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# Tempered exponential behavior for a dynamics in upper triangular form

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**Abstract.** We consider the problem of whether the existence of a tempered exponential dichotomy for a linear dynamics can be deduced from the same property for the dynamics restricted to each diagonal entry. More generally, we consider this problem for a dynamics in block upper triangular form. We also obtain corresponding results for a strong tempered exponential dichotomy and for a discrete time dynamics.

**Keywords:** block triangular equations, tempered exponential dichotomies.

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## 1 Introduction

Any linear dynamics, either autonomous or nonautonomous, can be transformed via a (possibly nonautonomous) coordinate change into one in upper triangular form. This is often quite convenient simply because it is easier to deal with a dynamics in upper triangular form. For example, in the case of continuous time this allows one to solve a linear equation by proceeding successively from the last component to the first one. More precisely, consider a sequence of $n \times n$ matrices $(A_m)_{m\in\mathbb{N}}$ and the associated dynamics

$$x_{m+1} = A_m x_m, \qquad m \in \mathbb{N}.$$

Then there exists a sequence of $n \times n$ orthogonal matrices $(U_m)_{m\in\mathbb{N}}$ such that the matrices $B_m = U_{m+1}^{-1} A_m U_m$ are upper triangular. In other words, the coordinate change $y_m = U_m^{-1} x_m$ given by the matrices $U_m$ leads to a dynamics

$$y_{m+1} = B_m y_m, \qquad m \in \mathbb{N},$$

where all matrices $B_m$ are upper triangular. Similarly, given $n \times n$ matrices $A(t)$ varying continuously with $t \geq 0$, consider the linear equation

$$x' = A(t)x. \tag{1.1}$$

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Then there exist matrices $U(t)$ varying differentiably with $t \geq 0$ such that the coordinate change $y(t) = U(t)^{-1}x(t)$ leads to the equation $y' = B(t)y$, where the matrices

$$B(t) = U(t)^{-1}A(t)U(t) - U(t)^{-1}U'(t)$$

are upper triangular for each $t \geq 0$. Both results are well known and follow from a simple application of the Gram–Schmidt process (see for example [1] for these and other constructions). When the dynamics is autonomous, it suffices to use the reduction to the Jordan canonical form, both for discrete and continuous time.

As a consequence, there is no loss of generality in considering only linear dynamics that are already in upper triangular form. Incidentally, one can ask whether it is possible to apply further coordinate changes in order to get rid of some elements above the diagonal, if possible bringing the dynamics to a diagonal form. This would certainly make many problems much simpler. Not surprisingly, this is not always possible (see for example [1]). On the other hand, one can still ask whether it is possible *for some specific property* to deduce that property solely from the information on the diagonal.

Here we consider the problem of whether the hyperbolicity of a dynamics in upper triangular form, and more generally in block upper triangular form, can be deduced from the hyperbolicity of the dynamics restricted to each diagonal entry. More precisely, we consider the notion of hyperbolicity corresponding to the existence of a tempered exponential dichotomy. The latter is the natural notion in the context of ergodic theory. In the particular case of a dynamics exhibiting only contraction, equation (1.1) is said to have a *tempered exponential contraction* if there exist $\lambda > 0$ and a function $D\colon \mathbb{R}_0^+ \to \mathbb{R}^+$ satisfying

$$\limsup_{t\to+\infty} \frac{1}{t} \log D(t) \leq 0, \tag{1.2}$$

such that

$$\|x(t)\| \leq D(s)e^{-\lambda(t-s)}\|x(s)\|, \quad \text{for } t \geq s,$$

where $x = x(t)$ is any solution of the equation (see Section 2 for the notion of a tempered exponential dichotomy). We recall that equation (1.1) is said to have an *exponential contraction* if there exist $\lambda, D > 0$ such that

$$\|x(t)\| \leq De^{-\lambda(t-s)}\|x(t)\|, \quad \text{for } t \geq s \tag{1.3}$$

and any solution $x = x(t)$ of the equation. For example, consider an autonomous equation $y' = f(y)$ whose flow $\varphi_t$ preserves a finite measure (such as any Hamiltonian flow restricted to a compact hypersurface, with respect to the Liouville measure). Then, for almost all initial conditions $y$, if the linear variational equation

$$x' = A_y(t)x, \quad \text{where } A_y(t) = d_{\varphi_t(y)}f,$$

has only negative Lyapunov exponents, then it has a tempered exponential contraction.

More generally, we consider the problem of whether the hyperbolicity of a dynamics in *block upper triangular form* can be deduced from the hyperbolicity of the dynamics restricted to each diagonal block. This includes the upper triangular case as a special case. For example, for continuous time this corresponds to consider the equation

$$\begin{aligned} x' &= A(t)x + C(t)y, \\ y' &= B(t)y. \end{aligned} \tag{1.4}$$

In Theorem 2.1 we show that the existence of a tempered exponential dichotomy for equation (1.4) yields the existence of tempered exponential dichotomies for the equations $x' = A(t)x$ and $y' = B(t)y$, which are associated with the blocks on the diagonal. On the other hand, Theorem 2.3 shows that under the condition

$$\limsup_{t\to+\infty} \frac{1}{t} \log \left(D(t)\|C(t)\|\right) \leq 0,$$

with $D = D(t)$ as in (1.2) or with a corresponding function in the case of a tempered exponential dichotomy, the converse of the statement in Theorem 2.1 holds. Corresponding results for the notion of an exponential dichotomy (which includes that of an exponential contraction in (1.3) as a particular case) were established by Battelli and Palmer in [2] (see also [4]). To the possible extent we follow their approach in the proofs of Theorems 2.1 and 2.3. However, we note that none of the mentioned results in the two papers follows from results in the other.

We also obtain corresponding results for a strong tempered exponential dichotomy (see Theorems 3.1 and 3.3). This correspond to consider both lower and upper bounds along the stable and unstable directions of a tempered exponential dichotomy. Finally, we establish versions of these results for discrete time (see Section 4). The arguments follow a similar path to those for continuous time although they require several nontrivial modifications.

## 2 Continuous time dynamics

Consider the linear equation (1.1) on $\mathbb{R}^n$, where $A\colon I \to \mathbb{R}^{n\times n}$ is a piecewise continuous function on some interval $I \subset \mathbb{R}$ (we shall consider the cases $I = \mathbb{R}_0^+$ and $I = \mathbb{R}_0^-$). We write the solutions in the form $x(t) = T(t,s)x(s)$, for $t, s \in I$, where $T(t,s)$ is the evolution family associated with (1.1).

We say that equation (1.1) has a *tempered exponential dichotomy on $I$* if there exist projections $P_t$ for $t \in I$ satisfying

$$P_t T(t,s) = T(t,s)P_s, \quad \text{for } t,s \in I, \tag{2.1}$$

and there exist $\lambda > 0$ and a function $D\colon I \to \mathbb{R}^+$ satisfying

$$\limsup_{|t|\to+\infty} \frac{1}{|t|} \log D(t) \leq 0 \tag{2.2}$$

such that

$$\|T(t,s)P_s\| \leq D(s)e^{-\lambda(t-s)}, \quad \text{for } t \geq s, \tag{2.3}$$

and

$$\|T(t,s)Q_s\| \leq D(s)e^{-\lambda(s-t)}, \quad \text{for } t \leq s, \tag{2.4}$$

where $Q_t = \mathrm{Id}_n - P_t$ for each $t$ (here $\mathrm{Id}_n$ is the identity on $\mathbb{R}^n$). The sets $P_s(\mathbb{R}^n)$ and $Q_s(\mathbb{R}^n)$ are called, respectively, *stable* and *unstable spaces* at time $s$. We note that

$$P_t = T(t,s)P_sT(s,t)$$

and so in particular $P_t = T(t,0)PT(0,t)$, where $P = P_0$. This shows that all the projections $P_t$ are determined by the projection at time 0.

Now we consider a block upper triangular equation (1.4), where $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$ for some integer $k \in (0,n)$. We write the corresponding evolution family in the form

$$T(t,s) = \begin{pmatrix} U(t,s) & W(t,s) \\ 0 & V(t,s) \end{pmatrix},$$

where $U(t,s)$ and $V(t,s)$ are the evolution families associated, respectively, with the equations

$$x' = A(t)x \quad \text{and} \quad y' = B(t)y. \tag{2.5}$$

It follows readily from the variation of constants formula that

$$W(t,s) = \int_s^t U(t,\tau)C(\tau)V(\tau,s)\,d\tau. \tag{2.6}$$

We first show that the existence of a tempered exponential dichotomy for equation (1.4) yields the existence of tempered exponential dichotomies for the equations associated with the blocks on the diagonal.

**Theorem 2.1.** *Assume that equation (1.4) has a tempered exponential dichotomy on $I = \mathbb{R}_0^+$ or $I = \mathbb{R}_0^-$ with constant $\lambda$. Then the equations (2.5) have tempered exponential dichotomies on $I$ with the same constant $\lambda$. Moreover, the projection $P_0$ associated with the tempered exponential dichotomy for equation (1.4) can be written in the form*

$$\begin{pmatrix} P^A & LP^B \\ 0 & P^B \end{pmatrix} \quad \text{if } I = \mathbb{R}_0^+ \tag{2.7}$$

*and*

$$\begin{pmatrix} P^A & L(\mathrm{Id}_{n-k} - P^B) \\ 0 & P^B \end{pmatrix} \quad \text{if } I = \mathbb{R}_0^-, \tag{2.8}$$

*where $P^A\colon \mathbb{R}^k \to \mathbb{R}^k$ and $P^B\colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ are, respectively, the projections at time 0 associated with the tempered exponential dichotomies for the equations in (2.5), and where $L\colon P^B(\mathbb{R}^{n-k}) \to P^A(\mathbb{R}^k)^\perp$ is the linear map given by*

$$Lv = -\int_0^{+\infty} (\mathrm{Id}_k - P^A)U(0,s)C(s)V(s,0)v\,ds \quad \text{if } I = \mathbb{R}_0^+ \tag{2.9}$$

*and $L\colon \ker P^B \to (\ker P^A)^\perp$ is the linear map given by*

$$Lv = -\int_{-\infty}^0 P^A U(0,s)C(s)V(s,0)v\,ds \quad \text{if } I = \mathbb{R}_0^-. \tag{2.10}$$

*Proof.* We start with an auxiliary result for $I = \mathbb{R}_0^+$. Let $U(t,s)$ be the evolution family associated with the equation $x' = A(t)x$.

