Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **92**, 1–28; doi: 10.14232/ejqtde.2018.1.92 http://www.math.u-szeged.hu/ejqtde/

Nonoscillatory solutions of planar half-linear differential systems: a Riccati equation approach

Jaroslav Jaroš[1], **Kusano Takaŝi**[2] and **Tomoyuki Tanigawa**[3]

[1]Department of Mathematical Analysis and Numerical Mathematics,
Faculty of Mathematics, Physics and Informatics, Comenius University,
Mlynská dolina, Bratislava, 842 48, Slovakia
[2]Department of Mathematics, Faculty of Science, Hiroshima University,
Higashi Hiroshima, 739-8526, Japan
[3] Department of Mathematical Sciences, Osaka Prefecture University, Osaka 599-8531, Japan

Received 30 July 2018, appeared 21 October 2018
Communicated by Josef Diblík

Abstract. In this paper an attempt is made to depict a clear picture of the overall structure of nonoscillatory solutions of the first order half-linear differential system

$$x' - p(t)\varphi_{1/\alpha}(y) = 0, \qquad y' + q(t)\varphi_\alpha(x) = 0, \tag{A}$$

where $\alpha > 0$ is a constant, $p(t)$ and $q(t)$ are positive continuous functions on $[0,\infty)$, and $\varphi_\gamma(u) = |u|^\gamma \operatorname{sgn} u$, $u \in \mathbb{R}$, $\gamma > 0$. A systematic analysis of the existence and asymptotic behavior of solutions of (A) is proposed for this purpose. A special mention should be made of the fact that all possible types of nonoscillatory solutions of (A) can be constructed by solving the Riccati type differential equations associated with (A). Worthy of attention is that all the results for (A) can be applied to the second order half-linear differential equation

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \tag{E}$$

to build automatically a nonoscillation theory for (E).

Keywords: half-linear differential systems, non-oscillatory solutions, Riccati equation.
2010 Mathematics Subject Classification: 34C10.

1 Introduction

We consider first order cyclic differential systems of the form

$$x' - p(t)\varphi_{1/\alpha}(y) = 0, \qquad y' + q(t)\varphi_\alpha(x) = 0, \tag{A}$$

where α is a positive constant, p and q are positive continuous functions on $[0,\infty)$, and φ_γ, $\gamma > 0$, denotes the odd function

$$\varphi_\gamma(u) = |u|^\gamma \operatorname{sgn} u, \qquad u \in \mathbb{R}. \tag{1.1}$$

Corresponding author. Email: jaros@fmph.uniba.sk

The nonlinearity of system (A) is referred to as *half-linear*. The qualitative study of half-linear differential systems was initiated by Elbert [4] and Mirzov [12], who showed that though system (A) with $\alpha \neq 1$ is nonlinear, it has several significant properties in common with the linear differential system $x' - p(t)y = 0$, $y' + q(t)x = 0$.

In this paper we are concerned exclusively with solutions of (A) which are defined and nontrivial on intervals of the form $[t_0, \infty)$, $t_0 \geq 0$. Such a solution (x, y) is called *oscillatory* or *nonoscillatory* according as both x and y are either oscillatory or nonoscillatory in the usual sense, respectively. Worthy of note is the fact that all solutions of (A) are either oscillatory or else nonoscillatory, that is, oscillatory solutions and nonoscillatory solutions cannot coexist for system (A); see Jaroš and Kusano [6]. System (A) is simply said to be oscillatory (or nonoscillatory) if all of its solutions are oscillatory (or nonoscillatory). We will focus our attention on the system (A) which is nonoscillatory and aim to acquire as much precise information as possible about the existence and asymptotic behavior at infinity of its solutions, thereby making it possible to depict a clear picture of the overall structure of the totality of nonoscillatory solutions of (A).

Let (x, y) be a nonoscillatory solution of (A) on $[t_0, \infty)$. Since both x and y are eventually one-signed, they are monotone for all large t so that there exist the limits $x(\infty) = \lim_{t\to\infty} x(t)$ and $y(\infty) = \lim_{t\to\infty} y(t)$ in the extended real numbers. It follows that $x(t)y(t) \neq 0$ on $[T, \infty)$ for some $T \geq t_0$. We say that (x, y) is a solution *of the first kind* (resp. *of the second kind*) if $x(t)y(t) > 0$ (resp. $x(t)y(t) < 0$) for $t \geq T$.

Based on the expectation that the behavior of solutions of (A) depends heavily upon the behavior of the coefficients p and q, more specifically, upon the convergence or divergence of the integrals

$$I_p = \int_0^\infty p(t)dt, \qquad I_q = \int_0^\infty q(t)dt, \tag{1.2}$$

we distinguish the four cases

$$I_p = \infty \wedge I_q = \infty, \quad I_p = \infty \wedge I_q < \infty, \quad I_p < \infty \wedge I_q = \infty, \quad I_p < \infty \wedge I_q < \infty, \tag{1.3}$$

in each of which an attempt is made to analyze how influential is the combination (I_p, I_q) on the determination of specific types of nonoscillatory solutions system (A) may possess.

Our nonoscillation theory of system (A) is presented in Sections 2 and 3. It is shown that all solutions of (A) are oscillatory in the first case of (1.3), and that nonoscillatory solutions of (A) really exist in the remaining three cases. In the last case of (1.3) it turns out that all solutions of (A) are bounded and their existence can be characterized with relative ease. So our efforts should be focused on the analysis of the two cases in the middle of (1.3). As is easily seen if (x, y) is a solution of (A), then $(-x, y)$ and $(x, -y)$ are solutions of the *dual* differential system

$$x' + p(t)\varphi_{1/\alpha}(y) = 0, \qquad y' - q(t)\varphi_\alpha(x) = 0, \tag{B}$$

and vice versa. Observe that systems (A) and (B) are structurally the same except that the roles of $\{x, y\}$, $\{p, q\}$ and $\{\alpha, 1/\alpha\}$ are interchanged. This self-evident fact is what we call the *duality principle* between (A) and (B). Suppose that the case $I_p = \infty \wedge I_q < \infty$ of (A) has been well analyzed. We now want to study the *new* system (A) in the case $I_p < \infty \wedge I_q = \infty$. Consider the dual system (B) of this new (A). Then, (B) can be regarded as the same *old* system (A) in the sense specified above. So, to each result for the old system (A) there corresponds its counterpart for the new system (A). The correspondence is automatic, and the new result thus obtained via the duality principle is correct if so is the old one. This is why our efforts are devoted for the most part to the analysis of system (A) in the case $I_p = \infty \wedge I_q < \infty$.

What is required of us in this paper is an in-depth study of the existence and asymptotic behavior of nonoscillatory solutions of system (A) with special emphasis on the cases $I_p = \infty \wedge I_q < \infty$ and $I_p < \infty \wedge I_q = \infty$. First, we classify the totality of nonoscillatory solutions of (A) into the three subsets, the classes of *maximal solutions, minimal solutions* and *intermediate solutions*, and then make attempts to characterize the membership of all the solution classes put in the classification list. As a result, the membership of the classes of maximal and minimal solutions can be completely characterized, in other words, necessary and sufficient conditions can be found for (A) to have maximal and minimal solutions. The results on maximal and minimal solutions are presented in Section 2 along with those on bounded solutions of system (A) in the case $I_p < \infty \wedge I_q < \infty$. As for intermediate solutions of (A), a few sufficient conditions for their existence are provided in Section 3. Examples are given to illustrate the main results.

Of central importance to the development of our theory for (A) is an effective utilization of the Riccati type differential equations

$$u' + \alpha p(t)|u|^{\frac{1}{\alpha}+1} + q(t) = 0, \tag{R1}$$

$$v' = \frac{1}{\alpha} q(t)|v|^{\alpha+1} + p(t), \tag{R2}$$

in establishing the existence of all types of nonoscillatory solutions of (A). Equations (R1) and (R2), which are known as the Riccati differential equations associated with system (A), were discovered by Mirzov [12], who proved that system (A) is nonoscillatory if and only if both (R1) and (R2) possess global solutions, that is, solutions which are continued to $t = \infty$. Mirzov used the Riccati equations to prove a comparison theorem regarding two half-linear differential systems of the form (A). To the best of our knowledge no attempts have been made to apply (R1) and (R2) to the solution of other qualitative problems for (A). The main aim of this paper is to demonstrate that all types of nonoscillatory solutions of system (A) can actually be constructed by means of suitably chosen global solutions of the Riccati equations.

An important by-product of Sections 2 and 3 is that the theory developed therein can exhaustively be applied to second order scalar half-linear differential equations of the form

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \tag{E}$$

where $\alpha > 0$ is a constant and p and q are positive continuous functions on $[0, \infty)$. Given a solution x of equation (E), we call the function $p(t)\varphi_\alpha(x')$ the *quasi-derivative* of x and denote it by Dx. By a nonoscillatory solution of (E) we mean a solution x which is defined in some neighborhood of infinity and satisfies $x(t)Dx(t) \neq 0$ for all large t. As a consequence of application of the results for system (A) the class of nonoscillatory solutions x of equation (E) is divided systematically into several subclasses according to the patterns of joint asymptotic behavior of x and Dx at infinity, and criteria for all of these solution subclasses to have members are established. In particular, the existence of the so-called *intermediate solutions* of (E) is ascertained. The contents of Section 4 seem to underscore the importance and effectiveness of the asymptotic analysis of simple first order half-linear differential systems such as (A).

We note that some results of Sections 2 and 4 are known (see [2, 3, 5, 7–10, 16]), but the derivation is essentially different. The results of Section 3 and their applications in Section 4 are new.

2 Maximal and minimal solutions of (A)

2.1 Classification of nonoscillatory solutions

We start with a rudimentary fact which is underlying throughout the subsequent discussions.

Proposition 2.1.

(i) *If $I_p = \infty$, then any nonoscillatory solution (x, y) of (A) is of the first kind such that $|x|$ is increasing and $|y|$ is decreasing for all large t.*

(ii) *If $I_q = \infty$, then any nonoscillatory solution (x, y) of (A) is of the second kind such that $|x|$ is decreasing and $|y|$ is increasing for all large t.*

Proof. Suppose first that $I_p = \infty$. Let (x, y) be a nonoscillatory of (A) such that $x(t)y(t) \neq 0$ on $[t_0, \infty)$. We may assume that $x(t) > 0$ for $t \geq t_0$. The second equation of (A) shows that $y(t)$ is decreasing for $t \geq t_0$. If there exists $T \geq t_0$ such that $y(t) \leq y(T) < 0$, $t \geq T$, then integrating the first equation of (A) on $[T, t]$, we obtain

$$x(t) - x(T) = -\int_T^t p(s)|y(s)|^{1/\alpha} ds \leq -|y(T)|^{1/\alpha} \int_T^t p(s) ds \to -\infty, \qquad t \to \infty,$$

which contradicts the assumed positivity of $x(t)$. Therefore, we see that $y(t)$ must be positive throughout $[t_0, \infty)$, concluding that (x, y) is a solution of the first kind. This proves the statement (i).

