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The Robin problem for singular $p(x)$-Laplacian equation in a cone

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Abstract. We study the behavior near the boundary angular or conical point of weak solutions to the Robin problem for an elliptic quasi-linear second-order equation with the variable $p(x)$-Laplacian.

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1 Introduction

The aim of our article is the investigation of the behavior of the weak solutions to the Robin problem for quasi-linear elliptic second-order equations with the variable $p(x)$-Laplacian in a neighborhood of an angular or a conical boundary point of the bounded cone. Boundary value problems for elliptic second order equations with a non-standard growth in function spaces with variable exponents have been an active investigations in recent years. We refer to [7] for an overview and the recent papers [1, 9, 10] and reference therein. Differential equations with variable exponents-growth conditions arise from the nonlinear elasticity theory, electrorheological fluids, etc. There are many essential differences between the variable exponent problems and the constant exponent problems. In the variable exponent problems, many singular phenomena occurred and many special questions were raised. V. Zhikov [11, 12] has gave examples of the Lavrentiev phenomenon for the variational problems with variable exponent.

Most of the works devoted to the quasi-linear elliptic second-order equations with the variable $p(x)$-Laplacian refers to the Dirichlet problem in smooth bounded domains (see [7]). Concerning the Robin problem for such equations we know only a few articles [2, 5, 6, 8], but in these works a domain is smooth and lower order terms depend only on (x, u) and do not depend on $|\nabla u|$. Our article [3] is deduced to the Robin problem **in a cone** for such equations with **a singular $p(x)$-power gradient lower order term**. The present article is the continuation of [3]. Here we describe qualitatively the behavior of the weak solution near

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a conical point, namely we derive the sharp estimate of the type $|u(x)| = O(|x|^{\varkappa})$ (cf. §3.1) for the weak solution modulus (for the solution decrease rate) of our problem near a conical boundary point. As well as, we establish the comparison principle for weak solutions.

The Robin boundary conditions appear in the solving Sturm–Liouville problems which are used in many contexts of science and engineering: for example, in electromagnetic problems, in heat transfer problems and for convection-diffusion equations (Fick's law of diffusion). The Robin problem plays a major role in the study of reflected shocks in transonic flow. Important applications of this problem is the capillary problem.

Let $\mathbb{C}$ be an open cone in $\mathbb{R}^n$, $n \geq 2$, with the vertex at the origin $\mathcal{O}$ and let B_r be an open ball with radius r centered at $\mathcal{O}$. We use the following standard notations:

- S^{n-1} : a unit sphere in $\mathbb{R}^n$ centered at $\mathcal{O}$;
- $(r, \omega), \omega = (\omega_1, \omega_2, \ldots, \omega_{n-1})$: the spherical coordinates of $x \in \mathbb{R}^n$ with pole $\mathcal{O}$:

$$\begin{aligned} x_1 &= r\cos\omega_1, \\ x_2 &= r\cos\omega_2 \sin\omega_1, \\ &\vdots \\ x_{n-1} &= r\cos\omega_{n-1}\sin\omega_{n-2}\ldots\sin\omega_1, \\ x_n &= r\sin\omega_{n-1}\sin\omega_{n-2}\ldots\sin\omega_1; \end{aligned}$$

- Ω : a domain on the unit sphere S^{n-1} with the smooth boundary $\partial\Omega$ obtained by the intersection of the cone $\mathbb{C}$ with the sphere S^{n-1};
- $\partial\Omega = \partial\mathbb{C} \cap S^{n-1}$;
- $G_0^d \equiv \mathbb{C} \cap B_d = \{(r, \omega) \mid 0 < r < d;\ \omega \in \Omega\}$;
- $\Gamma_0^d \equiv \partial\mathbb{C} \cap B_d = \{(r, \omega) \mid 0 < r < d;\ \omega \in \partial\Omega\}$;
- $\Omega_d = \overline{G_0^d} \cap \{|x| = d\}$.