**Lemma 2.2.** *Assume that the equation $x' = A(t)x$ on $\mathbb{R}^k$ has a tempered exponential dichotomy on $\mathbb{R}_0^+$ with constant $\lambda$ and projections $P_t$. Moreover, let $\overline{P}_t$ be another family of projections such that*

$$\overline{P}_t U(t,s) = U(t,s)\overline{P}_s, \quad \text{for } t,s \geq 0. \tag{2.11}$$

*Then the equation $x' = A(t)x$ has a tempered exponential dichotomy with projections $\overline{P}_t$ if and only if $P_0(\mathbb{R}^k) = \overline{P}_0(\mathbb{R}^k)$, in which case the equation has a tempered exponential dichotomy on $\mathbb{R}_0^+$ with projections $\overline{P}_t$, constant $\lambda$ and function $\overline{D}$ given by*

$$\overline{D}(s) = D(s) + D(0)D(s)\|P_0 - \overline{P}_0\|. \tag{2.12}$$

*Proof of the lemma.* To the possible extent we follow similar arguments in [3] for uniform exponential dichotomies. One can easily verify that if the equation has a tempered exponential dichotomy on $\mathbb{R}_0^+$ with projections $\overline{P}_t$, then

$$\overline{P}_0(\mathbb{R}^k) = \left\{ v \in \mathbb{R}^k : \sup_{t\geq 0} \|T(t,0)v\| < +\infty \right\} = P_0(\mathbb{R}^k)$$

(in other words, the stable space is uniquely determined and coincides with the set of all initial conditions leading to bounded solutions). Now assume that $\overline{P}_0(\mathbb{R}^k) = P_0(\mathbb{R}^k)$. Then

$$P_0\overline{P}_0 = \overline{P}_0 \quad \text{and} \quad \overline{P}_0 P_0 = P_0,$$

which implies that

$$P_0 - \overline{P}_0 = P_0(P_0 - \overline{P}_0) = (P_0 - \overline{P}_0)Q_0. \tag{2.13}$$

It follows from the existence of a tempered exponential dichotomy for the equation $x' = A(t)x$ that

$$\begin{aligned}
\|U(t,0)(P_0 - \overline{P}_0)v\| &= \|U(t,0)P_0(P_0 - \overline{P}_0)v\| \\
&\leq D(0)e^{-\lambda t}\|(P_0 - \overline{P}_0)v\| \\
&= D(0)e^{-\lambda t}\|(P_0 - \overline{P}_0)Q_0 v\| \\
&\leq D(0)e^{-\lambda t}\|P_0 - \overline{P}_0\| \cdot \|Q_0 v\| \\
&= D(0)e^{-\lambda t}\|P_0 - \overline{P}_0\| \cdot \|U(0,s)U(s,0)Q_0 v\| \\
&= D(0)e^{-\lambda t}\|P_0 - \overline{P}_0\| \cdot \|U(0,s)Q_s U(s,0)v\| \\
&\leq D(0)D(s)e^{-\lambda(s+t)}\|P_0 - \overline{P}_0\| \cdot \|U(s,0)v\|
\end{aligned} \tag{2.14}$$

for $t, s \geq 0$ and $v \in \mathbb{R}^k$. Therefore,

$$\begin{aligned}
\|U(t,s)\overline{P}_s v\| &\leq \|U(t,s)P_s v\| + \|U(t,s)(P_s - \overline{P}_s)v\| \\
&= \|U(t,s)P_s v\| + \|U(t,0)(P_0 - \overline{P}_0)U(0,s)v\| \\
&\leq D(s)e^{-\lambda(t-s)}\|v\| + D(0)D(s)e^{-\lambda(t-s)}\|P_0 - \overline{P}_0\| \cdot \|v\| \\
&= \overline{D}(s)e^{-\lambda(t-s)}\|v\|
\end{aligned}$$

whenever $t \geq s \geq 0$, with $\overline{D}$ as in (2.12). Similarly, letting $\overline{Q}_t = \mathrm{Id}_k - \overline{P}_t$ we obtain

$$\begin{aligned}
\|U(t,s)\overline{Q}_t v\| &\leq \|U(t,s)Q_s v\| + \|U(t,s)(P_s - \overline{P}_s)v\| \\
&= \|U(t,s)Q_s v\| + \|U(t,0)(P_0 - \overline{P}_0)U(0,s)v\| \\
&\leq D(s)e^{-\lambda(t-s)}\|v\| + D(0)D(s)e^{-\lambda(t-s)}\|P_0 - \overline{P}_0\| \cdot \|v\| \\
&= \overline{D}(s)e^{-\lambda(t-s)}\|v\|
\end{aligned} \tag{2.15}$$

whenever $s \geq t \geq 0$. This shows that the equation $x' = A(t)x$ has a tempered exponential dichotomy with projections $\overline{P}_t$. □

One can readily obtain a corresponding version of Lemma 2.2 for $I = \mathbb{R}_0^-$.

We proceed with the proof of the theorem. We first show that the equation $x' = A(t)x$ has a tempered exponential dichotomy on the interval $I$. Let $E_1 \subset \mathbb{R}^k$ be the vector space of all initial conditions at time 0 for which the solutions of $x' = A(t)x$ are bounded on $I$ and let

$E_2$ be any complement of $E_1$ in $\mathbb{R}^k$. Moreover, let $F_1 \subset \mathbb{R}^{n-k}$ be the vector space of all initial conditions $v$ at time 0 for which $V(t,0)v$ is bounded on $I$ and the equation

$$x' = A(t)x + C(t)V(t,0)v \tag{2.16}$$

has a bounded solution on the interval $I$. Finally, let $F_2$ be any complement of $F_1$ in $\mathbb{R}^{n-k}$.

We first show that given $v \in F_1$, there exists a unique bounded solution $x_v$ of equation (2.16) on $I$ with $x_v(0) \in E_2$. Let $x$ be a bounded solution of equation (2.16). We note that $\bar{x}$ is another bounded solution of (2.16) (for the same $v$) if and only if $x - \bar{x}$ is a bounded solution of $x' = A(t)x$, that is, if and only if $x(0) - \bar{x}(0) \in E_1$. This aside remark can be used to establish the existence and uniqueness of $x_v$, as follows.

Given bounded solutions $x$ and $\bar{x}$ of equation (2.16) with $x(0), \bar{x}(0) \in E_2$, it follows from the remark that $x - \bar{x}$ is a bounded solution of $x' = A(t)x$ with $x(0) - \bar{x}(0) \in E_2$. By the choice of $E_1$ and $E_2$, this yields that $x(0) = \bar{x}(0)$ and so $x = \bar{x}$. For the existence we take a bounded solution $x$ of equation (2.16) with $x(0) = u_1 + u_2$, where $u_1 \in E_1$ and $u_2 \in E_2$ (it exists since $v \in F_1$). Then for the solution $\bar{x}$ of equation (2.16) with $\bar{x}(0) = u_2 \in E_2$ we have $x(0) - \bar{x}(0) = u_1 \in E_1$ and so, by the remark, we conclude that $\bar{x}$ is bounded.

Using the solution $x_v$ we define a linear operator $L\colon F_1 \to E_2$ by $Lv = x_v(0)$. We note that $(u, v)$ is the initial condition of a bounded solution of equation (1.4) on $I$ if and only if

$$u - Lv \in E_1 \quad \text{and} \quad v \in F_1. \tag{2.17}$$

Moreover, for $I = \mathbb{R}_0^+$, let $P^A\colon \mathbb{R}^k \to \mathbb{R}^k$ and $Q\colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be, respectively, the projections onto the first components of the splittings $E_1 \oplus E_2$ and $F_1 \oplus F_2$. Finally, for $I = \mathbb{R}_0^-$, let $P^A\colon \mathbb{R}^k \to \mathbb{R}^k$ and $Q\colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be, respectively, the projections onto the second components of the splittings $E_1 \oplus E_2$ and $F_1 \oplus F_2$.

Now we consider the projection $\overline{P}$ given by

$$\overline{P} = \begin{pmatrix} P^A & LQ \\ 0 & Q \end{pmatrix} \quad \text{if } I = \mathbb{R}_0^+$$

and

$$\overline{P} = \begin{pmatrix} P^A & L(\mathrm{Id}_{n-k} - Q) \\ 0 & Q \end{pmatrix} \quad \text{if } I = \mathbb{R}_0^-.$$

It follows readily from the characterization of the initial conditions of the bounded solutions of equation (1.4) in (2.17) that $\overline{P}(\mathbb{R}^n)$ (both when $I = \mathbb{R}_0^+$ and $I = \mathbb{R}_0^-$) is the vector space of all initial conditions at time 0 leading to bounded solutions. In short, $\overline{P}(\mathbb{R}^n) = P_0(\mathbb{R}^n)$, where $P_t$ are the original projections with respect to which equation (1.4) has a tempered exponential dichotomy. It follows from Lemma 2.2 (and its corresponding version when $I = \mathbb{R}_0^-$) that equation (1.4) has a tempered exponential dichotomy on $I$ with respect to the projections

$$\overline{P}_t = T(t,0)\overline{P}T(0,t), \tag{2.18}$$

where $T(t,s)$ is the evolution family associated with equation (1.4), with constant $\lambda$ and function $\overline{D}$.

