k-1},y_{k-1})\|\qquad(k=1,2,\dots). \tag{3.15}
$$

For the coordinate functions of $\mathbf{f}$ and $\hat{\mathbf{f}}$ let us introduce the notations

$$\mathbf{f}(k,(x,y)) = (f^1(k,(x,y)), f^2(k,(x,y))), \qquad \hat{\mathbf{f}}(k,(x,y)) = (\hat{f}^1(k,(x,y)), \hat{f}^2(k,(x,y))).$$

With these notations, the Jacobian of $\mathbf{f}$ and $\hat{\mathbf{f}}$ are

$$\mathbf{f}'(k,(x,y)) = \begin{pmatrix} \operatorname{grad} f^1(k,(x,y)) \\ \operatorname{grad} f^2(k,(x,y)) \end{pmatrix}, \qquad \hat{\mathbf{f}}'(k,(x,y)) = \begin{pmatrix} \operatorname{grad} \hat{f}^1(k,(x,y)) \\ \operatorname{grad} \hat{f}^2(k,(x,y)) \end{pmatrix}.$$

With the aid of estimation (3.13) we can give the following estimate

$$\max\left\{\left\|\operatorname{grad} \hat{f}^1(k-1,\cdot)\right\|, \left\|\operatorname{grad} \hat{f}^2(k-1,\cdot)\right\|\right\} \le \sup_{0<\rho(\tau_{k-1})} \frac{\rho(\tau_k - 0)}{\rho(\tau_{k-1})} \le \exp\left[-\frac{\gamma_k}{n+1}\right], \tag{3.16}$$

where the supremum is taken over each solution where ρ is positive at time τ_{k-1}. It yields that

$$\begin{aligned} &\max\left\{\left\|\operatorname{grad} f^1(k-1,\cdot)\right\|, \left\|\operatorname{grad} f^2(k-1,\cdot)\right\|\right\} \\ &\quad\le \max\left\{\left\|\operatorname{grad} \hat{f}^1(k-1,\cdot)\right\|, \left\|\operatorname{grad} \hat{f}^2(k-1,\cdot)\right\|\right\} \max\left\{\frac{a_k}{a_{k+1}};1\right\} \\ &\quad\le \exp\left[-\frac{\gamma_k}{n+1}\right] \max\left\{\frac{a_k}{a_{k+1}};1\right\} \\ &\quad= \exp\left[-\frac{\gamma_k}{n+1} + \ln\max\left\{\frac{a_k}{a_{k+1}};1\right\}\right]. \end{aligned} \tag{3.17}$$

Let

$$\mathbf{F}(q,p;\cdot) = \mathbf{f}(q,\cdot) \circ \mathbf{f}(q-1,\cdot) \circ \cdots \circ \mathbf{f}(p,\cdot), \qquad q \ge p,\ p,q \in \mathbb{N}.$$

Then, according to the chain rule and according to condition (iii) the following holds for all $q \ge p$, $p,q \in \mathbb{N}$

$$\begin{aligned} &\max\left\{\left\|\operatorname{grad} F^1(q,p;\cdot)\right\|, \left\|\operatorname{grad} F^2(q,p;\cdot)\right\|\right\} \\ &\quad\le \prod_{i=p}^{q} \max\left\{\left\|\operatorname{grad} f^1(i,\cdot)\right\|, \left\|\operatorname{grad} f^2(i,\cdot)\right\|\right\} \\ &\quad\le \prod_{i=p}^{q} \exp\left[-\frac{\gamma_i}{n+1} + \ln\max\left\{\frac{a_i}{a_{i+1}};1\right\}\right] \\ &\quad= \exp\left[\sum_{i=p}^{q}\left(-\frac{\gamma_i}{n+1} + \ln\max\left\{\frac{a_i}{a_{i+1}};1\right\}\right)\right] \\ &\quad< K. \end{aligned} \tag{3.18}$$