Turning to the case $I_q = \infty$, let (x, y) be a solution of (A) such that $x(t)y(t) \neq 0$ on $[t_0, \infty)$. Consider the function $(-x, y)$. By the duality principle stated in the Introduction this function is a solution of the dual system (B), which is structurally the same system as (A) with $I_p = \infty$, and so from the statement (i) it follows that $-x(t)y(t) > 0$, i.e., $x(t)y(t) < 0$ for $t \geq t_0$. This means that (x, y) is a solution of the second kind of (A). Thus we are allowed to assert that the statement (ii) follows from (i) automatically via the duality principle (between (A) and (B)). This completes the proof. □

An immediate consequence of Proposition 2.1 is that if p and q satisfy $I_p = \infty$ and $I_q = \infty$, then system (A) admits no nonoscillatory solutions.

Theorem 2.2 (Mirzov [12])**.** *If $I_p = \infty \wedge I_q = \infty$, then all solutions of system (A) are oscillatory.*

Suppose now that p and q satisfy $I_p = \infty \wedge I_q < \infty$. By Proposition 2.1 all nonoscillatory solutions (x, y) of (A) are of the first kind, and both components x and y are eventually monotone, more precisely, $|x|$ are increasing and $|y|$ are decreasing for all large t. As is easily seen, the following three types or patterns of asymptotic behavior at infinity are possible for them:

I(i): $|x(\infty)| = \infty, \quad 0 < |y(\infty)| < \infty,$

I(ii): $|x(\infty)| = \infty, \quad y(\infty) = 0,$

I(iii): $0 < |x(\infty)| < \infty, \quad y(\infty) = 0.$

On the other hand, if p and q satisfy $I_p < \infty \wedge I_q = \infty$, then we see that all nonoscillatory solutions (x, y) of (A) are of the second kind, and $|x|$ are decreasing and $|y|$ are increasing for all large t, and that their asymptotic behaviors at infinity are restricted to the following three types

$$\text{II(i)}: \; 0 < |x(\infty)| < \infty, \quad |y(\infty)| = \infty,$$

$$\text{II(ii)}: \; x(\infty) = 0, \quad |y(\infty)| = \infty,$$

$$\text{II(iii)}: \; x(\infty) = 0, \quad 0 < |y(\infty)| < \infty.$$

Finally, if p and q satisfy $I_p < \infty$ and $I_q < \infty$, then the possible asymptotic behavior of any nonoscillatory solution of the first kind (resp. of the second kind) of (A) is either of the type I(iii) (resp. of the type II(iii)) or

$$\text{III}: \; 0 < |x(\infty)| < \infty, \quad 0 < |y(\infty)| < \infty.$$

Nonoscillatory solutions of the types I(i) and II(i) (resp. I(iii) and II(iii)) are called *maximal solutions* (resp. *minimal solutions*) of (A), while solutions of the types I(ii) and II(ii) are termed *intermediate solutions* of (A). Solutions with the terminal states of the type III are called *bounded solutions of non-minimal type* of (A).

2.2 Characterization of maximal and minimal solutions of (A)

It is shown that the situations for the existence of maximal and minimal nonoscillatory solutions of (A) can be completely characterized in both of the cases $I_p = \infty \wedge I_q < \infty$ and $I_p < \infty \wedge I_q = \infty$.

In what follows use is made of the functions

$$P(t) = \int_0^t p(s)ds \quad \text{if } I_p = \infty, \qquad \pi(t) = \int_t^\infty p(s)ds \quad \text{if } I_p < \infty, \tag{2.1}$$

$$Q(t) = \int_0^t q(s)ds \quad \text{if } I_q = \infty, \qquad \rho(t) = \int_t^\infty q(s)ds \quad \text{if } I_q < \infty. \tag{2.2}$$

Our main results of this section read as follows.

Theorem 2.3.

(i) *Assume that $I_p = \infty$. System (A) has maximal nonoscillatory solutions of the first kind if and only if*

$$\int_0^\infty q(t)P(t)^\alpha dt < \infty. \tag{2.3}$$

In this case, for any given constant $d \neq 0$ there exists a solution (x, y) such that

$$\lim_{t\to\infty} y(t) = d, \qquad \lim_{t\to\infty} \frac{x(t)}{P(t)} = \varphi_{1/\alpha}(d). \tag{2.4}$$

(ii) *Assume that $I_q < \infty$. System (A) has minimal nonoscillatory solutions of the first kind if and only if*

$$\int_0^\infty p(t)\rho(t)^{1/\alpha} dt < \infty. \tag{2.5}$$

In this case, for any given constant $c \neq 0$ there exists a solution (x, y) such that

$$\lim_{t\to\infty} x(t) = c, \qquad \lim_{t\to\infty} \frac{y(t)}{\rho(t)} = \varphi_\alpha(c). \tag{2.6}$$

Proof of the "only if" parts. (i) Suppose that (A) has a maximal solution (x, y) of the first kind. We may assume that $x(t) > 0$, $y(t) > 0$ for all large t, and $x(\infty) = \infty$ and $y(\infty) = d$ for some constant $d > 0$. Applying L'Hospital's rule to the first equation of (A), we see that

$$\lim_{t\to\infty} \frac{x(t)}{P(t)} = \lim_{t\to\infty} y(t)^{1/\alpha} = d^{1/\alpha}, \tag{2.7}$$

which implies that the precise asymptotic formula for (x, y) is known in advance. In view of (2.7) there exist constants $k > 0$ and $t_0 \geq 0$ such that $x(t) \geq kP(t)$ for $t \geq t_0$. Combining this inequality with

$$y(t_0) - y(t) = \int_{t_0}^{t} q(s)x(s)^{\alpha} ds, \qquad t \geq t_0,$$

following from the second equation of (A), we obtain

$$k^{\alpha} \int_{t_0}^{t} q(s)P(s)^{\alpha} ds \leq y(t_0), \qquad t \geq t_0.$$

This clearly implies the truth of (2.3).

(ii) Suppose that (A) has a minimal solution (x, y) of the first kind such that $x(t) > 0$, $y(t) > 0$ for all large t, and $x(\infty) = c$ and $y(\infty) = 0$ for some constant $c > 0$. L'Hospital's rule applied to the second equation of (A) implies that

$$\lim_{t\to\infty} \frac{y(t)}{\rho(t)} = \lim_{t\to\infty} x(t)^{\alpha} = c^{\alpha}. \tag{2.8}$$

Thus the precise asymptotic behavior of (x, y) at infinity is explicitly determined in advance. Because of (2.8) there exist constants $k > 0$ and $t_0 \geq 0$ such that $y(t) \geq k\rho(t)$ for $t \geq t_0$. This inequality combined with

$$c - x(t) = \int_{t}^{\infty} p(s)y(s)^{1/\alpha} ds, \qquad t \geq t_0,$$

following from the first equation of (A) gives

$$k^{1/\alpha} \int_{t}^{\infty} p(s)\rho(s)^{1/\alpha} ds \leq c, \qquad t \geq t_0,$$

which verifies the validity of (2.5).

Proof of the "if" parts. It suffices to prove the existence of *positive* maximal and minimal solutions of (A) under the conditions (2.3) and (2.5), respectively. The main tool we employ is the Riccati equations

$$u' + \alpha p(t)u^{\frac{1}{\alpha}+1} + q(t) = 0, \tag{R1}$$

and

$$v' = \frac{1}{\alpha} q(t)v^{\alpha+1} + p(t), \tag{R2}$$

whose positive solutions should give rise to the desired positive solutions of (A).

A close connection between system (A) and the associated Riccati equations (R1) and (R2) is explained below. Let (x, y) be a positive solution of system (A) on $J = [t_0, \infty)$. Then, the functions

$$u(t) = \frac{y(t)}{x(t)^{\alpha}} \quad \text{and} \quad v(t) = \frac{x(t)}{y(t)^{1/\alpha}}, \tag{2.9}$$

satisfy equations (R1) and (R2), respectively, on J. Conversely, assume that (R1) and (R2) have, respectively, positive solutions u and v on J. Then, from (2.9) combined with the first equation $x' = p(t)y^{1/\alpha}$ of (A) we obtain $x'(t)/x(t) = p(t)u(t)^{1/\alpha}$ and $x'(t)/x(t) = q(t)v(t)^{-1}$, and so x should be expressed in terms of u or v as follows

$$x(t) = a\exp\left(\int_{t_0}^{t} p(s)u(s)^{1/\alpha}ds\right) \quad \text{or} \quad x(t) = b\exp\left(\int_{t_0}^{t} q(s)v(s)^{-1}ds\right), \qquad t \in J, \tag{2.10}$$

where a and b are any positive constants. So, let us form the vector functions

$$\left(a\exp\left(\int_{t_0}^{t} p(s)u(s)^{1/\alpha}ds\right),\ a^{\alpha}u(t)\exp\left(\alpha\int_{t_0}^{t} p(s)u(s)^{1/\alpha}ds\right)\right), \tag{2.11}$$

and

$$\left(b\exp\left(\int_{t_0}^{t} q(s)v(s)^{-1}ds\right),\ b^{\alpha}v(t)^{-\alpha}\exp\left(\alpha\int_{t_0}^{t} q(s)v(s)^{-1}ds\right)\right). \tag{2.12}$$

Then, these functions become positive solutions of system (A) on J. The verification of this fact may be omitted. It should be remarked that if u is such that $p(t)u(t)^{1/\alpha}$ is integrable on J, then using the function

$$x(t) = \omega\exp\left(-\int_{t}^{\infty} p(s)u(s)^{1/\alpha}ds\right),$$

ω being any positive constant, instead of x in (2.10), one obtains a positive solution of (A)

$$\left(\omega\exp\left(-\int_{t}^{\infty} p(s)u(s)^{1/\alpha}ds\right),\ \omega^{\alpha}u(t)\exp\left(-\alpha\int_{t}^{\infty} p(s)u(s)^{1/\alpha}ds\right)\right). \tag{2.13}$$

A similar remark applies to the case where v has the property that $q(t)v(t)^{-1}$ is integrable on J.

An alternative way of finding solutions (x, y) of (A) by means of solutions u and v of (R1) and (R2) is to first construct y from u and v by using the relations $y'(t)/y(t) = -q(t)u(t)^{-1}$ and $y'(t)/y(t) = -q(t)v(t)^{\alpha}$. We then obtain

$$y(t) = a\exp\left(-\int_{t_0}^{t} q(s)u(s)^{-1}ds\right) \quad \text{or} \quad y(t) = \omega\exp\left(\int_{t}^{\infty} q(s)u(s)^{-1}ds\right),$$

from the first equation, and

$$y(t) = a\exp\left(-\int_{t_0}^{t} q(s)v(s)^{\alpha}ds\right) \quad \text{or} \quad y(t) = \omega\exp\left(\int_{t}^{\infty} q(s)v(s)^{\alpha}ds\right),$$

from the second equation, where a and ω are arbitrary positive constants. For example, if $q(t)v(t)^{\alpha}$ is integrable on $[t_0, \infty)$, then the function

$$\left(\omega^{1/\alpha}v(t)\exp\left(\frac{1}{\alpha}\int_{t}^{\infty} q(s)v(s)^{\alpha}ds\right),\ \omega\exp\left(\int_{t}^{\infty} q(s)v(s)^{\alpha}ds\right)\right), \tag{2.14}$$

gives a solution of system (A).