Let $\overline{Q}_s = \mathrm{Id}_n - \overline{P}_s$. For each $u \in \mathbb{R}^k$ and $s \in I$ we have

$$\overline{P}_s(u,0) = (P_s^A u, 0) \quad \text{and} \quad \overline{Q}_s(u,0) = (Q_s^A u, 0), \tag{2.19}$$

where

$$P_s^A = U(s,0)P^A U(0,s) \quad \text{and} \quad Q_s^A = \mathrm{Id}_k - P_s^A.$$

Therefore, for each $u \in \mathbb{R}^k$ and $t \geq s$ with $t, s \in I$ we have

$$\begin{aligned} \|U(t,s)P_s^A u\| &= \|T(t,s)(P_s^A u, 0)\| = \|T(t,s)\overline{P}_s(u,0)\| \\ &\leq \overline{D}(s)e^{-\lambda(t-s)}\|(u,0)\| = \overline{D}(s)e^{-\lambda(t-s)}\|u\|, \end{aligned} \tag{2.20}$$

using the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$ for $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^{n-k}$. Similarly, for each $u \in \mathbb{R}^k$ and $t \leq s$ with $t, s \in I$ we have

$$\begin{aligned} \|U(t,s)Q_s^A u\| &= \|T(t,s)Q_s^A(u,0)\| = \|T(t,s)\overline{Q}_s(u,0)\| \\ &\leq \overline{D}(s)e^{-\lambda(s-t)}\|(u,0)\| = \overline{D}(s)e^{-\lambda(s-t)}\|u\|. \end{aligned} \tag{2.21}$$

This shows that the equation $x' = A(t)x$ has a tempered exponential dichotomy on $I$ with projections $P_t^A$.

Before showing that the equation $y' = B(t)y$ has a tempered exponential dichotomy we obtain identity (2.9) for $v \in F_1$. By the variation of constants formula, for $t \geq 0$ and $v \in F_1$ we have

$$P_t^A x_v(t) = U(t,0)P_0^A x_v(0) + \int_0^t U(t,\tau)P_\tau^A C(\tau)V(\tau,0)v\,d\tau$$

and

$$Q_t^A x_v(t) = U(t,0)Q_0^A x_v(0) + \int_0^t U(t,\tau)Q_\tau^A C(\tau)V(\tau,0)v\,d\tau.$$

The last identity is equivalent to

$$Q_0^A x_v(0) = U(0,t)Q_t^A x_v(t) - \int_0^t U(0,\tau)Q_\tau^A C(\tau)V(\tau,0)v\,d\tau. \tag{2.22}$$

Since the function $x_v$ is bounded, we have $C = \sup_{t\geq 0}\|x_v(t)\| < +\infty$ and

$$\|U(0,t)Q_t^A x_v(t)\| \leq CDe^{-\lambda t}.$$

Hence, taking limits in (2.22) when $t \to +\infty$ we obtain

$$Q_0^A x_v(0) = -\int_0^{+\infty} U(0,\tau)Q_\tau^A C(\tau)V(\tau,0)v\,d\tau.$$

Recall that by construction we have $x_v(0) \in E_2$ and so $Q_0^A x_v(0) = x_v(0)$. Therefore,

$$Lv = x_v(0) = -\int_0^{+\infty} U(0,\tau)Q_\tau^A C(\tau)V(\tau,0)v\,d\tau, \tag{2.23}$$

which establishes identity (2.9). Identity (2.10) can be obtained in a similar manner.

Finally, we show that the equation $y' = B(t)y$ has a tempered exponential dichotomy on $I$. Consider the adjoint equation

$$\begin{aligned} x' &= -A(t)^* x \\ y' &= -C(t)^* x - B(t)^* y \end{aligned} \tag{2.24}$$

and write it in the form

$$\begin{aligned} z' &= -B(t)^* z - C(t)^* w, \\ w' &= -A(t)^* w, \end{aligned} \tag{2.25}$$

taking $(z,w) = (y,x)$. One knows from the theory that the adjoint equation (2.24) also has a tempered exponential dichotomy on $I$, with the same constant $\lambda$ and function $\overline{D}$, and with projections $\mathrm{Id}_n - \overline{P}_t^*$ (see (2.18)). This readily implies that equation (2.25) has a tempered exponential dichotomy on $I$ with projection at time 0 given by

$$\begin{pmatrix} \mathrm{Id}_{n-k} - Q^* & -Q^* L^* \\ 0 & \mathrm{Id}_k - (P^A)^* \end{pmatrix} \quad \text{if } I = \mathbb{R}_0^+$$

and

$$\begin{pmatrix} \mathrm{Id}_{n-k} - Q^* & (\mathrm{Id}_{n-k} - Q^*)L^* \\ 0 & \mathrm{Id}_k - (P^A)^* \end{pmatrix} \quad \text{if } I = \mathbb{R}_0^-,$$

with the same constant $\lambda$ and function $\overline{D}$. Thus, one can proceed as in (2.19), (2.20) and (2.21) to conclude that the equation $y' = -B(t)^* y$ has a tempered exponential dichotomy on $I$ with projections at time 0 equal to $\mathrm{Id}_{n-k} - Q^*$, with the same data. This implies that the equation $y' = B(t)y$ has also a tempered exponential dichotomy, with projection at time 0 equal to $P^B = Q$, thus leading to the projections in (2.7) and (2.8). In particular, identity (2.23) holds for

$$v \in F_1 = Q(\mathbb{R}^{n-k}) = P^B(\mathbb{R}^{n-k}),$$

with a corresponding remark for $I = \mathbb{R}_0^-$. □

Our second result gives a suficient condition for the converse of the statement in Theorem 2.1.

**Theorem 2.3.** *Assume that the equations $x' = A(t)x$ and $y' = B(t)y$ have tempered exponential dichotomies on $I = \mathbb{R}_0^+$ or $I = \mathbb{R}_0^-$ with constant $\lambda$, function $D$ and, respectively, projections $P^A$ and $P^B$ at time 0. If*

$$\limsup_{|t| \to +\infty} \frac{1}{t} \log \big( D(t) \| C(t) \| \big) \le 0, \tag{2.26}$$

*then equation (1.4) has a tempered exponential dichotomy on $I$ with any constant less than $\lambda$.*

*Proof.* Consider the projections

$$P_t = T(t,0) P T(0,t),$$

with $P$ as in (2.7) or (2.8), respectively, when $I = \mathbb{R}_0^+$ or $I = \mathbb{R}_0^-$. We claim that

$$P_t = \begin{pmatrix} P_t^A & R(t) \\ 0 & P_t^B \end{pmatrix},$$

where

$$P_t^A = U(t,0) P^A U(0,t) \quad \text{and} \quad P_t^B = V(t,0) P^B V(0,t),$$

with

$$\begin{aligned} R(t) = &- \int_0^t U(t,\tau) P_\tau^A C(\tau) (\mathrm{Id}_{n-k} - P_\tau^B) V(\tau,t)\, d\tau \\ &- \int_t^{+\infty} U(t,\tau) (\mathrm{Id}_k - P_\tau^A) C(\tau) P_\tau^B V(\tau,t)\, d\tau \end{aligned} \tag{2.27}$$

when $I = \mathbb{R}_0^+$ and

$$\begin{aligned} R(t) = &- \int_{-\infty}^t U(t,\tau) P_\tau^A C(\tau) (\mathrm{Id}_{n-k} - P_\tau^B) V(\tau,t)\, d\tau \\ &- \int_t^0 U(t,\tau) (\mathrm{Id}_k - P_\tau^A) C(\tau) P_\tau^B V(\tau,t)\, d\tau \end{aligned} \tag{2.28}$$

when $I = \mathbb{R}_0^-$. Clearly,

$$R(0) = \begin{cases} LP^B & \text{if } I = \mathbb{R}_0^+, \\ L(\mathrm{Id}_{n-k} - P^B) & \text{if } I = \mathbb{R}_0^-. \end{cases} \tag{2.29}$$

Identities (2.27) and (2.28) can be established as follows. For $I = \mathbb{R}_0^+$ it follows readily from (2.7) that

$$\begin{aligned} P_t &= T(t,0) \begin{pmatrix} P^A & LP^B \\ 0 & P^B \end{pmatrix} T(0,t) \\ &= \begin{pmatrix} U(t,0) & W(t,0) \\ 0 & V(t,0) \end{pmatrix} \begin{pmatrix} P^A & LP^B \\ 0 & P^B \end{pmatrix} \begin{pmatrix} U(0,t) & W(0,t) \\ 0 & V(0,t) \end{pmatrix} \\ &= \begin{pmatrix} P_t^A & P_t^A U(t,0)W(0,t) + [U(t,0)L + W(t,0)]V(0,t)P_t^B \\ 0 & P_t^B \end{pmatrix}. \end{aligned}$$

Using (2.6) we find that

$$\begin{aligned} R(t) &= P_t^A U(t,0)W(0,t) + [U(t,0)L + W(t,0)]V(0,t)P_t^B \\ &= U(t,0)R(0)V(0,t) + \int_0^t U(t,\tau)\left[C(\tau)P_\tau^B - P_\tau^A C(\tau)\right]V(\tau,t)\,d\tau. \end{aligned}$$

Using (2.8) one can readily obtain a corresponding property when $I = \mathbb{R}_0^-$. Identities (2.27) and (2.28) follow now in a straightforward manner from (2.9) and (2.10) together with (2.29).