We investigate the behavior in a neighborhood of the origin $\mathcal{O}$ of solutions to the Robin problem with the boundary condition on the lateral surface of the cone:

$$\begin{cases} -\triangle_{p(x)}u + b(u, \nabla u) = f(x), & x \in G_0^{d_0}, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \vec{n}} + \frac{\gamma}{|x|^{p(x)-1}}u|u|^{p(x)-2} = 0, & x \in \Gamma_0^{d_0}, \end{cases} \tag{RQL}$$

where $0 < d_0 \ll 1$ (d_0 is fixed) and

$$\triangle_{p(x)}u \equiv \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right). \tag{1.1}$$

We will work under the following assumptions:

(i) $1 < p_- \leq p(x) \leq p_+ = p(0) < n$, $\forall x \in \overline{G_0^{d_0}}$;

(ii) the Lipschitz condition: $p(x) \in C^{0,1}(\overline{G_0^{d_0}}) \implies 0 \leq p_+ - p(x) \leq L|x|$, $\forall x \in \overline{G_0^{d_0}}$; where L is the Lipschitz constant for $p(x)$.

(iii) $|f(x)| \leq f_0|x|^{\beta(x)}$, $f_0 \geq 0$, $\beta(x) > \frac{p_+-1}{p_+-1+\mu}(p(x)-1)\lambda - p(x)$; $\forall x \in \overline{G_0^{d_0}}$; $\gamma = const \geq 1$, $0 \leq \mu < 1$ and λ is the least positive eigenvalue of problem (*NEVP*) (see below);

(iv) the function $b(u,\xi)$ is differentiable with respect to the u, ξ variables in $\mathfrak{M} = \mathbb{R} \times \mathbb{R}^n$ and satisfy in $\mathfrak{M}$ the following inequalities:

(iv)$_\mathbf{a}$ $|b(u,\xi)| \leq \delta|u|^{-1}|\xi|^{p(x)} + b_0|u|^{p(x)-1}$, $0 \leq \delta < \mu$; if $\mu > 0$;

(iv)$_\mathbf{b}$ $b(u,\xi) \geq \nu|u|^{-1}|\xi|^{p(x)} - b_0|u|^{p(x)-1}$, $\nu > 0$; if $\mu = 0$;

(iv)$_\mathbf{c}$ $\sqrt{\sum_{i=1}^{n}\left|\frac{\partial b(u,\xi)}{\partial \xi_i}\right|^2} \leq b_1|u|^{-1}|\xi|^{p(x)-1}$; $\frac{\partial b(u,\xi)}{\partial u} \geq b_2|u|^{-2}|\xi|^{p(x)}$; $b_0 \geq 0$, $b_1 \geq 0$, $b_2 \geq 0$;

(iiv) the spherical region $\Omega \subset S^{n-1}$ is invariant with respect to rotations in S^{n-2}.

We consider the functions class

$$\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{d_0}) = \left\{ u \;\middle|\; u(x) \in L_\infty(G_0^{d_0}) \text{ and } \int_{G_0^{d_0}} \left\langle |x|^{-p(x)}|u|^{p(x)} + |u|^{-1}|\nabla u|^{p(x)} \right\rangle dx < \infty \right\}$$

which was introduced in [4]. It is obvious that $\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{d_0}) \subset W^{1,p(x)}(G_0^{d_0})$.

Definition 1.1. The function u is called a weak bounded solution of problem (*RQL*) provided that $u(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{d_0})$ and satisfies the integral identity

$$\begin{aligned} Q(u,\eta) :\equiv & \int_{G_0^{d_0}} \left\langle |\nabla u|^{p(x)-2} u_{x_i}\eta_{x_i} + b\left(u, \nabla u\right)\eta \right\rangle dx + \gamma \int_{\Gamma_0^{d_0}} r^{1-p(x)} u|u|^{p(x)-2}\eta dS \\ & - \int_{\Omega_{d_0}} |\nabla u|^{p(x)-2}\frac{\partial u}{\partial r}\eta d\Omega_d = \int_{G_0^{d_0}} f(x)\eta(x)dx \end{aligned} \qquad (II)$$

for all $\eta(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^{d_0})$.