Next, to apply Theorem 2.1 we need to estimate the change of the phase volume during the dynamics of system (3.14). To do this, first, using Hadamard's inequality and inequality (3.16) we estimate the determinant of the Jacobian of $\hat{\mathbf{f}}(k-1,\cdot)$

$$\begin{aligned} \left|\det \hat{\mathbf{f}}'(k-1,(x,y))\right| &\le \left(\max\left\{\left\|\operatorname{grad} \hat{f}^1(k-1,(x,y))\right\|, \left\|\operatorname{grad} \hat{f}^2(k-1,(x,y))\right\|\right\}\right)^2 \\ &\le \|(x,y)\|^2 \left(\max\left\{\left\|\operatorname{grad} \hat{f}^1(k-1,\cdot)\right\|, \left\|\operatorname{grad} \hat{f}^2(k-1,\cdot)\right\|\right\}\right)^2 \\ &\le \|(x,y)\|^2 \exp\left[-\frac{2\gamma_k}{n+1}\right], \end{aligned} \tag{3.19}$$

which holds for all points (x, y) of the Minkowski plane. Let H_0 be the unit circle in the Minkowski plane and let $(x_0, y_0) \in H_0$. Then, applying again the chain rule and inequalities (3.15) and (3.16), we get

$$\begin{aligned}
\left|\det \mathbf{F}'(k-1, 0; (x_0, y_0))\right| &= \prod_{i=1}^{k} \left|\det \mathbf{f}'(i-1, (x_{i-1}, y_{i-1}))\right| \\
&= \prod_{i=1}^{k} \frac{a_i}{a_{i+1}} \left|\det \hat{\mathbf{f}}'(i-1, (x_{i-1}, y_{i-1}))\right| \\
&\leq \prod_{i=1}^{k} \frac{a_i}{a_{i+1}} \cdot \|(x_{i-1}, y_{i-1})\|^2 \exp\left[-\frac{2\gamma_i}{n+1}\right] \\
&= \prod_{i=1}^{k} \|(x_{i-1}, y_{i-1})\|^2 \times \prod_{i=1}^{k} \exp\left[-\frac{2\gamma_i}{n+1} + \ln\frac{a_i}{a_{i+1}}\right] \\
&\leq \prod_{i=1}^{k} \exp\left[-\frac{2\gamma_i}{n+1} + \ln\frac{a_i}{a_{i+1}}\right] \\
&= \exp\left[\sum_{i=1}^{k}\left(-\frac{2\gamma_i}{n+1} + \ln\frac{a_i}{a_{i+1}}\right)\right].
\end{aligned}$$

One can easily see that due to condition (ii)

$$\lim_{k\to\infty} \left|\det \mathbf{F}'(k, 0; (x_0, y_0))\right| = 0 \qquad ((x_0, y_0) \in H_0),$$

thus with the application of Theorem 2.1 we conclude our proof. □

Remark 3.3. With the aid of the examples in Remark 7 in [23] one can easily see that conditions (ii) and (iii) in Theorem 3.2 are independent of each other.

Remark 3.4. Theorem 3.2 is a direct generalization of Theorem 1.3, the slight differences between them are due to the fact, that in the proof of the linear case some trigonometric identities were used, which (to the authors' best knowledge) do not apply for the half-linear case. According to Remark 8 in [23] Theorem 3.2 is also a generalization of Theorem 1.2.

In the case without damping, that is $c_k = 0$ $(k = 1, 2, \ldots)$, Theorem 3.2 takes the following form.

Theorem 3.5. *Assume that conditions (3.2) are satisfied, furthermore let $c_k = 0$ $(k = 1, 2, \ldots)$. If*

$$\lim_{k\to\infty} a_k = \infty$$

and there is a number K such that for every $p, q \in \mathbb{N}$, $(1 \leq p \leq q)$

$$\sum_{i=p}^{q} \ln \max\left\{\frac{a_i}{a_{i+1}}; 1\right\} < K$$

holds. Then equation (3.1) has at least one small solution.

Remark 3.6. Using the inequality $x - 1 \geq \ln x$ one can easily see that the theorem of Hatvani (Theorem 1.1) implies Theorem 3.5.

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