We use the former identities to show that the equation $x' = A(t)x$ has a tempered exponential dichotomy on $I$ with projections $P_t$. First we show that (2.3) holds for $t \geq s$ with $t, s \in I$, for some constant $\lambda$ and some function $D$ satisfying (2.2). Note that

$$T(t,s)P_s = \begin{pmatrix} U(t,s)P_s^A & W(t,s)P_s^B + U(t,s)R(s) \\ 0 & V(t,s)P_s^B \end{pmatrix}. \tag{2.30}$$

We have

$$\begin{aligned} W(t,s)P_s^B &= \int_s^t U(t,\tau)P_\tau^A C(\tau)P_\tau^B V(\tau,s)\,d\tau \\ &\quad + \int_s^t U(t,\tau)(\mathrm{Id}_k - P_\tau^A)C(\tau)P_\tau^B V(\tau,s)\,d\tau. \end{aligned}$$

When $I = \mathbb{R}_0^+$ we obtain

$$\begin{aligned} U(t,s)R(s) &= -\int_0^s U(t,\tau)P_\tau^A C(\tau)(\mathrm{Id}_{n-k} - P_\tau^B)V(\tau,s)\,d\tau \\ &\quad - \int_s^{+\infty} U(t,\tau)(\mathrm{Id}_k - P_\tau^A)C(\tau)P_\tau^B V(\tau,s)\,d\tau \end{aligned}$$

and hence,

$$\begin{aligned} W(t,s)P_s^B + U(t,s)R(s) &= \int_s^t U(t,\tau)P_\tau^A C(\tau)P_\tau^B V(\tau,s)\,d\tau \\ &\quad - \int_t^{+\infty} U(t,\tau)(\mathrm{Id}_k - P_\tau^A)C(\tau)P_\tau^B V(\tau,s)\,d\tau \\ &\quad - \int_0^s U(t,\tau)P_\tau^A C(\tau)(\mathrm{Id}_{n-k} - P_\tau^B)V(\tau,s)\,d\tau. \end{aligned} \tag{2.31}$$

Similarly, when $I = \mathbb{R}_0^-$ we obtain

$$\begin{aligned} W(t,s)P_s^B + U(t,s)R(s) = &\int_s^t U(t,\tau)P_\tau^A C(\tau)P_\tau^B V(\tau,s)\,d\tau \\ &- \int_t^0 U(t,\tau)(\mathrm{Id}_k - P_\tau^A)C(\tau)P_\tau^B V(\tau,s)\,d\tau \\ &- \int_{-\infty}^s U(t,\tau)P_\tau^A C(\tau)(\mathrm{Id}_{n-k} - P_\tau^B)V(\tau,s)\,d\tau. \end{aligned}$$

By (2.26), given $\varepsilon > 0$, there exists $d > 0$ such that

$$D(t)\|C(t)\| \le de^{\varepsilon|t|}, \quad \text{for } t \in I. \tag{2.32}$$

Hence, whenever $t \ge s \ge 0$ we have

$$\begin{aligned} \|W(t,s)P_s^B + U(t,s)R(s)\| \le &\int_s^t D(\tau)e^{-\lambda(t-\tau)}\|C(\tau)\|D(s)e^{-\lambda(\tau-s)}\,d\tau \\ &+ \int_t^{+\infty} D(\tau)e^{-\lambda(\tau-t)}\|C(\tau)\|D(s)e^{-\lambda(\tau-s)}\,d\tau \\ &+ \int_0^s D(\tau)e^{-\lambda(t-\tau)}\|C(\tau)\|D(s)e^{-\lambda(s-\tau)}\,d\tau \\ \le &\, dD(s)\left(ce^{-\mu(t-s)} + \frac{e^{-(\lambda-\varepsilon)(t-s)+\varepsilon s}}{2\lambda-\varepsilon} + \frac{e^{-(\lambda-\varepsilon)(t-s)+\varepsilon s}}{2\lambda-\varepsilon}\right) \\ \le &\, dD(s)e^{\varepsilon s}e^{-\mu(t-s)}\left(c + \frac{2}{2\lambda-\varepsilon}\right), \end{aligned} \tag{2.33}$$

for some constants $c > 0$ and $\mu < \lambda$ (independent of $\varepsilon$) provided that $\varepsilon$ is sufficiently small so that $\mu < \lambda - \varepsilon$. The first term follows from noting that

$$\begin{aligned} \int_s^t e^{\varepsilon\tau}e^{-\lambda(t-\tau)}e^{-\lambda(\tau-s)}\,d\tau &= \int_s^t e^{\varepsilon\tau}e^{-\lambda(t-s)}\,d\tau \\ &\le (t-s)e^{\varepsilon t}e^{-\lambda(t-s)} \le ce^{-\mu(t-s)+\varepsilon s}, \end{aligned}$$

for some constants as above. It follows from (2.2) and (2.33) that

$$\limsup_{s\to+\infty} \frac{1}{s}\log\left(\|W(t,s)P_s^B + U(t,s)R(s)\|e^{\mu(t-s)}\right) \le \varepsilon. \tag{2.34}$$

In view of identity (2.30), it follows from (2.34) and the arbitrariness of $\varepsilon$ that (2.3) holds whenever $t \ge s \ge 0$ with $\lambda$ replaced by $\mu$ and $D$ replaced by the function

$$s \mapsto \sup_{t\ge s}\left(\|W(t,s)P_s^B + U(t,s)R(s)\|e^{\mu(t-s)}\right).$$

The case when $I = \mathbb{R}_0^-$ can be treated similarly (it only requires replacing in the integrals the lower limit 0 by $-\infty$ and the upper limit $+\infty$ by 0).

Now we show that (2.4) holds for $t \le s$ with $t, s \in I$, for some constant $\lambda$ and some function $D$ satisfying (2.2). First note that

$$T(t,s)Q_s = \begin{pmatrix} (\mathrm{Id}_k - P_t^A)U(t,s) & (\mathrm{Id}_k - P_t^A)W(t,s) - R(t)V(t,s) \\ 0 & (\mathrm{Id}_{n-k} - P_t^B)V(t,s) \end{pmatrix}.$$

Again, for simplicity of the exposition, we consider only $I = \mathbb{R}_0^+$. We have

$$R(t)V(t,s) = -\int_0^t U(t,\tau)P_\tau^A C(\tau)(\mathrm{Id}_{n-k} - P_\tau^B)V(\tau,s)\,d\tau \\ -\int_t^{+\infty} (\mathrm{Id}_k - P_\tau^A)C(\tau)P_\tau^B V(\tau,s)\,d\tau$$

and

$$(\mathrm{Id}_k - P_t^A)W(t,s) = \int_s^t U(t,\tau)(\mathrm{Id}_k - P_\tau^A)C(\tau)V(\tau,s)\,d\tau.$$

Hence,

$$\begin{aligned}(\mathrm{Id}_k - P_t^A)W(t,s) - R(t)V(t,s) = &\int_s^{+\infty} U(t,\tau)(\mathrm{Id}_k - P_\tau^A)C(\tau)P_\tau^B V(\tau,s)\,d\tau \\ &- \int_s^t U(t,\tau)(\mathrm{Id}_k - P_\tau^A)C(\tau)(\mathrm{Id}_{n-k} - P_\tau^B)V(\tau,s)\,d\tau \\ &+ \int_0^t U(t,\tau)P_\tau^A C(\tau)(\mathrm{Id}_{n-k} - P_\tau^B)V(\tau,s)\,d\tau,\end{aligned}$$

which implies that whenever $0 \le t \le s$ we have

$$\begin{aligned}&\|(\mathrm{Id}_k - P_t^A)W(t,s) - R(t)V(t,s)\| \\ &\quad\le dD(s)e^{\varepsilon s}\left(\frac{e^{(\lambda-\varepsilon)(t-s)+\varepsilon s}}{2\lambda-\varepsilon} + ce^{\mu(t-s)} + \frac{e^{(\lambda-\varepsilon)(t-s)+\varepsilon s}}{2\lambda-\varepsilon}\right) \\ &\quad\le dD(s)e^{\varepsilon s}\left(\frac{2}{2\lambda-\varepsilon} + c\right)e^{\lambda(t-s)}\end{aligned} \tag{2.35}$$

for some constants $c > 0$ and $\mu < \lambda$ (independent of $\varepsilon$) provided that $\varepsilon$ is sufficiently small so that $\mu < \lambda - \varepsilon$. Proceeding as in (2.34), it follows from (2.35) that (2.4) holds whenever $0 \le t \le s$, with $\lambda$ replaced by $\mu$ and $D$ replaced by some other function. This completes the proof of the theorem. □

Now we give a counterexample to the converse of Theorem 2.3 when condition (2.26) is not satisfied.

**Example 2.4.** Consider the triangular equation

$$x' = 2x + ye^{at}, \qquad y' = -2y \tag{2.36}$$

with $a > 0$. Both linear equations

$$x' = 2x, \qquad y' = -2y$$

have a tempered exponential dichotomy on $\mathbb{R}_0^+$ with constant function $D$ and projections, respectively, $P^A = 0$ and $P^B = 1$. Since $C(t) = e^{at}$, we have

$$\limsup_{t\to+\infty} \frac{1}{t}\log(D(t)\|C(t)\|) = \limsup_{t\to+\infty} \frac{1}{t}e^{at} = a > 0$$

and so (2.26) is not satisfied. Now we show that the triangular equation (2.36) has no tempered exponential dichotomy on $\mathbb{R}_0^+$. The solutions are

$$x(t) = \left(x(0) - \frac{y(0)}{a-3}\right)e^{2t} + y(0)e^{(a-2)t}a - 3, \qquad y(t) = y(0)e^{-2t}$$

if $a \neq 3$ and

$$x(t) = x(0)e^t + y(0)te^t, \qquad y(t) = y(0)e^{-2t}$$

if $a = 3$. It follows from the proof of Theorem 2.1 that if equation (2.36) has a tempered exponential dichotomy, then the projection $P$ has rank 1. In view of the first component this happens if and only if $a \leq 2$ and the initial condition is a scalar multiple of $(1, a-3)$. On the other hand, by Theorem 2.1 we have $P = \left(\begin{smallmatrix} 0 & c \\ 0 & 1 \end{smallmatrix}\right)$ for some $c \in \mathbb{R}$. For $a \leq 2$ the matrix $U(t) = T(t, 0)$ associated with (2.36) is

$$U(t) = \begin{pmatrix} e^{2t} & \frac{e^{2t} - e^{(a-2)t}}{3-a} \\ 0 & e^{-2t} \end{pmatrix}$$

and so

$$U(t)PU(t)^{-1} = \begin{pmatrix} 0 & \frac{[(3-a)c+1]e^{4t} - e^{at}}{3-a} \\ 0 & 1 \end{pmatrix}.$$

Clearly,

$$\liminf_{t \to +\infty} \frac{1}{t} \log \left| [(3-a)c + 1]e^{4t} - e^{at} \right| > 0$$

for all $a \in (0, 2]$ and so, for $a > 0$ equation (2.36) does not have a tempered exponential dichotomy on $\mathbb{R}_0^+$.

## 3 Strong tempered exponential dichotomies

We say that equation (1.1) has a *strong tempered exponential dichotomy* on an interval $I$ if there exist projections $P_t$ for $t \in I$ satisfying (2.1) and there exist constants $\mu > \lambda > 0$ and a function $D \colon I \to \mathbb{R}^+$ satisfying (2.2) such that

$$\|T(t,s)P_s\| \leq D(s)e^{-\lambda(t-s)}, \qquad \|T(s,t)P_t\| \leq D(t)e^{\mu(t-s)} \tag{3.1}$$

and

$$\|T(s,t)Q_t\| \leq D(t)e^{-\lambda(t-s)}, \qquad \|T(t,s)Q_s\| \leq D(s)e^{\mu(t-s)} \tag{3.2}$$

for $t \geq s$, where $Q_t = \mathrm{Id}_n - P_t$ for each $t$.