It should be remarked that in constructing a solution of (A) we do not need to solve both of (R1) and (R2) since they are interdependent ($v = u^{-1/\alpha}$). We need only to select either one of them whichever is convenient and look for its solution u or v which gives rise to a solution (x, y) of (A) with the desired asymptotic behavior at infinity.

Before proving the "if" parts of Theorem 2.3 we mention an important necessary condition for nonoscillation of system (A) which is derived from the Riccati equation (R1).

Proposition 2.4. *Let $I_p = \infty \wedge I_q < \infty$. If system (A) is nonoscillatory, then it holds that*

$$\int_0^\infty p(t)\rho(t)^{\frac{1}{\alpha}+1}dt < \infty. \tag{2.15}$$

Proof. In fact, suppose that (A) has a nonoscillatory solution (x, y) on $[t_0, \infty)$. We may assume that $x(t) > 0$ and $y(t) > 0$ for $t \geq t_0$. The function $u(t) = y(t)/x(t)^\alpha$ satisfies (R1) on $[t_0, \infty)$. Since $u(t) > 0$ is nonincreasing and tends to a finite limit $u(\infty) \geq 0$ as $t \to \infty$, integration of (R1) on $[t, \infty)$ gives

$$u(t) = u(\infty) + \alpha \int_t^\infty p(s)u(s)^{\frac{1}{\alpha}+1}ds + \rho(t), \qquad t \geq t_0.$$

We must have $u(\infty) = 0$, since otherwise we would have

$$\int_{t_0}^t p(s)u(s)^{\frac{1}{\alpha}+1}ds \geq u(\infty)^{\frac{1}{\alpha}+1} \int_{t_0}^t p(s)ds \to \infty, \qquad t \to \infty,$$

which contradicts the integrability of $p(t)u(t)^{\frac{1}{\alpha}+1}$ on $[t_0, \infty)$. It follows that $u(t)$ satisfies the integral equation

$$u(t) = \alpha \int_t^\infty p(s)u(s)^{\frac{1}{\alpha}+1}ds + \rho(t), \qquad t \geq t_0. \tag{2.16}$$

Noting that the inequality $u(t) \geq \rho(t)$ follows from (2.16), we conclude that

$$\infty > \int_{t_0}^\infty p(s)u(s)^{\frac{1}{\alpha}+1}ds \geq \int_{t_0}^\infty p(s)\rho(s)^{\frac{1}{\alpha}+1}ds,$$

which clearly implies (2.15). □

The "if" part of (i) is proved if one assumes (2.3) and confirms the existence of a positive solution (x, y) of (A) which satisfies (2.4) for any given constant $d > 0$. To this end we need a positive solution v of (R2) satisfying $\lim_{t\to\infty} v(t)/P(t) = 1$ to be obtained as a solution of the integral equation

$$v(t) = \frac{1}{\alpha} \int_T^t q(s)v(s)^{\alpha+1}ds + P(t), \tag{2.17}$$

on some interval $[T, \infty)$. We are going to solve the equation

$$w(t) = 1 + \frac{1}{\alpha P(t)} \int_T^t q(s)(P(s)w(s))^{\alpha+1}ds, \tag{2.18}$$

to which (2.17) is reduced by the substitution $v = P(t)w$.

Let A be any constant such that $1 < A < 1 + \frac{1}{\alpha}$. Choose $T > 0$ so large that

$$\int_T^\infty q(s)P(s)^\alpha ds \leq \alpha(A-1)A^{-\alpha-1}, \tag{2.19}$$

and consider the set $\mathcal{W}$ defined by

$$W = \{w \in C_b[T, \infty) : 1 \leq w(t) \leq A, \ \ t \geq T\}, \tag{2.20}$$

where $C_b[T, \infty)$ denotes the Banach space of all bounded continuous functions on $[T, \infty)$ with the sup-norm $\|w\|_b = \sup\{|w(t)| : t \geq T\}$. Consider the integral operator

$$Fw(t) = 1 + \frac{1}{\alpha P(t)} \int_T^t q(s)(P(s)w(s))^{\alpha+1}ds, \qquad t \geq T, \tag{2.21}$$

and let it act on the closed set W. Using (2.19)–(2.21), we see that if $w \in W$, then

$$1 \le Fw(t) \le 1 + A^{\alpha+1} \frac{1}{\alpha P(t)} \int_T^t q(s)P(s)^{\alpha+1} ds$$
$$\le 1 + A^{\alpha+1} \frac{1}{\alpha} \int_T^t q(s)P(s)^{\alpha} ds \le 1 + (A-1) = A,$$

for $t \ge T$. This shows that $Fw \in W$, that is, F maps W into itself. Moreover, if $w_2,\ w_2 \in W$, then using the inequality

$$|w_1(t)^{\alpha+1} - w_2(t)^{\alpha+1}| \le (\alpha+1)A^{\alpha}|w_1(t) - w_2(t)|, \qquad t \ge T,$$

we find that

$$|Fw_1(t) - Fw_2(t)| \le \frac{1}{\alpha P(t)} \int_T^t q(s)P(s)^{\alpha+1}|w_1(s)^{\alpha+1} - w_2(s)^{\alpha+1}| ds$$
$$\le \left(\frac{1}{\alpha} + 1\right) A^{\alpha} \int_T^t q(s)P(s)^{\alpha} ds \|w_1 - w_2\|_b \le \gamma \|w_1 - w_2\|_b, \qquad t \ge T,$$

where

$$\gamma = (1+\alpha)\left(1 - \frac{1}{A}\right) < 1.$$

This implies that

$$\|Fw_1 - Fw_2\|_b \le \gamma \|w_1 - w_2\|_b \quad \text{for any } w_1,\ w_2 \in W,$$

so that F is a contraction on W. Therefore by the contraction principle F has a unique fixed point w in W which gives a positive solution of (2.18) on $[T, \infty)$. It remains to show that $w(\infty) = 1$. Let $\varepsilon > 0$ be given arbitrarily. Choose $t_\varepsilon > T$ so that

$$\frac{A^{\alpha+1}}{\alpha} \int_{t_\varepsilon}^{\infty} q(s)P(s)^{\alpha} ds < \frac{\varepsilon}{2}. \tag{2.22}$$

Since $P(t) \to \infty$ as $t \to \infty$, there exists $T_\varepsilon > t_\varepsilon$ such that

$$\frac{A^{\alpha+1}}{\alpha P(t)} \int_T^{t_\varepsilon} q(s)P(s)^{\alpha+1} ds < \frac{\varepsilon}{2}, \qquad t \ge T_\varepsilon. \tag{2.23}$$

We then see that

$$1 \le w(t) \le 1 + \frac{A^{\alpha+1}}{\alpha P(t)} \int_T^t q(s)P(s)^{\alpha+1} ds$$
$$\le 1 + \frac{A^{\alpha+1}}{\alpha P(t)} \int_T^{t_\varepsilon} q(s)P(s)^{\alpha+1} ds + \frac{A^{\alpha+1}}{\alpha} \int_{t_\varepsilon}^{\infty} q(s)P(s)^{\alpha} ds < 1 + \varepsilon,$$

for $t \ge T_\varepsilon$. Since ε is arbitrary, it follows that $1 \le \liminf_{t\to\infty} w(t) \le \limsup_{t\to\infty} w(t) \le 1$, i.e., $\lim_{t\to\infty} w(t) = 1$.

We now put $v(t) = P(t)w(t)$, $t \ge T$. Then, v is a solution of (2.17), and hence of the Riccati equation (R2) on $[T, \infty)$ and satisfies $v(t)/P(t) \to 1$ as $t \to \infty$. Observing that $q(t)v(t)^{\alpha}$ is integrable over $[T, \infty)$, form the vector function (x, y) by

$$(x(t), y(t)) = \left(d^{1/\alpha} v(t) \exp\left(\frac{1}{\alpha} \int_t^{\infty} q(s)v(s)^{\alpha} ds\right), d \exp\left(\int_t^{\infty} q(s)v(s)^{\alpha} ds\right)\right). \tag{2.24}$$

Then, it is confirmed that (x, y) is a solution of system (A) satisfying (2.4): $x(t)/P(t) \to d^{1/\alpha}$ and $y(t) \to d$ as $t \to \infty$. This finishes the proof of the "if" part of the statement (i) of Theorem 2.3.

The "if" part of the statement (ii) is proved with the help of the Riccati equation (R1). Assume that (2.5) holds. Let B be any constant such that $1 < B < 1 + \alpha$ and choose $T > 0$ so that

$$\int_T^\infty p(s)\rho(s)^{1/\alpha} ds \le \frac{1}{\alpha}(B-1)B^{-(\frac{1}{\alpha}+1)}. \tag{2.25}$$

Consider the set U and the integral operator G defined by

$$U = \{u \in C_b[T,\infty) : \rho(t) \le u(t) \le B\rho(t), \ \ t \ge T\}, \tag{2.26}$$

and

$$Gu(t) = \alpha \int_t^\infty p(s)u(s)^{\frac{1}{\alpha}+1} ds + \rho(t), \qquad t \ge T. \tag{2.27}$$

Using (2.25)–(2.27), we see that if $u \in U$, then

$$\begin{aligned}\rho(t) \le Gu(t) &\le \alpha B^{\frac{1}{\alpha}+1} \int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}+1} ds + \rho(t) \\ &\le \left(\alpha B^{\frac{1}{\alpha}+1} \int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}} ds + 1\right)\rho(t) \le B\rho(t), \qquad t \ge T,\end{aligned}$$

and that if $u_1,\ u_2 \in U$, then

$$\begin{aligned}|Gu_1(t) - Gu_2(t)| &\le \alpha \int_t^\infty p(s)|u_1(s)^{\frac{1}{\alpha}+1} - u_2(s)^{\frac{1}{\alpha}+1}| ds \\ &\le (\alpha+1)B^{\frac{1}{\alpha}} \int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}} |u_1(s) - u_2(s)| ds \\ &\le (\alpha+1)B^{\frac{1}{\alpha}} \int_T^\infty p(s)\rho(s)^{\frac{1}{\alpha}} ds \|u_1 - u_2\|_b \le \delta \|u_1 - u_2\|_b,\end{aligned}$$

where

$$\delta = \left(1 + \frac{1}{\alpha}\right)\left(1 - \frac{1}{B}\right) < 1.$$

This shows that G is a contraction on the closed subset U of $C_b[T,\infty)$. Therefore, there exists a unique fixed point $u \in U$ which satisfies

$$u(t) = \alpha \int_t^\infty p(s)u(s)^{\frac{1}{\alpha}+1} ds + \rho(t), \qquad t \ge T, \tag{2.28}$$

and hence gives a solution of the Riccati equation (R1) on $[T,\infty)$. With this u construct the function (cf. (2.13))

$$(x(t), y(t)) = \left(c \exp\left(-\int_t^\infty p(s)u(s)^{\frac{1}{\alpha}} ds\right),\ c^\alpha u(t) \exp\left(-\alpha \int_t^\infty p(s)u(s)^{\frac{1}{\alpha}} ds\right)\right), \tag{2.29}$$

for $t \ge T$, where $c > 0$ is any given constant. Then, (x, y) is a solution of system (A) on $[T,\infty)$. From (2.29) we see that $\lim_{t\to\infty} x(t) = c$ and $\lim_{t\to\infty} y(t)/\rho(t) = c^\alpha$. The latter follows immediately from

$$1 \le \frac{u(t)}{\rho(t)} \le 1 + \alpha B^{\frac{1}{\alpha}+1} \int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}} ds, \quad t \ge T, \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{u(t)}{\rho(t)} = 1.$$

Thus (x, y) given by (2.29) satisfies (2.6) and gives a minimal solution of (A) on $[T,\infty)$. This completes the proof of Theorem 2.3. □

As regards maximal and minimal solutions the second kind of (A), owing to the duality principle, necessary and sufficient conditions for their existence can be formulated automatically from Theorem 2.3.