Remark 1.2. It is easy to verify that the above assumptions **(i), (iii), (iv)** ensure the existence of integrals over G_0^d and Γ_0^d. Therefore, Definition 1.1 is correct.

Main result is the following statement.

Theorem 1.3. *Let u be a weak bounded solution of problem (RQL), $M_0 = \sup_{x\in G_0^{d_0}} |u(x)|$ (see [3]) and let λ be the least positive eigenvalue of problem (NEVP) (see Section 2). Suppose that* **(i)–(iiv)** *hold. Then there exist $\widetilde{d} \in (0, d_0)$ and a constant $C_0 > 0$ depending only on $\lambda, d_0, M_0, p_+, p_-, L, n$, $(\mu - \delta), \nu, b_0, f_0$ and such that*

$$|u(x)| \leq C_0|x|^{\varkappa}, \quad \varkappa = \frac{p_+ - 1}{p_+ - 1 + \mu}\lambda; \quad \forall x \in G_0^{\widetilde{d}}. \qquad (1.2)$$

2 Nonlinear eigenvalue problem

To prove the main result we shall consider the nonlinear eigenvalue problem for $\psi(\omega) \in C^2(\Omega) \cap C^1(\overline{\Omega})$:

$$\begin{cases} -\mathrm{div}_\omega\left((\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}\nabla_\omega\psi\right) \\ \quad = \lambda\left(\lambda(p_+ - 1) + n - p_+\right)(\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}\psi, & \omega \in \Omega, \\ (\lambda^2\psi^2 + |\nabla_\omega\psi|^2)^{(p_+-2)/2}\frac{\partial\psi}{\partial\overrightarrow{\nu}} + \gamma\left(\frac{p_+-1+\mu}{p_+-1}\right)^{p_+-1}\cdot\psi|\psi|^{p_+-2} = 0, & \omega \in \partial\Omega, \end{cases} \qquad (NEVP)$$

where $|\nabla_\omega \psi|$ denotes the projection of the vector $\nabla\psi$ onto the tangent plane to the unit sphere at the point ω :

$$\nabla_\omega \psi = \left\{ \frac{1}{\sqrt{q_1}} \frac{\partial \psi}{\partial \omega_1}, \dots, \frac{1}{\sqrt{q_{n-1}}} \frac{\partial \psi}{\partial \omega_{n-1}} \right\},$$

$$|\nabla_\omega \psi|^2 = \sum_{i=1}^{n-1} \frac{1}{q_i} \left(\frac{\partial \psi}{\partial \omega_i} \right)^2, \quad q_1 = 1,\ q_i = (\sin\omega_1 \cdots \sin\omega_{i-1})^2,\ i \geq 2$$

and $\overrightarrow{\nu}$ denotes the exterior normal to $\partial\mathcal{C}$ at points of $\partial\Omega$.

If we rename $\omega = \omega_1$, $\omega' = (\omega_2, \dots, \omega_{n-1})$, then, by assumption **(iiv)**, we can assume that $\psi(\omega_1, \omega')$ does not depend on ω'. Therefore, our problem (*NEVP*) is equivalent to the following

$$\begin{cases} \left(\lambda^2\psi^2 + (p_+ - 1)\psi'^2\right) \psi''(\omega) + (n-2)\cot\omega \left(\lambda^2\psi^2 + \psi'^2\right) \psi'(\omega) \\ \quad +\lambda \left(\lambda(2p_+ - 3) + n - p_+\right) \psi'^2 \psi(\omega) \\ \quad +\lambda^3 \left(\lambda(p_+ - 1) + n - p_+\right) \psi^3(\omega) = 0, \qquad \omega \in \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right), \\ \pm(\lambda^2\psi^2 + \psi'^2)^{(p_+-2)/2}\psi'(\omega) + \gamma \left(\frac{p_+-1+\mu}{p_+-1}\right)^{p_+-1} \cdot \psi(\omega)|\psi(\omega)|^{p_+-2}\Big|_{\omega=\pm\frac{\omega_0}{2}} = 0. \end{cases} \tag{OEVP}$$

2.1 Properties of the (*OEVP*) eigenvalue and corresponding eigenfunction

First of all, we note that any two eigenfunctions are scalar multiples of each other if they solve problem for the same λ. Therefore, without loss of generality we can assume $\psi\left(\frac{\omega_0}{2}\right) = 1$.