**Theorem 3.1.** *Assume that equation (1.4) has a strong tempered exponential dichotomy on $I = \mathbb{R}_0^+$ or $I = \mathbb{R}_0^-$ with constants $\lambda$ and $\mu$. Then the equations $x' = A(t)x$ and $y' = B(t)y$ have strong tempered exponential dichotomies on $I$ with the same constants $\lambda$ and $\mu$.*

*Proof.* For simplicity of the exposition we consider only the case of $I = \mathbb{R}_0^+$. Let $U(t,s)$ be the evolution family associated with the equation $x' = A(t)x$.

**Lemma 3.2.** *Assume that the equation $x' = A(t)x$ on $\mathbb{R}^k$ has a strong tempered exponential dichotomy on $\mathbb{R}_0^+$ with constants $\lambda, \mu$ and projections $P_t$. Moreover, let $\overline{P}_t$ be another family of projections satisfying (2.11). Then the equation $x' = A(t)x$ has a strong tempered exponential dichotomy with projections $\overline{P}_t$ if and only if $P_0(\mathbb{R}^k) = \overline{P}_0(\mathbb{R}^k)$, in which case the equation has a strong tempered exponential dichotomy on $\mathbb{R}_0^+$ with projections $\overline{P}_t$, constants $\lambda, \mu$ and function $\overline{D}$ given by (2.12).*

*Proof of the lemma.* The lemma can be established along the lines of the proof of Lemma 2.2 and so we only outline the differences. Namely, we only need to obtain the second bounds in (3.1) and (3.2). Proceeding as in the proof of Lemma 2.2 (see (2.14)) we have

$$\begin{aligned}\|U(s,t)\overline{P}_t v\| &\le \|U(s,t)P_t v\| + \|U(s,t)(P_t - \overline{P}_t)v\| \\ &= \|U(s,t)P_t v\| + \|U(s,0)(P_0 - \overline{P}_0)U(0,t)v\| \\ &\le D(t)e^{\mu(t-s)}\|v\| + D(0)D(t)e^{-\lambda(t+s)}\|P_0 - \overline{P}_0\| \cdot \|v\| \\ &= \overline{D}(t)e^{\mu(t-s)}\|v\|\end{aligned}$$

for $t \ge s$ and $v \in \mathbb{R}^k$, with $\overline{D}$ as in (2.12). Similarly, letting $\overline{Q}_t = \mathrm{Id}_k - \overline{P}_t$ we obtain

$$\begin{aligned}\|U(s,t)\overline{Q}_t v\| &\le \|U(s,t)Q_t v\| + \|U(s,t)(P_t - \overline{P}_t)v\| \\ &= \|U(s,t)Q_t v\| + \|U(s,0)(P_0 - \overline{P}_0)U(0,t)v\| \\ &\le D(t)e^{\mu(t-s)}\|v\| + D(0)D(t)e^{\lambda(t+s)}\|P_0 - \overline{P}_0\| \cdot \|v\| \\ &= \overline{D}(t)e^{\mu(t-s)}\|v\|\end{aligned}$$

for $t \le s$. This shows that the equation $x' = A(t)x$ has a strong tempered exponential dichotomy with projections $\overline{P}_t$. □

We proceed with the proof of the theorem. In view of Theorem 2.1 (see (2.20) and (2.21)), for the equation $x' = A(t)x$ it remains to obtain the second bounds in (3.1) and (3.2). For each $u \in \mathbb{R}^k$ and $t \ge s \ge 0$ we have

$$\begin{aligned}\|U(s,t)P_t^A u\| &= \|T(s,t)(P_t^A u, 0)\| = \|T(s,t)\overline{P}_t(u,0)\| \\ &\le \overline{D}(t)e^{\mu(t-s)}\|(u,0)\| = \overline{D}(t)e^{\mu(t-s)}\|u\|,\end{aligned}$$

using again the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$ for $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^{n-k}$. Similarly, for each $u \in \mathbb{R}^k$ and $s \ge t \ge 0$ we have

$$\begin{aligned}\|U(s,t)Q_t^A u\| &= \|T(s,t)Q_t^A(u,0)\| = \|T(s,t)\overline{Q}_t(u,0)\| \\ &\le \overline{D}(t)e^{\mu(s-t)}\|(u,0)\| = \overline{D}(t)e^{\mu(s-t)}\|u\|.\end{aligned}$$

This shows that the equation $x' = A(t)x$ has a strong tempered exponential dichotomy on $\mathbb{R}_0^+$ with projections $P_t^A$. Proceeding analogously with the adjoint equation (see the proof of Theorem 2.1), we conclude that the equation $y' = B(t)y$ has a strong tempered exponential dichotomy on $\mathbb{R}_0^+$. □

We note that the projection $P_0$ associated with the strong tempered exponential dichotomy for equation (1.4) can be written in the form (2.7) and (2.8), respectively, where $P^A$ and $P^B$ are, respectively, the projections at time 0 associated with the strong tempered exponential dichotomies for the equations in (2.5), and where $L$ is the linear map given by (2.9) and (2.10), respectively.

The following result is a partial converse of Theorem 3.1.

**Theorem 3.3.** *Assume that the equations $x' = A(t)x$ and $y' = B(t)y$ have strong tempered exponential dichotomies on $I = \mathbb{R}_0^+$ or $I = \mathbb{R}_0^-$ with constants $\lambda, \mu$, function $D$ and, respectively, projections $P^A$ and $P^B$ at time 0. If condition (2.26) holds, then equation (1.4) has a strong tempered exponential dichotomy on $I$ with any constants, respectively, less than $\lambda$ and greater than $\mu$.*

*Proof.* The statement can be established as in the proof of Theorem 2.3 and so we only outline the differences. For simplicity of the exposition we assume that $I = \mathbb{R}_0^+$ (the case when $I = \mathbb{R}_0^-$ can be treated similarly). Using (2.32), for $t \geq s \geq 0$ the quantity $W(s,t)P_t^B + U(s,t)R(t)$ satisfies (see (2.31)).

It follows in a similar manner as in (2.31) that for $t \geq s$ we have

$$\begin{aligned} W(s,t)P_t^B + U(s,t)R(t) = &-\int_s^t U(s,\tau)P_\tau^A C(\tau)P_\tau^B V(\tau,t)\,d\tau \\ &-\int_s^{+\infty} U(s,\tau)(\mathrm{Id}_k - P_\tau^A)C(\tau)P_\tau^B V(\tau,t)\,d\tau \\ &-\int_0^t U(s,\tau)P_\tau^A C(\tau)(\mathrm{Id}_{n-k} - P_\tau^B)V(\tau,t)\,d\tau. \end{aligned}$$

Now using (3.1) and (3.2) we readily get

$$\begin{aligned} &\|W(s,t)P_t^B + U(s,t)R(t)\| \\ &\quad\leq \int_s^t D(\tau)e^{\mu(\tau-s)}\|C(\tau)\|D(t)e^{\mu(t-\tau)}\,d\tau \\ &\qquad+ \int_s^t D(\tau)e^{-\lambda(\tau-s)}\|C(\tau)\|D(t)e^{\mu(t-\tau)}\,d\tau \\ &\qquad+ \int_t^{+\infty} D(\tau)e^{-\lambda(\tau-s)}\|C(\tau)\|D(t)e^{-\lambda(\tau-t)}\,d\tau \\ &\qquad+ \int_0^s D(\tau)e^{\mu(\tau-s)}\|C(\tau)\|D(t)e^{-\lambda(\tau-t)}\,d\tau \\ &\qquad+ \int_s^t D(\tau)e^{-\lambda(s-\tau)}\|C(\tau)\|D(t)e^{-\lambda(\tau-t)}\,d\tau \\ &\quad\leq dD(t)e^{\mu(t-s)}\int_s^t e^{\varepsilon\tau}\,d\tau + dD(t)e^{\mu t+\lambda s}\int_s^t e^{-(\mu+\lambda)\tau+\varepsilon\tau}\,d\tau \\ &\qquad+dD(t)e^{\lambda(t+s)}\int_t^{+\infty} e^{-2\lambda\tau+\varepsilon\tau}\,d\tau + dD(t)e^{\lambda t-\mu s}\int_0^s e^{(\mu-\lambda)\tau+\varepsilon\tau}\,d\tau \\ &\qquad+dD(t)e^{\lambda(t-s)}\int_s^t e^{\varepsilon\tau}\,d\tau \\ &\quad\leq dD(t)\left(ce^{\nu(t-s)+\varepsilon t} + \frac{e^{\mu(t-s)+\varepsilon s}}{\mu+\lambda-\varepsilon} + \frac{e^{-\lambda(t-s)+\varepsilon t}}{2\lambda-\varepsilon} + \frac{e^{\lambda(t-s)+\varepsilon s}}{\mu-\lambda+\varepsilon} + ce^{\nu(t-s)+\varepsilon t}\right) \\ &\quad\leq dD(t)e^{\varepsilon t}\left(2ce^{\nu(t-s)} + \frac{e^{\mu(t-s)}}{\mu+\lambda-\varepsilon} + \frac{e^{-\lambda(t-s)}}{2\lambda-\varepsilon} + \frac{e^{\lambda(t-s)}}{\mu-\lambda+\varepsilon}\right) \\ &\quad\leq dD(t)e^{\varepsilon t}e^{\nu(t-s)}\left(2c + \frac{2}{2\lambda-\varepsilon} + \frac{1}{\mu-\lambda+\varepsilon}\right), \end{aligned}$$

for some constants $c > 0$ and $\nu > \mu$ (independent of $\varepsilon$) provided that $\varepsilon$ is sufficiently small so that $2\lambda > \varepsilon$.