Theorem 2.5.

(i) *Assume that $I_q = \infty$. System (A) has maximal nonoscillatory solutions of the second kind if and only if*

$$\int_0^\infty p(t)Q(t)^{1/\alpha}dt < \infty. \tag{2.30}$$

In this case, for any given constant $c \neq 0$ there exists a solution (x, y) such that

$$\lim_{t\to\infty} x(t) = c, \qquad \lim_{t\to\infty} \frac{y(t)}{Q(t)} = -\varphi_\alpha(c). \tag{2.31}$$

(ii) *Assume that $I_p < \infty$. System (A) has minimal nonoscillatory solutions of the second kind if and only if*

$$\int_0^\infty q(t)\pi(t)^\alpha dt < \infty. \tag{2.32}$$

In this case, for any given constant $d \neq 0$ there exists a solution (x, y) such that

$$\lim_{t\to\infty} y(t) = d, \qquad \lim_{t\to\infty} \frac{x(t)}{\pi(t)} = -\varphi_{1/\alpha}(d). \tag{2.33}$$

2.3 Bounded nonoscillatory solutions of the non-minimal type

Suppose that $I_p < \infty \wedge I_q < \infty$. Then it is easy to see that both condition (2.5) of Theorem 2.3(ii) and condition (2.32) of Theorem 2.5(ii) are satisfied, and so system (A) always has minimal nonoscillatory solutions of the first and of the second kinds. Moreover, as is demonstrated below, under the assumption of the convergence of both integrals I_p and I_q system (A) possesses also bounded nonoscillatory solutions of the non-minimal type. Thus, in this case the structure of the set of nonoscillatory solutions of (A) is simple because all subclasses of bounded solutions appearing in its "apriori" classification scheme are always nonempty.

Theorem 2.6. *If $I_p < \infty \wedge I_q < \infty$, then all nonoscillatory solutions of system (A) are bounded, and for any given constants c and d with $cd \neq 0$ there exists a solution (x, y) of (A) satisfying $x(\infty) = c$ and $y(\infty) = d$.*

Proof. We need only to prove the second half of the theorem. First we deal with solutions of the first kind of (A). Given any pair of constants (c, d) such that $cd > 0$, we construct a solution (x, y) of (A) such that $x(t)y(t) > 0$ for all large t and tends to (c, d) as $t \to \infty$. The Riccati equation (R1) is used for this purpose. Remembering that (R1) is the differential equation to be satisfied by $u = y/\varphi_\alpha(x)$, it is natural to expect that a positive solution u of (R1) satisfying $u(\infty) = d/\varphi_\alpha(c)$ should give rise to a solution (x, y) of the first kind of (A) such that $(x(\infty), y(\infty)) = (c, d)$. To verify the truth of this expectation we proceed as follows.

For simplicity we put

$$\omega = \frac{d}{\varphi_\alpha(c)}. \tag{2.34}$$

Choose $T > 0$ so that

$$\int_T^\infty p(s)ds \le \frac{1}{2\omega^{1/\alpha}}(\alpha+1)^{\frac{1}{\alpha}+1}, \qquad \int_T^\infty q(s)ds \le \frac{\alpha\omega}{2}. \tag{2.35}$$

Define the integral operator

$$Hw(t) = \omega + \alpha \int_t^\infty p(s) w(s)^{\frac{1}{\alpha}+1} ds + \rho(t), \qquad t \geq T, \tag{2.36}$$

and let it act on the set

$$\mathcal{W} = \{w \in C_b[T, \infty) : \omega \leq w(t) \leq (1+\alpha)\omega, \;\; t \geq T\}. \tag{2.37}$$

Since it easy to show that $H(\mathcal{W}) \subset \mathcal{W}$ and that $w_1, w_2 \in \mathcal{W}$ implies

$$\|Hw_1 - Hw_2\|_b \leq \frac{1}{2} \|w_1 - w_2\|_b,$$

by the contraction mapping principle there exists a unique $w \in \mathcal{W}$ such that $w = Hw$, i.e.,

$$w(t) = \omega + \alpha \int_t^\infty p(s) w(s)^{\frac{1}{\alpha}+1} ds + \rho(t), \qquad t \geq T. \tag{2.38}$$

Differentiating (2.38), we see that w is a solution of (R1) on $[T, \infty)$ satisfying $w(\infty) = \omega > 0$. We now form the function (cf. (2.13))

$$(x(t), y(t)) = \left(c \exp\left(- \int_t^\infty p(s) w(s)^{\frac{1}{\alpha}} ds \right), \; \varphi_\alpha(c) w(t) \exp\left(-\alpha \int_t^\infty p(s) w(s)^{\frac{1}{\alpha}} ds \right) \right), \tag{2.39}$$

for $t \geq T$. Then, (x, y) is a solution of the first kind of system (A) such that $(x(t), y(t)) \to (c, d)$ as $t \to \infty$.

We next deal with solutions of the second kind of (A). For any given pair (c, d) such that $cd < 0$ we have to confirm the existence of solutions (x, y) of (A) tending to (c, d) as $t \to \infty$. As the duality principle shows this problem is equivalent to proving that the dual system (B) possesses a solution (X, Y) of the first kind such that $(X(\infty), Y(\infty)) = (-c, d)$ (or $(X(\infty), Y(\infty)) = (c, -d)$). The latter problem, however, has already been resolved above by solving the differential equation

$$W' + \frac{1}{\alpha} q(t) W^{\alpha+1} + p(t) = 0, \tag{2.40}$$

which is the Riccati equation of the type (R1) for system (B). This completes the proof. □

3 Existence of intermediate solutions of (A)

We turn to the question of constructing intermediate nonoscillatory solutions of system (A) via the Riccati equations. This question seems to be more difficult than we imagine, and we have to be content with giving a partial answer, eight theorems ensuring the existence of intermediate solutions of (A), presented below.

Theorem 3.1. *Assume that $I_p = \infty \wedge I_q < \infty$. System (A) possesses intermediate nonoscillatory solutions of the first kind if the conditions*

$$\int_0^\infty p(t) \rho(t)^{1/\alpha} dt = \infty \quad \text{and} \quad \int_0^\infty q(t) P(t)^\alpha dt < \infty \tag{3.1}$$

are satisfied.

Proof. We intend to solve the Riccati equation (R1) so that the obtained solution u gives birth to an intermediate solution (x, y) of (A). This time we employ the Schauder–Tychonoff fixed point theorem instead of the contraction mapping principle.

Let $T > 0$ be large enough so that

$$\int_T^\infty q(s)P(s)^\alpha ds \leq (A-1)^\alpha A^{-\alpha-1}, \tag{3.2}$$

where $A > 1$ is any fixed constant. This is possible by the second condition of (3.1). We use the abbreviation

$$r(t) = \int_t^\infty q(s)P(s)^\alpha ds. \tag{3.3}$$

It is clear that $r(t) \leq 1$ for $t \geq T$. Noting that

$$\rho(t) = \int_t^\infty q(s)P(s)^\alpha \cdot P(s)^{-\alpha} ds \leq r(t)P(t)^{-\alpha}, \qquad t \geq T,$$

we define

$$\mathcal{U} = \{u \in C[T, \infty) : \rho(t) \leq u(t) \leq Ar(t)P(t)^{-\alpha},\ \ t \geq T\}, \tag{3.4}$$

and prove that the integral operator

$$Gu(t) = \alpha \int_t^\infty p(s)u(s)^{\frac{1}{\alpha}+1} ds + \rho(t), \qquad t \geq T, \tag{3.5}$$

is a continuous self-map of $\mathcal{U}$ and sends $\mathcal{U}$ into a relatively compact subset of the locally convex space $C[T, \infty)$.

(i) If $u \in \mathcal{U}$, then

$$\begin{aligned}\alpha \int_t^\infty p(s)u(s)^{\frac{1}{\alpha}+1} ds &\leq \alpha A^{\frac{1}{\alpha}+1} \int_t^\infty p(s)r(s)^{\frac{1}{\alpha}+1} P(s)^{-\alpha-1} ds \\ &\leq \alpha A^{\frac{1}{\alpha}+1} r(t)^{\frac{1}{\alpha}+1} \int_t^\infty p(s)P(s)^{-\alpha-1} ds = A^{\frac{1}{\alpha}+1} r(t)^{\frac{1}{\alpha}+1} P(t)^{-\alpha},\end{aligned}$$

for $t \geq T$, and so we have

$$\begin{aligned}\rho(t) \leq Gu(t) &\leq A^{\frac{1}{\alpha}+1} r(t)^{\frac{1}{\alpha}+1} P(t)^{-\alpha} + r(t)P(t)^{-\alpha} \\ &= \left(A^{\frac{1}{\alpha}+1} r(t)^{\frac{1}{\alpha}} + 1\right) r(t)P(t)^{-\alpha} \leq \left(A^{\frac{1}{\alpha}+1} r(T)^{\frac{1}{\alpha}} + 1\right) r(t)P(t)^{-\alpha} \\ &\leq Ar(t)P(t)^{-\alpha}, \qquad t \geq T,\end{aligned}$$

where we have used (3.2) at the first step. This means that G is a self-map of $\mathcal{U}$.