In a similar manner as we did in (2.35) we get, for $t \geq s$

$$\|(\mathrm{Id}_k - P_t^A)W(t,s) - R(t)V(t,s)\| \leq \tilde{c}D(t)e^{\varepsilon t}e^{\nu(t-s)}$$

for some constants $\tilde{c} > 0$ and $\nu > \mu$ (independent of $\varepsilon$) provided that $\varepsilon$ is sufficiently small. □

## 4 Discrete time dynamics

In this section we obtain corresponding results to those in Section 2 for a dynamics with discrete time. We first introduce the notion of a tempered exponential dichotomy. Let $(A_m)_{m\in I}$ be a sequence of invertible $n \times n$ matrices, where $I = \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^-$. For each $m, l \in I$ we define

$$\mathcal{A}(m,l) = \begin{cases} A_{m-1} \cdots A_l & \text{if } m > l, \\ \text{Id} & \text{if } m = l, \\ A_m^{-1} \cdots A_{l-1}^{-1} & \text{if } m < l. \end{cases}$$

We say that the sequence $(A_m)_{m\in I}$ has a *tempered exponential dichotomy on I* if there exist projections $P_m$ for $m \in I$ satisfying

$$P_m \mathcal{A}(m,l) = \mathcal{A}(m,l) P_l, \quad \text{for } m, l \in I, \tag{4.1}$$

and there exist $\lambda > 0$ and a sequence $(D_m)_{m\in I} \subset \mathbb{R}^+$ satisfying

$$\limsup_{|m|\to+\infty} \frac{1}{|m|} \log D_m \leq 0 \tag{4.2}$$

such that

$$\|\mathcal{A}(m,l)P_l\| \leq D_l e^{-\lambda(m-l)}, \quad \text{for } m \geq l, \tag{4.3}$$

and

$$\|\mathcal{A}(m,l)Q_l\| \leq D e^{-\lambda(l-m)}, \quad \text{for } m \leq l, \tag{4.4}$$

where $Q_m = \text{Id}_n - P_m$ for each $m$ (as before, $\text{Id}_n$ is the identity on $\mathbb{R}^n$). The sets $P_l(\mathbb{R}^n)$ and $Q_l(\mathbb{R}^n)$ are called, respectively, *stable* and *unstable* spaces at time $l$. In a similar manner to that in the case of continuous time, all the projections are determined by the projection at time 0.

Now we consider an upper triangular sequence

$$H_m = \begin{pmatrix} A_m & C_m \\ 0 & B_m \end{pmatrix} \tag{4.5}$$

acting on $\mathbb{R}^k \times \mathbb{R}^{n-k}$ for some integer $k \in (0, n)$. For the dynamics $x_{m+1} = H_m x_m$, we have $x_m = \mathcal{H}(m,l)x_l$ for $m, l \in I$ where

$$\mathcal{H}(m,l) = \begin{pmatrix} \mathcal{A}(m,l) & \mathcal{C}(m,l) \\ 0 & \mathcal{B}(m,l) \end{pmatrix},$$

with

$$\mathcal{A}(m,l) = \begin{cases} A_{m-1} \cdots A_l & \text{if } m > l, \\ \text{Id} & \text{if } m = l, \\ A_m^{-1} \cdots A_{l-1}^{-1} & \text{if } m < l \end{cases} \qquad \mathcal{B}(m,l) = \begin{cases} B_{m-1} \cdots B_l & \text{if } m > l, \\ \text{Id} & \text{if } m = l, \\ B_m^{-1} \cdots B_{l-1}^{-1} & \text{if } m < l \end{cases}$$

and

$$\mathcal{C}(m,l) = \begin{cases} \sum_{j=l}^{m-1} \mathcal{A}(m, j+1) C_j \mathcal{B}(j,l) & \text{if } m > l, \\ 0 & \text{if } m = l, \\ -\sum_{j=m}^{l-1} \mathcal{A}(m, j+1) C_j \mathcal{B}(j,l) & \text{if } m < l. \end{cases} \tag{4.6}$$

In the following result we show that the existence of a tempered exponential dichotomy for the sequence in (4.5) yields the existence of a tempered exponential dichotomies for the sequences $A_m$ and $B_m$.

**Theorem 4.1.** *Assume that the sequence in* (4.5) *has a tempered exponential dichotomy on $I = \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^-$ with constant $\lambda$. Then the sequences $(A_m)_{m\in I}$ and $(B_m)_{m\in I}$ have tempered exponential dichotomies on $I$ with the same constant $\lambda$. Moreover, the projection $P_0$ associated with the tempered exponential dichotomy for the sequence $(H_m)_{m\in I}$ can be written in the form*

$$\begin{pmatrix} P^A & LP^B \\ 0 & P^B \end{pmatrix} \quad \text{if } I = \mathbb{Z}_0^+ \tag{4.7}$$

*and*

$$\begin{pmatrix} P^A & L(\mathrm{Id}_{n-k} - P^B) \\ 0 & P^B \end{pmatrix} \quad \text{if } I = \mathbb{Z}_0^-, \tag{4.8}$$

*where $P^A\colon \mathbb{R}^k \to \mathbb{R}^k$ and $P^B\colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ are, respectively, the projections at time 0 associated with the tempered exponential dichotomies for the sequences $(A_m)_{m\in I}$ and $(B_m)_{m\in I}$, and where $L\colon P^B(\mathbb{R}^{n-k}) \to P^A(\mathbb{R}^k)^\perp$ is the linear map given by*

$$Lv = -\sum_{j=0}^{+\infty} (\mathrm{Id}_k - P^A)\mathcal{A}(0,j)C_j\mathcal{B}(j,0)v \quad \text{if } I = \mathbb{Z}_0^+ \tag{4.9}$$

*and $L\colon \ker P^B \to (\ker P^A)^\perp$ is the linear map given by*

$$Lv = -\sum_{j=-\infty}^{0} P^A\mathcal{A}(0,j)C_j\mathcal{B}(j,0)v \quad \text{if } I = \mathbb{Z}_0^-. \tag{4.10}$$

*Proof.* We start with an auxiliary result that is a discrete-time version of Lemma 2.2.

**Lemma 4.2.** *Assume that the sequence $(A_m)_{m\geq 0}$ of $k\times k$ matrices has a tempered exponential dichotomy on $\mathbb{Z}_0^+$ with projections $P_m$. Moreover, let $\overline{P}_m$ be another family of projections such that*

$$\overline{P}_m\mathcal{A}(m,l) = \mathcal{A}(m,l)\overline{P}_l, \quad \text{for } m,l \in \mathbb{Z}_0^+.$$

*Then the sequence $(A_m)_{m\geq 0}$ has a tempered exponential dichotomy with projections $\overline{P}_m$ if and only if $P_0(\mathbb{R}^k) = \overline{P}_0(\mathbb{R}^k)$, in which case $(A_m)_{m\in I}$ has a tempered exponential dichotomy on $\mathbb{Z}_0^+$ with projections $\overline{P}_m$, constant $\lambda$ and sequence $\overline{D}_m$ given by*

$$\overline{D}_m = D_m + D_0D_m\|P_0 - \overline{P}_0\|. \tag{4.11}$$

*Proof of the lemma.* One can easily verify that if the sequence $(A_m)_{m\in\mathbb{N}}$ has a tempered exponential dichotomy on $\mathbb{Z}_0^+$ with projections $\overline{P}_m$, then

$$\overline{P}_0(\mathbb{R}^k) = \left\{ v \in \mathbb{R}^k : \sup_{m\geq 0}\|\mathcal{A}(m,0)v\| < +\infty \right\} = P_0(\mathbb{R}^k).$$

Now assume that $P_0(\mathbb{R}^k) = \overline{P}_0(\mathbb{R}^k)$. Then (2.13) holds and so it follows from (4.3) and (4.4) that

$$\|\mathcal{A}(m,0)(P_0 - \overline{P}_0)v\| \leq D_0D_le^{-\lambda(l+m)}\|P_0 - \overline{P}_0\| \cdot \|\mathcal{A}(l,0)v\|$$

for $m,l \in \mathbb{Z}_0^+$ and $v \in \mathbb{R}^k$ (see (2.14)). Therefore,

$$\begin{aligned}
\|\mathcal{A}(m,l)\overline{P}_lv\| &\leq \|\mathcal{A}(m,l)P_lv\| + \|\mathcal{A}(m,l)(P_l - \overline{P}_l)v\| \\
&= \|\mathcal{A}(m,l)P_lv\| + \|\mathcal{A}(l,0)(P_0 - \overline{P}_0)\mathcal{A}(0,l)v\| \\
&\leq D_le^{-\lambda(m-l)}\|v\| + D_0D_le^{-\lambda(m-l)}\|P_0 - \overline{P}_0\| \cdot \|v\| \\
&= \overline{D}_le^{-\lambda(m-l)}\|v\|
\end{aligned}$$

for $m \geq l$, with $\overline{D}_l$ given by (4.11). Similarly, letting $\overline{Q}_m = \text{Id}_k - \overline{P}_m$ we obtain

$$\|\mathcal{A}(m,l)\overline{Q}_l v\| \leq \overline{D}_l e^{-\lambda(m-l)}\|v\|$$

for $m \leq l$ (see (2.15)). This shows that the sequence $(A_m)_{m\geq 0}$ has a tempered exponential dichotomy with projections $\overline{P}_m$. □

To establish the statement in the theorem we first show that the sequence $(A_m)_{m\in I}$ has a tempered exponential dichotomy. Let $E_1 \subset \mathbb{R}^k$ be the vector space of all initial conditions at time 0 for which the sequence $x_{m+1} = A_m x_m$ is bounded on $I$ and let $E_2$ be any complement of $E_1$ in $\mathbb{R}^k$. Moreover, let $F_1 \subset \mathbb{R}^{n-k}$ be the vector space of all initial conditions $v$ at time 0 for which the sequence $\mathcal{B}(n,0)v$ is bounded on $I$ and the dynamics

$$x_{m+1} = A_m x_m + C_m \mathcal{B}(m,0)v \tag{4.12}$$

has a bounded solution on $I$. Finally, let $F_2$ be any complement of $F_1$ in the space $\mathbb{R}^{n-k}$.