(ii) Let $\{u_n\}_{n=1}^\infty$ be a sequence in $\mathcal{U}$ converging to $u \in \mathcal{U}$ uniformly on compact subintervals of $[T, \infty)$. Noting that

$$|u_n(t)^{\frac{1}{\alpha}+1} - u(t)^{\frac{1}{\alpha}+1}| \leq 2\left(\frac{1}{\alpha}+1\right) A^{\frac{1}{\alpha}+1} r(t)^{\frac{1}{\alpha}+1} P(t)^{-\alpha-1} \leq 2\left(\frac{1}{\alpha}+1\right) A^{\frac{1}{\alpha}+1} P(t)^{-\alpha-1}$$

for all n, we conclude from the Lebesgue convergence theorem that $Gu_n(t) \to Gu(t)$, $n \to \infty$, uniformly on compact subintervals of $[T, \infty)$. This shows that G is a continuous map.

(iii) The inclusion $G(\mathcal{U}) \subset \mathcal{U}$ implies the local uniform boundedness of $G(\mathcal{U})$ on $[T, \infty)$. The local equicontinuity of $G(\mathcal{U})$ follows from the fact that any $u \in \mathcal{U}$ satisfies

$$0 \leq -(Gu)'(t) = \alpha p(t)u(t)^{\frac{1}{\alpha}+1} + q(t) \leq \alpha A^{\frac{1}{\alpha}+1} p(t)P(t)^{-\alpha-1} + q(t),$$

for $t \geq T$. It follows that $\mathcal{U}$ is relatively compact in $C[T, \infty)$.

Thus all the hypotheses of the Schauder–Tychonoff fixed point theorem are fulfilled (see e.g. Coppel [1]), and hence there exists $u \in \mathcal{U}$ such that $u = Gu$, that is,

$$u(t) = \alpha \int_t^\infty p(s) u(s)^{\frac{1}{\alpha}+1} ds + \rho(t), \qquad t \geq T,$$

which means that $u(t)$ is a positive solution of (R1) on $[T, \infty)$. With this $u(t)$ construct the function (cf. (2.11))

$$(X(t), Y(t)) = \left(a \exp\left(\int_T^t p(s) u(s)^{\frac{1}{\alpha}} ds \right), \; a^\alpha u(t) \exp\left(\alpha \int_T^t p(s) u(s)^{\frac{1}{\alpha}} ds \right) \right), \tag{3.6}$$

where $a > 0$ is any constant. The x-component of (3.6) satisfies

$$X(t) \geq a \exp\left(\int_T^t p(s) \rho(s)^{\frac{1}{\alpha}} ds \right) \to \infty, \qquad t \to \infty,$$

because of the first condition of (3.1). We notice, however, that

$$\limsup_{t \to \infty} \frac{X(t)}{P(t)} < \infty.$$

In fact, choosing $T_1 > T$ so that $P(T_1) \geq 1$ and

$$u(t) \leq A r(t) P(t)^{-\alpha} \leq P(t)^{-\alpha}, \qquad t \geq T_1,$$

we obtain

$$\int_{T_1}^t p(s) u(s)^{\frac{1}{\alpha}} ds \leq \int_{T_1}^t p(s) P(s)^{-1} ds \leq \log P(t), \qquad t \geq T_1.$$

The boundedness of $X(t)/P(t)$ then follows from (3.6) immediately. Using (3.4), we see that the y component of (3.6) satisfies

$$Y(t) = u(t) X(t)^\alpha \leq A r(t) \left(\frac{X(t)}{P(t)} \right)^\alpha \to 0, \qquad t \to \infty.$$

Thus it is assured that the function (X, Y) given by (3.6) is an intermediate solution of the first kind of system (A) on $[T, \infty)$. This completes the proof. $\square$

Remark 3.2. From the generalization of Fubini's theorem obtained by Došlá et al. [2] it follows that integral conditions in (3.1) are consistent only if $\alpha > 1$.

The counterpart of Theorem 3.1 in the case $I_p < \infty \wedge I_q = \infty$ is formulated as follows. Its truth is ensured by the duality principle.

Theorem 3.3. *Assume that $I_p < \infty \wedge I_q = \infty$. System (A) possesses intermediate nonoscillatory solutions of the second kind if the following conditions are satisfied:*

$$\int_0^\infty q(t) \pi(t)^\alpha dt = \infty \quad \text{and} \quad \int_0^\infty p(t) Q(t)^{1/\alpha} dt < \infty. \tag{3.7}$$

System (A) may have intermediate solutions in the case where p and q satisfy

$$\int_0^\infty p(t) \rho(t)^{\frac{1}{\alpha}} dt = \infty \quad \text{and} \quad \int_0^\infty q(t) P(t)^\alpha dt = \infty. \tag{3.8}$$

Theorem 3.4. *Assume that* $I_p = \infty \wedge I_q < \infty$. *System* (A) *possesses intermediate nonoscillatory solutions of the first kind if in addition to* (3.8) *the condition*

$$\int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}+1}ds = o(\rho(t)), \qquad t \to \infty, \tag{3.9}$$

is satisfied.

Proof. Because of (3.9) there is a positive continuous function ω such that $\omega(t) \to 0$ as $t \to \infty$ and

$$\int_t^\infty q(s)\rho(s)^{\frac{1}{\alpha}+1}ds = \omega(t)\rho(t), \tag{3.10}$$

for all large t. Let $B > 1$ be any given constant and choose $T > 0$ so that

$$\omega(t) \le \frac{1}{\alpha}(B-1)B^{-\frac{1}{\alpha}-1}, \qquad t \ge T. \tag{3.11}$$

We now let the integral operator G defined by (3.5) act on the set

$$\mathcal{U} = \{u \in C[T,\infty) : \rho(t) \le u(t) \le B\rho(t), \ \ t \ge T\}. \tag{3.12}$$

If $u \in \mathcal{U}$, then from (3.10) and (3.11) one easily sees that

$$\begin{aligned}\rho(t) \le Gu(t) &\le \alpha B^{\frac{1}{\alpha}+1}\int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}+1}ds + \rho(t)\\ &= \alpha B^{\frac{1}{\alpha}+1}\omega(t)\rho(t) + \rho(t) \le B\rho(t), \qquad t \ge T.\end{aligned}$$

This shows that G maps $\mathcal{U}$ into itself. The continuity of G and the relative compactness of $G(\mathcal{U})$ can be proved routinely (as in the proof of Theorem 3.1). It follows that G has a fixed point $u \in \mathcal{U}$ which gives a global solution of equation (R1) on $[T,\infty)$. Using this u we define the function (X,Y) by (3.6). Then, it is a solution of the first kind of (A) on $[T,\infty)$. It is clear that $X(t) \to \infty$ as $t \to \infty$. It can be shown that the growth order of X as $t \to \infty$ is smaller than that of P, i.e.,

$$\lim_{t\to\infty}\frac{X(t)}{P(t,T)} = 0, \quad \text{where} \quad P(t,T) = \int_T^t p(s)ds. \tag{3.13}$$

In fact, from the equation

$$\left(\frac{X(t)}{P(t,T)}\right)' = \frac{p(t)}{P(t,T)^2}(P(t,T)Y(t)^{\frac{1}{\alpha}} - X(t)),$$

and the inequality

$$X(t) \ge \int_T^t p(s)Y(s)^{\frac{1}{\alpha}}ds \ge P(t,T)Y(t)^{\frac{1}{\alpha}},$$

holding for $t > T$, we see that $X(t)/P(t,T)$ is nonincreasing for $t > T$ and tends to a finite limit $k \ge 0$ as $t \to \infty$. If $k > 0$, then

$$\lim_{t\to\infty}\frac{X(t)}{P(t,T)} = \lim_{t\to\infty}\frac{X'(t)}{p(t)} = \lim_{t\to\infty}Y(t)^{\frac{1}{\alpha}} = k^{\frac{1}{\alpha}} > 0.$$

This implies that (X,Y) is a maximal solution of (A). By Theorem 2.3 such a situation occurs only if $\int_0^\infty q(t)P(t)^\alpha dt < \infty$, which contradicts the second condition of (3.8). Thus we must have (3.13).

It remains to verify that $Y(t) \to 0$ as $t \to \infty$. Let u be the solution of (R1) obtained as a fixed point of G. Put $v = u^{-1/\alpha}$. Then, v is a solution of the second Riccati equation (R2). Integrating (R2) on $[T,t]$ gives

$$v(t) = v(T) + \frac{1}{\alpha}\int_T^t q(s)v(s)^{\alpha+1}ds + P(t,T), \qquad t \geq T,$$

which implies in particular that

$$v(t) \geq P(t,T), \ \ t \geq T, \quad \Longrightarrow \quad u(t)P(t,T)^{\alpha} \leq 1, \ \ t \geq T.$$

Using the last inequality along with (3.13), we find that

$$Y(t) = u(t)X(t)^{\alpha} = u(t)P(t,T)^{\alpha}\left(\frac{X(t)}{P(t,T)}\right)^{\alpha} \to 0, \qquad t \to \infty.$$

It is concluded therefore that (X,Y) is an intermediate solution of system (A). This completes the proof. $\square$

The duality principle applied to Theorem 3.4 guarantees the validity of the following result.

Theorem 3.5. *Assume that $I_p < \infty \wedge I_q = \infty$. System (A) possesses intermediate nonoscillatory solutions of the second kind if the following conditions are satisfied:*

$$\int_0^{\infty} q(t)\pi(t)^{\alpha}dt = \infty, \qquad \int_0^{\infty} p(t)Q(t)^{\frac{1}{\alpha}}dt = \infty, \tag{3.14}$$

and

$$\int_0^{\infty} q(t)\pi(t)^{\alpha+1}dt = o(\pi(t)), \qquad t \to \infty. \tag{3.15}$$

Example 3.6. Consider the half-linear system (A) in which p and q are continuous regularly varying functions given by

$$p(t) = t^{\lambda}l(t), \qquad q(t) = t^{\mu}m(t),$$

where λ and μ are constants and l and m are slowly varying functions on $(0,\infty)$. We assume that λ and μ satisfy

$$\lambda > -1, \qquad \mu < -1, \qquad \lambda + 1 + \frac{1}{\alpha}(\mu+1) = 0. \tag{3.16}$$

Such a system is referred to as system (A_1). Note that this system (A_1) is in the case $I_p = \infty \wedge I_q < \infty$ for which we have

$$P(t) \sim \frac{t^{\lambda+1}l(t)}{\lambda+1}, \qquad \rho(t) \sim \frac{t^{\mu+1}m(t)}{-(\mu+1)}, \qquad t \to \infty. \tag{3.17}$$

Here the symbol $\sim$ is used to denote the asymptotic equivalence between two positive functions

$$f(t) \sim g(t), \ \ t \to \infty \quad \Longleftrightarrow \quad \lim_{t\to\infty}\frac{f(t)}{g(t)} = 1.$$

For the definition and basic properties of slowly and regularly varying functions the reader is referred to Marić [11, Appendix].