Given $v \in F_1$, there exists a unique bounded solution $(x_m^v)_{m\in I}$ of (4.12) on $I$ with $x_0^v \in E_2$. Indeed, let $(x_m)_{m\in I}$ be a bounded solution of (4.12). We note that $(\bar{x}_m)_{m\in I}$ is another bounded solution of (4.12) (for the same $v$) if and only if $(x_m - \bar{x}_m)_{m\in I}$ is a bounded solution of $x_{m+1} = A_m x_m$, that is, if and only if $x_0 - \bar{x}_0 \in E_1$. This remark will be used to establish the existence and uniqueness of $x^v$. Given bounded solutions $(x_m)_{m\in I}$ and $(\bar{x}_m)_{m\in I}$ of (4.12) with $x_0, \bar{x}_0 \in E_2$, it follows from the remark that $(x_m - \bar{x}_m)_{m\in I}$ is a bounded solution of $x_{m+1} = A_m x_m$ with $x_0 - \bar{x}_0 \in E_2$. By the choice of $E_1$ and $E_2$, this implies that $x_0 = \bar{x}_0$ and so $x_m = \bar{x}_m$ for all $m \in I$. For the existence we take a bounded solution $(x_m)_{m\in I}$ of (4.12) with $x_0 = u + v$, where $u \in E_1$ and $v \in E_2$ (it exists because $v \in F_1$). Then for the solution $(\bar{x}_m)_{m\in I}$ of (4.12) with $\bar{x}_0 = v \in E_2$ we have $x_0 - \bar{x}_0 = u \in E_1$ and so we conclude by the remark that $(\bar{x}_m)_{m\in I}$ is bounded.

Using the sequence $x^v$ we define a linear map $L\colon F_1 \to E_2$ by $Lv = x_0^v$. We note that $(u,v)$ is the initial condition of a bounded solution of $x_{m+1} = H_m x_m$ on $I$ if and only if

$$u - Lv \in E_1 \quad \text{and} \quad v \in F_1.$$

Moreover, for $I = \mathbb{Z}_0^+$ let $P^A\colon \mathbb{R}^k \to \mathbb{R}^k$ and $Q\colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be, respectively, the projections onto the first components of the splittings $E_1 \oplus E_2$ and $F_1 \oplus F_2$. Finally, for $I = \mathbb{Z}_0^-$ let $P^A\colon \mathbb{R}^k \to \mathbb{R}^k$ and $Q\colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be, respectively, the projections onto the second components of the splittings $E_1 \oplus E_2$ and $F_1 \oplus F_2$.

We also consider the projection $\overline{P}$ given by

$$\overline{P} = \begin{pmatrix} P^A & LQ \\ 0 & Q \end{pmatrix} \quad \text{if } I = \mathbb{Z}_0^+$$

and

$$\overline{P} = \begin{pmatrix} P^A & L(\text{Id}_{n-k} - Q) \\ 0 & Q \end{pmatrix} \quad \text{if } I = \mathbb{Z}_0^-.$$

Then $\overline{P}(\mathbb{R}^n)$ (both when $I = \mathbb{Z}_0^+$ and $I = \mathbb{Z}_0^-$) is the vector space of all initial conditions at time 0 leading to bounded solutions. It follows from Lemma 4.2 (and its corresponding version when $I = \mathbb{Z}_0^-$) that the sequence $(H_m)_{m\in I}$ has a tempered exponential dichotomy on $I$ with respect to the projections

$$\overline{P}_m = \mathcal{H}(m,0)\overline{P}\mathcal{H}(0,m),$$

with constant $\lambda$ and sequence $\overline{D}_m$. One can now proceed as in (2.20) and (2.21) to show that the sequence $(A_m)_{m\in\mathbb{N}}$ has a tempered exponential dichotomy on $I$ with projections $P_m^A = \mathcal{A}(m,0)P^A\mathcal{A}(0,m)$. Moreover, for $m \geq 0$ and $v \in F_1$ we have

$$P_m^A x_m^v = \mathcal{A}(m,0)P_0^A x_0^v + \sum_{j=0}^{m-1} \mathcal{A}(m,j+1)P_j^A C_j \mathcal{B}(j,0)$$

and

$$Q_m^A x_m^v = \mathcal{A}(m,0)Q_0^A x_0^v + \sum_{j=0}^{m-1} \mathcal{A}(m,j+1)Q_j^A C_j \mathcal{B}(j,0).$$

The last identity is equivalent to

$$Q_0^A x_0^v = \mathcal{A}(0,m)Q_m^A x_m^v - \sum_{j=0}^{m-1} \mathcal{A}(0,j+1)Q_{j+1}^A C_j \mathcal{B}(j,0). \tag{4.13}$$

Since the sequence $x^v$ is bounded, we have $C = \sup_{m\geq 0}\|x_m^v\| < +\infty$ and

$$\|\mathcal{A}(0,m)Q_m^A x_m^v\| \leq CDe^{-\lambda m}.$$

Hence, taking limits in (4.13) when $m \to +\infty$ we obtain

$$Q_0^A x_0^v = -\sum_{j=0}^{+\infty} \mathcal{A}(0,j+1)Q_{j+1}^A C_j \mathcal{B}(j,0)v.$$

Since $x_0^v \in E_2$, we have $Q_0^A x_0^v = x_0^v$ and so

$$Lv = x_0^v = -\sum_{j=0}^{+\infty} \mathcal{A}(0,j+1)Q_{j+1}^A C_j \mathcal{B}(j,0)v.$$

To show that the sequence $(B_m)_{m\in I}$ has a tempered exponential dichotomy on $I$ one can proceed in a similar manner to that in the proof of Theorem 2.1 using the adjoint dynamics. □

We also give a sufficient condition for the converse of the statement in Theorem 4.1.

**Theorem 4.3.** *Assume that the sequences $(A_m)_{m\in I}$ and $(B_m)_{m\in I}$ have tempered exponential dichotomies on $I = \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^-$ with constant $\lambda$ and sequence $D_m$ and, respectively, projections $P^A$ and $P^B$ at time 0. If*

$$\limsup_{|m|\to+\infty} \frac{1}{m} \log\left(D_{m+1}\|C_m\|\right) \leq 0, \tag{4.14}$$

*then the sequence $(H_m)_{m\in I}$ has a tempered exponential dichotomy with any constant less than $\lambda$.*

*Proof.* Consider the projections

$$P_m = \mathcal{H}(m,0)P\mathcal{H}(0,m),$$

with $P$ as in (4.7) or (4.8), respectively, when $I = \mathbb{Z}_0^+$ and $I = \mathbb{Z}_0^-$. We claim that

$$P_m = \begin{pmatrix} P_m^A & R_m \\ 0 & P_m^B \end{pmatrix},$$

where

$$P_m^A = \mathcal{A}(m,0)P^A\mathcal{A}(0,m) \quad \text{and} \quad P_m^B = \mathcal{B}(m,0)P^B\mathcal{B}(0,m),$$

with

$$\begin{aligned} R_m = & -\sum_{j=0}^{m-1} \mathcal{A}(m,j+1)P_{j+1}^A C_j(\mathrm{Id}_{n-k} - P_j^B)\mathcal{B}(j,m) \\ & -\sum_{j=m}^{+\infty} \mathcal{A}(m,j+1)(\mathrm{Id}_k - P_{j+1}^A)C_j P_j^B \mathcal{B}(j,m) \end{aligned} \tag{4.15}$$

when $I = \mathbb{Z}_0^+$ and

$$\begin{aligned} R_m = & -\sum_{j=-\infty}^{m-1} \mathcal{A}(m,j+1)P_{j+1}^A C_j(\mathrm{Id}_{n-k} - P_j^B)\mathcal{B}(j,m) \\ & -\sum_{j=m}^{0} \mathcal{A}(m,j+1)(\mathrm{Id}_k - P_{j+1}^A)C_j P_j^B \mathcal{B}(j,m) \end{aligned}$$

when $I = \mathbb{Z}_0^-$. Clearly

$$R_0 = \begin{cases} LP^B & \text{if } I = \mathbb{Z}_0^+, \\ L(\mathrm{Id}_{n-k} - P^B) & \text{if } I = \mathbb{Z}_0^-. \end{cases}$$

Identity (4.15) can be obtained as follows. For $I = \mathbb{Z}_0^+$ it follows from (4.7) that

$$\begin{aligned} P_m &= \mathcal{H}(m,0)\begin{pmatrix} P^A & LP^B \\ 0 & P^B \end{pmatrix}\mathcal{H}(0,m) \\ &= \begin{pmatrix} \mathcal{A}(m,0) & \mathcal{C}(m,0) \\ 0 & \mathcal{B}(m,0) \end{pmatrix}\begin{pmatrix} P^A & LP^B \\ 0 & P^B \end{pmatrix}\begin{pmatrix} \mathcal{A}(0,m) & \mathcal{C}(0,m) \\ 0 & \mathcal{B}(0,m) \end{pmatrix} \\ &= \begin{pmatrix} P_m^A & P_m^A\mathcal{A}(m,0)\mathcal{C}(0,m) + [\mathcal{A}(m,0)L + \mathcal{C}(m,0)]\mathcal{B}(0,m)P_m^B \\ 0 & P_m^B \end{pmatrix}. \end{aligned}$$

Using (4.6) we find that

$$\begin{aligned} R_m &= P_m^A\mathcal{A}(m,0)\mathcal{C}(0,m) + [\mathcal{A}(m,0)L + \mathcal{C}(m,0)]\mathcal{B}(0,m)P_m^B \\ &= \mathcal{A}(m,0)R_0\mathcal{B}(0,m) + \sum_{j=0}^{m-1} \mathcal{A}(m,j+1)\,[C_jP_j^B - P_{j+1}^A C_j]\mathcal{B}(j,m). \end{aligned}$$

Using (4.8) one can readily obtain the corresponding result for $I = \mathbb{Z}_0^-$. Identity (4.15) follow now in a straightforward manner from (4.9) (and the corresponding when $I = \mathbb{Z}_0^-$ follows from (4.10)).