Using the Karamata integration theorem, we see that the following asymptotic relations hold:

$$\int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}+1}ds \sim \frac{t^{\mu+1}l(t)m(t)^{\frac{1}{\alpha}+1}}{\big(-(\mu+1)\big)^{\frac{1}{\alpha}+2}}, \tag{3.18}$$

$$\int_0^t p(s)\rho(s)^{\frac{1}{\alpha}}ds \sim \frac{1}{(-(\mu+1))^{\frac{1}{\alpha}}}\int_a^t s^{-1}l(s)m(s)^{\frac{1}{\alpha}}ds, \tag{3.19}$$

$$\int_0^t q(s)P(s)^\alpha ds \sim \frac{1}{(\lambda+1)^\alpha}\int_a^t s^{-1}l(s)^\alpha m(s)ds, \tag{3.20}$$

where $a > 0$ is any fixed constant.

We see that (3.18) implies (3.15), and that (3.19) and (3.20) imply (3.1) if both

$$\int_a^\infty t^{-1}l(t)m(t)^{\frac{1}{\alpha}}dt = \infty, \tag{3.21}$$

and

$$\int_a^\infty t^{-1}l(t)^\alpha m(t)dt < \infty \tag{3.22}$$

are satisfied. Further, since (3.18) is rewritten as $\int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}+1}ds \sim \omega(t)\rho(t)$ with $\omega(t) = l(t)m(t)^{\frac{1}{\alpha}}/(-(\mu+1)^{\frac{1}{\alpha}+1})$, condition (3.9) is fulfilled if

$$\lim_{t\to\infty} l(t)m(t)^{\frac{1}{\alpha}} = 0. \tag{3.23}$$

Therefore, under (3.16) system (A_1) possesses an intermediate solution of the first kind if (3.21) and (3.22) are satisfied (by Theorem 3.1), or if (3.21) and (3.23), plus the condition

$$\int_a^\infty t^{-1}l(t)^\alpha m(t)dt = \infty, \tag{3.24}$$

are satisfied (by Theorem 3.4).

The condition (3.9) in Theorem 3.4 requires the integral $\int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}+1}ds$ to decrease faster than $\rho(t)$ as $t \to \infty$. This requirement can be relaxed to a significant degree as follows.

Theorem 3.7. *Assume that $I_p = \infty \wedge I_q < \infty$. System (A) possesses intermediate nonoscillatory solutions of the first kind if in addition to (3.8) the condition*

$$\int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}+1}ds \le \gamma\rho(t) \quad \textit{for all large } t, \tag{3.25}$$

holds for some positive constant γ such that

$$\gamma \le (\alpha+1)^{-\frac{1}{\alpha}-1}. \tag{3.26}$$

Proof. Let $T > 0$ be so large that (3.25) holds for $t \geq T$. Because of (3.26) there exists a constant $C > 1$ such that

$$\gamma \leq \frac{1}{\alpha}(C-1)C^{-\frac{1}{\alpha}-1}. \tag{3.27}$$

This follows from the fact that $(\alpha+1)^{-\frac{1}{\alpha}-1} = \max_{C>1} \frac{1}{\alpha}(C-1)C^{-\frac{1}{\alpha}-1}$. (Note that the maximum is attained at $C = \alpha+1$.) With this C define the set $\mathcal{U}$ by

$$\mathcal{U} = \Big\{ u \in C[T,\infty) : \rho(t) \leq u(t) \leq C\rho(t), \ \ t \geq T \Big\},$$

and let the integral operator G given by (3.5) act on $\mathcal{U}$. G is a self-map of $\mathcal{U}$ since $u \in \mathcal{U}$ implies that

$$\begin{aligned} Gu(t) &\leq \alpha C^{\frac{1}{\alpha}+1} \int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}+1} ds + \rho(t) \\ &\leq \Big(\alpha C^{\frac{1}{\alpha}+1}\gamma + 1\Big)\rho(t) \leq C\rho(t), \qquad t \geq T. \end{aligned}$$

Since it is shown routinely that G is continuous and sends $\mathcal{U}$ into a relatively compact subset of $C[T,\infty)$, the Schauder–Tychonoff theorem guarantees the existence of a fixed point $u \in \mathcal{U}$ of G which gives a global solution of the Riccati equation (R1). From this point on proceeding exactly as in the proof of Theorem 3.4, we can construct the intermediate solution (X,Y) of (A) according to the formula (3.6) by making use of the solution u of (R1) mentioned above. This completes the proof. □

Example 3.8. Consider the differential system

$$x' - p(t)\varphi_{1/\alpha}(y) = 0, \qquad y' + \frac{kp(t)}{P(t)^{\alpha+1}}\varphi_\alpha(x) = 0, \tag{A$_2$}$$

where $I_p = \infty$, $P(t) = \int_0^t p(s)ds$ and k is a positive constant. For this system we have

$$\begin{aligned} \rho(t) &= \int_t^\infty \frac{kp(s)}{P(s)^{\alpha+1}} ds = \frac{k}{\alpha}P(t)^{-\alpha}, \\ \int_0^\infty p(s)\rho(s)^{\frac{1}{\alpha}} ds &= \left(\frac{k}{\alpha}\right)^{\frac{1}{\alpha}} \int_0^\infty p(s)P(s)^{-1} ds = \infty, \\ \int_0^\infty q(s)P(s)^\alpha ds &= k\int_0^\infty p(s)P(s)^{-1} ds = \infty, \end{aligned}$$

and

$$\int_t^\infty p(s)\rho(s)^{\frac{1}{\alpha}+1} ds = \frac{1}{\alpha}\left(\frac{k}{\alpha}\right)^{\frac{1}{\alpha}} \rho(t).$$

From Theorem 3.7 it follows that system (A$_2$) possesses an intermediate solution of the first kind if

$$\frac{1}{\alpha}\left(\frac{k}{\alpha}\right)^{\frac{1}{\alpha}} \leq (\alpha+1)^{-\frac{1}{\alpha}-1}, \quad \text{i.e.,} \quad k \leq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

If in particular $k = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$, then (A$_2$) has an exact intermediate solution (x,y) with

$$x(t) = P(t)^{\frac{\alpha}{\alpha+1}}, \qquad y(t) = \left(\frac{\alpha}{\alpha+1}\right)^\alpha P(t)^{-\frac{\alpha}{\alpha+1}}.$$

The duality principle ensures the truth of the following theorem as a counterpart of Theorem 3.7.

Theorem 3.9. *Assume that $I_p < \infty \wedge I_q = \infty$. System (A) possesses intermediate nonoscillatory solutions of the second kind if in addition to (3.14) the condition*

$$\int_t^\infty q(s)\pi(s)^{\alpha+1} ds \leq \delta\pi(t) \quad \text{for all large } t, \tag{3.28}$$

is satisfied for some positive constant δ such that

$$\delta \leq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{3.29}$$

An alternative way to formulate and prove the existence results on intermediate solutions under condition $I_p = \infty \wedge I_q < \infty$ is to use the function $P(t)$ and the Riccati equation (R2) instead of $\rho(t)$ and (R1), respectively.

Theorem 3.10. *Assume that $I_p = \infty \wedge I_q < \infty$. System (A) possesses intermediate nonoscillatory solutions of the first kind if in addition to (3.8) the condition*

$$\int_0^t q(s)P(s)^{\alpha+1} ds = o(P(t)), \qquad t \to \infty, \tag{3.30}$$

is satisfied.

Proof. The relation (3.30) means that there exists a positive continuous function ε such that $\varepsilon(t) \to 0$ as $t \to \infty$ and

$$\int_0^t q(s)P(s)^{\alpha+1} ds = \varepsilon(t)P(t) \tag{3.31}$$

for all large t. For any given constant $A > 1$ we can choose $T > 0$ so large that

$$\varepsilon(t) \leq \alpha(A-1)A^{-\alpha-1} \tag{3.32}$$

for $t \geq T$. We now define the set $\mathcal{V}$ and the mapping $\mathcal{G}$ by

$$\mathcal{V} = \{v \in C[T,\infty) : P(t) \leq v(t) \leq AP(t),\ t \geq T\}$$

and

$$Gv(t) = P(t) + \frac{1}{\alpha}\int_T^t q(s)v(s)^{\alpha+1} ds, \qquad t \geq T. \tag{3.33}$$

It is clear that $\mathcal{V}$ is a closed convex subset of the Fréchet space $C[T,\infty)$ with the topology of uniform convergence of functions on compact subintervals of $[T,\infty)$.

If $v \in \mathcal{V}$, then from (3.31) and (3.32) it follows that

$$P(t) \leq Gv(t) \leq P(t) + \frac{1}{\alpha}A^{\alpha+1}\int_T^t q(s)P(s)^{\alpha+1} ds \leq P(t) + \frac{1}{\alpha}A^{\alpha+1}\varepsilon(t)P(t) \leq AP(t)$$

for $t \geq T$, which proves that G maps $\mathcal{V}$ into itself. It can be shown routinely that G is a continuous mapping and that $G(\mathcal{V})$ is a relatively compact subset of $C[T,\infty)$. Therefore it follows from the Schauder–Tychonoff fixed point theorem that there is an element $v \in \mathcal{V}$ such that $v = Gv$. In view of the definition of G this function $v = v(t)$ satisfies the integral equation

$$v(t) = P(t) + \frac{1}{\alpha}\int_T^t q(s)v(s)^{\alpha+1} ds, \quad t \geq T.$$

Using this v we define the function (X,Y) by

$$(X(t),Y(t)) = \left(b^{\frac{1}{\alpha}}v(t)\exp\left(-\frac{1}{\alpha}\int_T^t q(s)v(s)^\alpha ds\right),\ b\exp\left(-\int_T^t q(s)v(s)^\alpha ds\right)\right), \tag{3.34}$$

where $t \geq T$ and b is any positive constant. Then, it is a nonoscillatory solution of the first kind of (A) on $[T,\infty)$. It is clear that $Y(t) \to 0$ as $t \to \infty$. It remains to prove that $X(t) \to \infty$ as $t \to \infty$. For this purpose we first show that

$$\lim_{t\to\infty} \frac{Y(t)}{\rho(t)} = \infty. \tag{3.35}$$

In fact, from the equation

$$\left[\frac{Y(t)}{\rho(t)}\right]' = \frac{q(t)}{\rho(t)^2}\left[Y(t) - \rho(t)X(t)^\alpha\right],$$

and the inequality

$$Y(t) = \int_t^\infty q(s)X(s)^\alpha ds \geq X(t)^\alpha \rho(t),$$

which holds for $t > T$, we see that the function $Y(t)/\rho(t)$ is nondecreasing for $t \geq T$ and tends to a positive limit $K(\leq \infty)$ as $t \to \infty$. If $K < \infty$, then

$$\lim_{t\to\infty} \frac{Y(t)}{\rho(t)} = \lim_{t\to\infty} \frac{Y'(t)}{-q(t)} = \lim_{t\to\infty} X(t)^\alpha < \infty.$$

This implies that (X,Y) is a minimal solution of (A). But this contradicts the first condition of (3.8) (cf. Theorem 2.3(ii)). Thus we must have (3.35).

Now, let v be the function obtained as a fixed point of G and put $u = v^{-\alpha}$. Then, u is a solution of the first Riccati equation (R1) and integration of (R1) from t to ∞ yields

$$u(t) = \alpha \int_t^\infty p(s)u(s)^{\frac{1}{\alpha}+1} ds + \rho(t), \qquad t \geq T,$$

which implies, in particular, that

$$u(t) \geq \rho(t),\ t \geq T \implies v(t)\rho(t)^{\frac{1}{\alpha}} \leq 1, \qquad t \geq T.$$

Using this inequality along with (3.35) we get that

$$X(t) = v(t)Y(t)^{\frac{1}{\alpha}} = v(t)\rho(t)^{\frac{1}{\alpha}}\left[\frac{Y(t)}{\rho(t)}\right]^{\frac{1}{\alpha}} \to \infty, \qquad t \to \infty.$$

Thus, (X,Y) is an intermediate solution of (A). This completes the proof. □

Theorem 3.11. *Assume that $I_p = \infty \wedge I_q < \infty$. System (A) possesses intermediate nonoscillatory solutions of the first kind if (3.8) is satisfied and there exist constants $T > 0$ and γ_T such that*

$$0 < \gamma_T \leq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{3.36}$$

and

$$\int_T^t q(s)P(s)^{\alpha+1} ds \leq \gamma_T P(t) \tag{3.37}$$

holds for $t \geq T$.

Proof. Choose $T > 0$ so large that (3.37) holds for $t \geq T$ and a positive constant γ_T satisfying (3.36). From (3.36) it follows that there exists a constant $C > 1$ such that

$$\gamma_T \leq \alpha(C-1)C^{-\alpha-1}. \tag{3.38}$$

(This follows from the fact that

$$\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} = \max_{C>1} \alpha(C-1)C^{-\alpha-1}.$$

Note that the maximum is attained at $C = \frac{1}{\alpha} + 1$.) With this C define the set $\mathcal{V}$ by

$$\mathcal{V} = \{v \in C[T,\infty) : \; P(t) \leq v(t) \leq CP(t), \; t \geq T\},$$

and let the integral operator G given by (3.33) act on $\mathcal{V}$. G is a self-map of $\mathcal{V}$ since $v \in \mathcal{V}$ implies that

$$P(t) \leq Gv(t) \leq P(t) + \frac{1}{\alpha}C^{\alpha+1}\int_T^t q(s)P(s)^{\alpha+1}ds \leq \left(1 + \frac{1}{\alpha}C^{\alpha+1}\gamma_T\right) \leq CP(t), \qquad t \geq T.$$

It can be shown routinely that G is continuous and sends $\mathcal{V}$ into a relatively compact subset of $C[T,\infty)$, and so the Schauder–Tychonoff fixed point theorem guarantees the existence of a fixed point $v \in \mathcal{V}$ of G which gives a global solution of the Riccati equation (R2). From this point on proceeding exactly as in the proof of Theorem 3.10, we can construct the intermediate solution (X, Y) of (A) according to the formula (3.34) using the solution v of (R2) mentioned above. This completes the proof. □

Analogs of Theorems 3.10 and 3.11 concerning the case $I_p < \infty \wedge I_q = \infty$ which can be obtained easily using the duality principle are omitted here.

Example 3.12. Consider the system

$$x' = p(t)\varphi_{1/\alpha}(y), \qquad y' + \lambda \frac{p(t)}{P(t)^{\alpha+1}\big(\log P(t)\big)^{\beta}}\varphi_\alpha(x) = 0, \tag{3.39}$$

where $\alpha \neq 1, \beta$ and λ are positive constants and $P(t) := \int_0^t p(s)ds \to \infty$ as $t \to \infty$.

If $\alpha > 1$ and $\beta \in (1, \alpha]$, then conditions (3.1) are satisfied (cf. with Example 4.1 in [5]) and according to Theorem 2.3(ii) and Theorem 3.1 system (A) has both minimal and intermediate nonoscillatory solutions of the first kind.

If $\beta \leq \min\{1, \alpha\}$, then the condition (3.8) holds and system (A) has neither minimal nor maximal nonoscillatory solutions. But if we choose $T > 0$ so large that $P(T) > 1$ and

$$\lambda\big(\log P(T)\big)^{-\beta} \leq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, \tag{3.40}$$

then

$$\begin{aligned}\int_T^t q(s)P(s)^{\alpha+1}ds &= \int_T^t \frac{\lambda p(s)}{P(s)^{\alpha+1}\big(\log P(s)\big)^{\beta}}P(s)^{\alpha+1}ds \\ &= \lambda \int_T^t p(s)\big(\log P(s)\big)^{-\beta}ds \leq \lambda\big(\log P(T)\big)^{-\beta}\int_T^t p(s)ds \leq \lambda\big(\log P(T)\big)^{-\beta}P(t)\end{aligned}$$

for $t \geq T$, and Theorem 3.11 guarantees the existence of intermediate solutions of the first kind for system (3.39).

Remark 3.13. As an earlier work more or less related to ours we refer to Li and Cheng [10] in which asymptotic analysis is made of the first order cyclic differential system $x' = p(t)f(y)$, $y' = -q(t)g(x)$, which are structurally more general than our (A), from the viewpoint of nonoscillation.

4 Application to scalar second order half-linear differential equations

The final section is devoted to the application of our theory for system (A) to scalar second order half-linear differential equations of the type

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \tag{E}$$

where α is a positive constant, and p and q are positive continuous functions on $[0, \infty)$.

By a solution we here mean a function x which is defined in a neighborhood of infinity and satisfies (E) at all large t. We call $p(t)\varphi_\alpha(x')$ the *quasi-derivative* of x and denote it by Dx. A solution x of (E) is called nonoscillatory if it satisfies $x(t)Dx(t) \neq 0$ for all large t. A nonoscillatory solutions is said to be *of the first kind* (resp. *of the second kind*) if $x(t)Dx(t) > 0$ (resp. $x(t)Dx(t) < 0$) for all large t.

The key to the systematic analysis of equation (E) is a simple fact that the substitution $y = Dx$ transforms this equation into the first order cyclic differential system

$$x' - p(t)^{-\frac{1}{\alpha}}\varphi_{\frac{1}{\alpha}}(y) = 0, \qquad y' + q(t)\varphi_\alpha(x) = 0, \tag{4.1}$$

which can be regarded as a special case of system (A). This fact makes it possible to translate a nonoscillation problem to be solved for (E) into a problem for (4.1) to which a general nonoscillation theory already established for (A) applies. Thus, in order to study (E) from the viewpoint of nonoscillation it is not necessary for us to analyze the second order equation (E) per se, but it suffices just to formulate automatically the desired results for (E) from the corresponding theorems for system (A) specialized to (4.1). The following notations are needed for this purpose:

$$\begin{aligned}
P_\alpha(t) &= \int_0^t p(s)^{-\frac{1}{\alpha}} ds \qquad && \text{if } \int_0^\infty p(t)^{-\frac{1}{\alpha}} dt = \infty, \\
\pi_\alpha(t) &= \int_t^\infty p(s)^{-\frac{1}{\alpha}} ds \qquad && \text{if } \int_0^\infty p(t)^{-\frac{1}{\alpha}} dt < \infty, \\
Q(t) &= \int_0^t q(s) ds \qquad && \text{if } \int_0^\infty q(t) dt = \infty, \\
\rho(t) &= \int_t^\infty q(s) ds \qquad && \text{if } \int_0^\infty q(t) dt < \infty.
\end{aligned}$$

We first state a result which follows from Theorem 2.2 specialized to (4.1).

Theorem 4.1. *All solutions of equation* (E) *are oscillatory if*

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}} dt = \infty \quad \textit{and} \quad \int_0^\infty q(t) dt = \infty. \tag{4.2}$$

To analyze the structure of nonoscillatory solutions of (E) it is necessary to distinguish the three cases:

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}}dt = \infty \quad \text{and} \quad \int_0^\infty q(t)dt < \infty, \tag{4.3}$$

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}}dt < \infty \quad \text{and} \quad \int_0^\infty q(t)dt = \infty, \tag{4.4}$$

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}}dt < \infty \quad \text{and} \quad \int_0^\infty q(t)dt < \infty. \tag{4.5}$$

According to the classification of solutions of system (A) applied to (4.1) we see that if (4.3) holds, then a nonoscillatory solution x of (E) is of the first kind ($x(t)Dx(t) > 0$, $|x(t)|$ is increasing and $|Dx(t)|$ is decreasing) and exhibits one of the following three patterns of asymptotic behavior at infinity:

I(i) $|x(\infty)| = \infty, \quad 0 < |Dx(\infty)| < \infty,$

I(ii) $|x(\infty)| = \infty, \quad Dx(\infty) = 0,$

I(iii) $0 < |x(\infty)| < \infty, \quad Dx(\infty) = 0,$

and that if (4.4) holds, then a nonoscillatory solution x of (E) is of the second kind ($x(t)Dx(t) < 0$, $|x(t)|$ is decreasing and $|Dx(t)|$ is increasing) and exhibits one of the following three patterns of asymptotic behavior at infinity:

II(i) $0 < |x(\infty)| < \infty, \quad |Dx(\infty)| = \infty,$

II(ii) $x(\infty) = 0, \quad |Dx(\infty)| = \infty,$

II(iii) $x(\infty) = 0, \quad 0 < |Dx(\infty)| < \infty.$

Finally, if (4.5) holds, then a nonoscillatory solution x of (A) is either of the first kind and satisfies either I(iii) or

III: $0 < |x(\infty)| < \infty, \quad 0 < |Dx(\infty)| < \infty,$

or is of the second kind and satisfies either II(iii) or III.

Solutions of the types I(i) and II(i) (resp. I(iii) and II(iii)) are said to be *maximal solutions* (resp. *minimal solutions*) of (E), while solutions of the types I(ii) and II(ii) are called *intermediate solutions* of (E). Solutions of the last type III are termed *bounded solutions of the non-minimal type*.

The existence of maximal and minimal solutions of (E) can be characterized completely, but intermediate solutions are put in a different situation.

Sharp existence theorems of maximal and minimal solutions of (E) are obtained as a result of specialization of Theorems 2.3 and 2.5 to system (4.1).

Theorem 4.2.

(i) *Suppose that* $\int_0^\infty p(t)^{-\frac{1}{\alpha}}dt = \infty$. *Equation* (E) *possesses maximal solutions of the first kind if and only if*

$$\int_0^\infty q(t)P_\alpha(t)^\alpha dt < \infty. \tag{4.6}$$

In this case, for any constant $d \neq 0$ *there exists a solution* x *of* (E) *such that*

$$\lim_{t\to\infty} \frac{x(t)}{P_\alpha(t)} = d, \qquad \lim_{t\to\infty} Dx(t) = \varphi_\alpha(d).$$

(ii) *Suppose that* $\int_0^\infty q(t)dt < \infty$. *Equation* (E) *possesses minimal solutions of the first kind if and only if*

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}} \rho(t)^{\frac{1}{\alpha}} dt < \infty. \tag{4.7}$$

In this case, for any constant $c \neq 0$ *there exists a solution* x *of* (E) *such that*

$$\lim_{t\to\infty} x(t) = c, \qquad \lim_{t\to\infty} \frac{Dx(t)}{\rho(t)} = \varphi_\alpha(c).$$

Theorem 4.3.