We use the former identities to show that the sequence $(A_m)_{m\in I}$ has a tempered exponential dichotomy on $I$ with projections $P_m$. First we show that (4.3) holds for $m \geq l$ with $m, l \in I$, for some constant $\lambda$ and some sequence $D_l$ satisfying (4.2). Note that

$$\mathcal{H}(m,l)P_l = \begin{pmatrix} \mathcal{A}(m,l)P_l^A & \mathcal{C}(m,l)P_l^B + \mathcal{A}(m,l)R_l \\ 0 & \mathcal{B}(m,l)P_l^B \end{pmatrix}. \tag{4.16}$$

For simplicity of the exposition we consider only the case when $I = \mathbb{Z}_0^+$ (the case when $I = \mathbb{Z}_0^-$ requires only replacing in the sums the lower limit 0 by $-\infty$ and the upper limit $+\infty$

by 0). We have

$$\mathcal{C}(m,l)P_l^B = \sum_{j=l}^{m-1} \mathcal{A}(m,j+1)P_{j+1}^A C_j P_j^B \mathcal{B}(j,l) + \sum_{j=l}^{m-1} \mathcal{A}(m,j+1)(\mathrm{Id}_k - P_{j+1}^A)C_j P_j^B \mathcal{B}(j,l).$$

When $I = \mathbb{Z}_0^+$ we obtain

$$\mathcal{A}(m,l)R_l = -\sum_{j=0}^{l-1} \mathcal{A}(m,j+1)P_{j+1}^A C_j(\mathrm{Id}_{n-k} - P_j^B)\mathcal{B}(j,l) - \sum_{j=l}^{+\infty} \mathcal{A}(m,j+1)(\mathrm{Id}_k - P_{j+1}^A)C_j P_j^B \mathcal{B}(j,l)$$

and hence,

$$\mathcal{C}(m,l)P_l^B + \mathcal{A}(m,l)R_l = \sum_{j=l}^{m-1} \mathcal{A}(m,j+1)P_{j+1}^A C_j P_j^B \mathcal{B}(j,l) - \sum_{j=m}^{+\infty} \mathcal{A}(m,j+1)(\mathrm{Id}_k - P_{j+1}^A)C_j P_j^B \mathcal{B}(j,l) - \sum_{j=0}^{l-1} \mathcal{A}(m,j+1)P_{j+1}^A C_j(\mathrm{Id}_{n-k} - P_j^B)\mathcal{B}(j,l).$$

By (4.14), given $\varepsilon > 0$, there exists $d > 0$ such that

$$D_{m+1}\|C_m\| \le de^{\varepsilon m}, \quad \text{for } m \in I.$$

Hence, for $m \ge l \ge 0$ we have

$$\begin{aligned} \|\mathcal{C}(m,l)P_l^B + \mathcal{A}(m,l)R_l\| &\le \sum_{j=l}^{m-1} D_{j+1}e^{-\lambda(m-j-1)}\|C_j\|D_l e^{-\lambda(j-l)} \\ &\quad + \sum_{j=m}^{+\infty} D_{j+1}e^{-\lambda(j+1-m)}\|C_j\|D_l e^{-\lambda(j-l)} \\ &\quad + \sum_{j=0}^{l-1} D_{j+1}e^{-\lambda(m-j-1)}\|C_j\|D_l e^{-\lambda(l-j)} \\ &\le dD_l e^{\lambda}\left(ce^{-\mu(m-l)} + \frac{e^{-(\lambda-\varepsilon)(m-l)+\varepsilon l}}{1-e^{-2\lambda+\varepsilon}} + \frac{e^{-(\lambda-\varepsilon)(m-l)+\varepsilon l}}{1-e^{-2\lambda+\varepsilon}}\right) \\ &\le dD_l e^{\lambda}e^{\varepsilon l}e^{-\mu(m-l)}\left(c + \frac{2}{1-e^{-2\lambda+\varepsilon}}\right), \end{aligned} \tag{4.17}$$

for some constants $c > 0$ and $\mu < \lambda$ (independent of $\varepsilon$) provided that $\varepsilon$ is sufficiently small. The first term is justified by the inequalities

$$\sum_{j=n}^{m-1} e^{\varepsilon j}e^{-\lambda(m-j-1)}e^{-\lambda(j-l)} = e^{\lambda}\sum_{j=n}^{m-1} e^{\varepsilon j}e^{-\lambda(m-l)} \le (m-n)e^{\lambda}e^{\varepsilon m}e^{-\lambda(m-l)} \le ce^{-\mu(m-l)+\varepsilon l},$$

for some constants as above. It follows from (4.2) and (4.17) that

$$\limsup_{l\to+\infty}\frac{1}{l}\log\big(\|\mathcal{C}(m,l)P_l^B+\mathcal{A}(m,l)R_l\|e^{\mu(m-l)}\big)\le\varepsilon. \tag{4.18}$$

In view of identity (4.16), it follows from (4.18) and the arbitrariness of $\varepsilon$ that (4.3) holds for $m\ge l$, with $\lambda$ replaced by $\mu$ and $D$ replaced by the sequence

$$l\mapsto\sup_{m\ge l}\big(\|\mathcal{C}(m,l)P_l^B+\mathcal{A}(m,l)R_l\|e^{\mu(m-l)}\big).$$

Finally we show that (4.4) holds for $m\le l$ with $m,l\in I$, for some constant $\lambda$ and some function $D$ satisfying (4.2). First note that

$$\mathcal{H}(m,l)Q_l=\begin{pmatrix}(\mathrm{Id}_k-P_m^A)\mathcal{A}(m,l) & (\mathrm{Id}_k-P_m^A)\mathcal{C}(m,l)-R_m\mathcal{B}(m,l)\\ 0 & (\mathrm{Id}_{n-k}-P_m^B)\mathcal{B}(m,l)\end{pmatrix}.$$

We have

$$\begin{aligned}R_m\mathcal{B}(m,n)=&-\sum_{j=0}^{m-1}\mathcal{A}(m,j+1)P_{j+1}^AC_j(\mathrm{Id}_{n-k}-P_j^B)\mathcal{B}(j,l)\\&-\sum_{j=m}^{+\infty}\mathcal{A}(m,j+1)(\mathrm{Id}_k-P_{j+1}^A)C_jP_j^B\mathcal{B}(j,l)\end{aligned}$$

and

$$(\mathrm{Id}_k-P_m^A)\mathcal{C}(m,l)=\sum_{j=n}^{m-1}\mathcal{A}(m,j+1)(\mathrm{Id}_k-P_{j+1}^A)C_j\mathcal{B}(j,l).$$

Hence,

$$\begin{aligned}(\mathrm{Id}_k-P_m^A)\mathcal{C}(m,l)-R_m\mathcal{B}(m,l)=&\sum_{j=l}^{+\infty}\mathcal{A}(m,j+1)(\mathrm{Id}_k-P_{j+1}^A)C_jP_j^B\mathcal{B}(j,l)\\&-\sum_{j=l}^{m-1}\mathcal{A}(m,j+1)(\mathrm{Id}_k-P_{j+1}^A)C_j(\mathrm{Id}_{n-k}-P_j^B)\mathcal{B}(j,l)\\&+\sum_{j=0}^{m-1}\mathcal{A}(m,j+1)P_{j+1}^AC_j(\mathrm{Id}_{n-k}-P_j^B)\mathcal{B}(j,l),\end{aligned}$$

which implies that for $m\le l$ we have

$$\begin{aligned}&\|(\mathrm{Id}_k-P_m^A)\mathcal{C}(m,l)-R_m\mathcal{B}(m,l)\|\\&\le dD_le^{\lambda}e^{\varepsilon l}\left(\frac{e^{(\lambda-\varepsilon)(m-l)+\varepsilon l}}{1-e^{2\lambda-\varepsilon}}+ce^{\mu(m-l)}+\frac{e^{(\lambda-\varepsilon)(m-l)+\varepsilon l}}{1-e^{2\lambda-\varepsilon}}\right)\\&\le dD_le^{\lambda}e^{\varepsilon l}\left(\frac{2}{1-e^{2\lambda-\varepsilon}}+c\right)e^{\lambda(m-l)}\end{aligned} \tag{4.19}$$

for some constants $c>0$ and $\mu<\lambda$ as above. Proceeding as in (4.18), it follows from (4.19) that (4.4) holds for $m\le l$, with $\lambda$ replaced by $\mu$ and $D$ replaced by some other sequence. This completes the proof of the theorem. □

We also formulate corresponding results for the notion of a strong tempered exponential dichotomy. We say that the sequence $(A_m)_{m\in I}$ of invertible $n \times n$ matrices has a *strong tempered exponential dichotomy on I* if there exist projections $P_m$ for $m \in I$ satisfying (4.1) and there exist constants $\mu > \lambda > 0$ and a sequence $(D_m)_{m\in I} \subset \mathbb{R}^+$ satisfying (4.2) such that

$$\|\mathcal{A}(m,l)P_l\| \le D_l e^{-\lambda(m-l)}, \qquad \|\mathcal{A}(l,m)P_m\| \le D_m e^{\mu(m-l)}$$

and

$$\|\mathcal{A}(l,m)Q_m\| \le D_m e^{-\lambda(m-l)}, \qquad \|\mathcal{A}(m,l)Q_l\| \le D_l e^{\mu(m-l)}$$

for $m \ge l$, where $Q_m = \mathrm{Id}_n - P_m$ for each $m$. We have the following two results (the proofs are similar to the previous ones and so we omit them).

**Theorem 4.4.** *Assume that the sequence in (4.5) has a strong tempered exponential dichotomy on $I = \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^-$ with constants $\lambda$ and $\mu$. Then the sequences $(A_m)_{m\in I}$ and $(B_m)_{m\in I}$ have strong tempered exponential dichotomies on I with the same constants $\lambda$ and $\mu$.*

The following result is a partial converse of Theorem 4.4.

**Theorem 4.5.** *Assume that the sequences $(A_m)_{m\in I}$ and $(B_m)_{m\in I}$ have strong tempered exponential dichotomies on $I = \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^-$ with constants $\lambda, \mu$, sequence $(D_m)_{m\in I}$ and, respectively, projections $P^A$ and $P^B$ at time* 0*. If condition (4.14) holds, then the sequence $(H_m)_{m\in I}$ has a strong tempered exponential dichotomy with any constants, respectively, less than $\lambda$ and greater than $\mu$.*

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