(i) *Suppose that* $\int_0^\infty q(t)dt = \infty$. *Equation* (E) *possesses maximal solutions of the second kind if and only if*

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}} Q(t)^{\frac{1}{\alpha}} dt < \infty. \tag{4.8}$$

In this case, for any constant $c \neq 0$ *there exists a solution* x *of* (E) *such that*

$$\lim_{t\to\infty} x(t) = c, \qquad \lim_{t\to\infty} \frac{Dx(t)}{Q(t)} = -\varphi_\alpha(c).$$

(ii) *Suppose that* $\int_0^\infty p(t)^{-\frac{1}{\alpha}} dt < \infty$. *Equation* (E) *possesses minimal solutions of the second kind if and only if*

$$\int_0^\infty q(t)\pi_\alpha(t)^\alpha dt < \infty. \tag{4.9}$$

In this case, for any constant $d \neq 0$ *there exists a solution* x *of (E) such that*

$$\lim_{t\to\infty} \frac{x(t)}{\pi_\alpha(t)} = d, \qquad \lim_{t\to\infty} Dx(t) = -\varphi_\alpha(d).$$

In the case (4.5) the following theorem can be considered as a corollary of Theorem 2.6.

Theorem 4.4. *Suppose that* (4.5) *holds. Then, for any given constants* c *and* d *with* $cd \neq 0$ *there exists a solution* x *of* (E) *satisfying* $x(\infty) = c$ *and* $Dx(\infty) = d$.

The closing topic is the existence of intermediate solutions for equation (E). Recall that an intermediate solution of the first kind (resp. of the second kind) of (E) may exist only in the case where p and q satisfy (4.3) (resp. (4.4)), and that such a solution x behaves like

$$\lim_{t\to\infty} |x(t)| = \infty, \qquad \lim_{t\to\infty} \frac{x(t)}{P_\alpha(t)} = 0, \qquad \lim_{t\to\infty} Dx(t) = 0, \qquad \lim_{t\to\infty} \frac{|Dx(t)|}{\rho(t)} = \infty,$$

if it is of the first kind, and like

$$\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} \frac{|x(t)|}{\pi_\alpha(t)} = \infty, \qquad \lim_{t\to\infty} |Dx(t)| = \infty, \qquad \lim_{t\to\infty} \frac{Dx(t)}{Q(t)} = 0,$$

if it is of the second kind. All existence theorems of intermediate solutions of (E) produced from the eight theorems of Section 3 can be summarized as follows.

Theorem 4.5.

(i) *Assume that* (4.3) *holds. Equation* (E) *possesses an intermediate solution of the first kind if the following conditions are satisfied*

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}}\rho(t)^{\frac{1}{\alpha}}dt = \infty, \qquad \int_0^\infty q(t)P_\alpha(t)^\alpha dt < \infty. \tag{4.10}$$

(ii) *Assume that* (4.4) *holds. Equation* (E) *possesses an intermediate solution of the second kind if the following conditions are satisfied*

$$\int_0^\infty q(t)\pi_\alpha(t)^\alpha dt = \infty, \qquad \int_0^\infty p(t)^{-\frac{1}{\alpha}}Q(t)^{\frac{1}{\alpha}}dt < \infty. \tag{4.11}$$

Theorem 4.6.

(i) *Assume that* (4.3) *holds. Equation* (E) *possesses an intermediate solution of the first kind if the following conditions are satisfied*

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}}\rho(t)^{\frac{1}{\alpha}}dt = \infty, \qquad \int_0^\infty q(t)P_\alpha(t)^\alpha dt = \infty, \tag{4.12}$$

$$\int_t^\infty p(s)^{-\frac{1}{\alpha}}\rho(s)^{\frac{1}{\alpha}+1}ds = o(\rho(t)), \qquad t \to \infty. \tag{4.13}$$

(ii) *Assume that* (4.4) *holds. Equation* (E) *possesses an intermediate solution of the second kind if the following conditions are satisfied*

$$\int_0^\infty q(t)\pi_\alpha(t)^\alpha dt = \infty, \quad \int_0^\infty p(t)^{-\frac{1}{\alpha}}Q(t)^{\frac{1}{\alpha}}dt = \infty. \tag{4.14}$$

$$\int_t^\infty q(s)\pi_\alpha(s)^{\alpha+1}ds = o(\pi_\alpha(t)), \qquad t \to \infty. \tag{4.15}$$

Theorem 4.7.

(i) *Assume that* (4.3) *holds. Equation* (E) *possesses an intermediate solution of the first kind if in addition to* (4.12) *the following condition is satisfied*

$$\int_t^\infty p(s)^{-\frac{1}{\alpha}}\rho(s)^{\frac{1}{\alpha}+1}ds \leq \delta\rho(t) \quad \textit{for all large} \ \ t, \tag{4.16}$$

where δ is a positive constant such that $\delta \leq (\alpha+1)^{-(\alpha+1)/\alpha}$.

(ii) *Assume that* (4.4) *holds. Equation* (E) *possesses an intermediate solution of the second kind if in addition to* (4.14) *the the following condition is satisfied*

$$\int_t^\infty q(s)\pi_\alpha(s)^{\alpha+1}ds \leq \gamma\pi_\alpha(t) \quad \textit{for all large} \ \ t, \tag{4.17}$$

where γ is a positive constant satisfying (3.26).

Theorem 4.8. *Assume that* (4.3) *holds. Equation* (E) *possesses an intermediate solution of the first kind if in addition to* (4.12) *the condition*

$$\int_0^t q(s)P_\alpha(s)^{\alpha+1}ds = o(P_\alpha(t)), \qquad t \to \infty \tag{4.18}$$

is satisfied.

Theorem 4.9. *Assume that (4.3) holds. Equation (E) possesses an intermediate solution of the first kind if (4.12) is satisfied and there exist constants $T > 0$ and γ_T such that (3.36) and*

$$\int_T^t q(s)P_\alpha(s)^{\alpha+1}ds \le \gamma_T P_\alpha(t) \tag{4.19}$$

holds for $t \ge T$.

Example 4.10. Consider the half-linear differential equation

$$(e^{-\alpha t}\varphi_\alpha(x'))' + Ke^{-\beta t}\varphi_\alpha(x) = 0, \qquad t \ge 0, \tag{4.20}$$

where α, β and K are positive constants. The functions $p(t) = e^{-\alpha t}$ and $q(t) = Ke^{-\beta t}$ satisfy the condition (4.4). Since $P_\alpha(t) = e^t - 1$ and $\rho(t) = (K/\beta)e^{-\beta t}$, one obtains

$$p(t)^{-\frac{1}{\alpha}}\rho(t)^{\frac{1}{\alpha}} = \left(\frac{K}{\beta}\right)^{\frac{1}{\alpha}} e^{(1-\frac{\beta}{\alpha})t}, \qquad q(t)P_\alpha(t)^\alpha = Ke^{(\alpha-\beta)t}(1-e^{-t})^\alpha,$$

which implies that both maximal and minimal solutions of (4.20) exist if $\alpha < \beta$, more precisely, for any given positive constants c and d there exist solutions x and X of (4.20) such that

$$\lim_{t\to\infty} x(t) = c, \qquad \lim_{t\to\infty} e^{\beta t}Dx(t) = c^\alpha,$$

and

$$\lim_{t\to\infty} \frac{X(t)}{e^t} = d, \qquad \lim_{t\to\infty} DX(t) = d^{\frac{1}{\alpha}},$$

and that (4.20) is deprived of those solutions if $\alpha \ge \beta$.

To gain an insight into intermediate solutions of (4.20) we have to limit our attention to the case $\alpha \ge \beta$ and compute the integral of $p(s)^{-\frac{1}{\alpha}}\rho(s)^{\frac{1}{\alpha}+1}$ on $[t,\infty)$. As is easily checked, this integral converges if $\beta > \alpha/(\alpha+1)$, in which case we obtain

$$\begin{aligned}\int_t^\infty p(s)^{-\frac{1}{\alpha}}\rho(s)^{\frac{1}{\alpha}+1}ds &= \left(\frac{K}{\beta}\right)^{\frac{1}{\alpha}+1}\frac{\alpha}{(\alpha+1)\beta-\alpha}\, e^{(1-\frac{(\alpha+1)\beta}{\alpha})t} \\ &= \left(\frac{K}{\beta}\right)^{\frac{1}{\alpha}}\frac{\alpha}{(\alpha+1)\beta-\alpha}e^{(1-\frac{\beta}{\alpha})t}\rho(t).\end{aligned}$$

This shows that only the case $\alpha = \beta$ should be examined. In this case, since

$$\int_t^\infty p(s)^{-\frac{1}{\alpha}}\rho(s)^{\frac{1}{\alpha}+1}ds = \frac{K^{\frac{1}{\alpha}}}{\alpha^{\frac{1}{\alpha}+1}}\rho(t),$$

we conclude by (i) of Theorem 4.7 that equation (4.20) possesses an intermediate solution if

$$K \le \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

Notice that equation (4.20) with $K = (\alpha/(\alpha+1))^{\alpha+1}$ has an exact intermediate solution $x_0(t) = e^{\alpha t/(\alpha+1)}$ whose quasi-derivative is $Dx_0(t) = (\alpha/(\alpha+1))^\alpha e^{-\alpha t/(\alpha+1)}$.

Concluding remarks

(1) Qualitative theory of second order half-linear differential equations of the form (E) has been the subject of intensive investigations in recent years. The reader is referred to Došlý and Řehák [3] for a systematic exposition of a variety of topics regarding (E) and related differential equations.

(2) The Riccati differential equations have a long history; see e.g., Sansone [14]. With time their importance became known in connection with oscillation theory of second order linear differential equations. A number of fundamental oscillation criteria for the linear equations obtained through the analysis of the Riccati equations can be found in Swanson [15]. It has turned out that a portion of oscillation theory for second order half-linear differential equations can be developed with the help of the associated Riccati-type differential equations; see e.g., Kusano and Naito [7], Kusano and Yoshida [8], Li and Yeh [9], Opluštil [13] and Yang and Lo [16].

(3) The present work was motivated by the question as to the possibility of constructing all nonoscillatory solutions (x, y) of system (A) by utilizing global solutions u or v of the associated Riccati-type equation (R1) or (R2). As far as we know, this kind of question has never been raised in the existing literature on differential equations. Given that u (or v) is a compound of x and y, at first it seemed difficult to draw accurate information on x and y separately from u (or v), but a careful and close analysis of the half-linear law (A) governing (x, y) and the nonlinear law (R1) (or (R2)) governing u (or v) has eventually enabled us to establish the explicit representation formulas for almost all (x, y) in terms of u (or v).

Acknowledgements

The first author was supported by the Slovak Grant Agency VEGA-MŠ, project No.1/0078/17.

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