Next, we observe that the following two cases are possible: either $\psi(-\omega) = -\psi(\omega)$ or $\psi(-\omega) = \psi(\omega)$. In Section 2 [4], it was shown that we obtain the least positive eigenvalue λ^* if $\psi(-\omega) = \psi(\omega)$; then $\psi'(-\omega) = -\psi'(\omega) \implies \psi'(0) = 0,\ \psi(0) \neq 0$, as well as the following inequalities for eigenvalue and for the corresponding eigenfunction:

$$\lambda \left(\lambda(p_+ - 1) + n - p_+\right) > 0; \quad 0 < \lambda^* < \frac{\pi}{\omega_0}; \tag{2.1}$$

$$1 \leq \psi(\omega) \leq \psi_0 = const(n, p, \lambda, \omega_0) \tag{2.2}$$

hold.

Now we define the function $y(\omega) = \frac{\psi'(\omega)}{\psi(\omega)}$, $\psi(0) \neq 0$ and let $y_0 = y\left(\frac{\omega_0}{2}\right)$. From (*OEVP*) we obtain the Cauchy problem

$$\begin{cases} \left((p_+ - 1)y^2 + \lambda^2\right) y'(\omega) + (p_+ - 1)y^4 + (n-2)\cot\omega \left(y^2 + \lambda^2\right) y(\omega) \\ \quad +\lambda \left(2\lambda(p_+ - 1) + n - p_+\right) y^2 + \lambda^3 \left(\lambda(p_+ - 1) + n - p_+\right) = 0, \qquad \omega \in \left(0, \frac{\omega_0}{2}\right), \\ y(0) = 0. \end{cases} \tag{CP}$$

and the following equation for λ :

$$y_0 \left\langle \lambda^2 + y_0^2 \right\rangle^{\frac{p_+-2}{2}} = -\gamma \left(\frac{p_+ - 1 + \mu}{p_+ - 1} \right)^{p_+-1}. \tag{λ}$$

Since

$$(p_+ - 1)y^4 + \lambda \left(2\lambda(p_+ - 1) + n - p_+\right) y^2 + \lambda^3 \left(\lambda(p_+ - 1) + n - p_+\right)$$
$$= (p_+ - 1)(y^2 + \lambda^2) \left(y^2 + \lambda^2 + \frac{n - p_+}{p_+ - 1}\lambda \right),$$

the (CP) equation can be rewritten as follows

$$\frac{y'(\omega)}{y^2+\lambda^2} = -\frac{(n-2)\cot\omega}{(p_+-1)y^2+\lambda^2}\cdot y(\omega) - \frac{(p_+-1)(y^2+\lambda^2)+(n-p_+)\lambda}{(p_+-1)y^2+\lambda^2}, \quad \omega \in \left(0, \frac{\omega_0}{2}\right). \tag{2.3}$$

By Lemma 2.2 [4], we have

$$y(\omega) \le 0, \quad |y(\omega)| \le z_0 = const(n,\lambda,\omega_0,p_+), \quad \forall\omega \in [0,\omega_0/2]. \tag{2.4}$$

Proposition 2.1. *If assumption* **(i)** *is satisfied and* $\gamma \ge 1$ *(see assumption* **(iii)**)*, then*

$$\left(\frac{\varkappa}{\lambda}\sqrt{\lambda^2+y_0^2}\right)^{p(x)-p(0)} \le 1, \quad \forall x \in \Gamma_0^d, \tag{2.5}$$

where $\varkappa$ *is defined by* (1.2).

Proof. We rewrite (λ) with regard to (1.2):

$$\begin{cases} |y_0| = \frac{\gamma\lambda}{\varkappa}, & \text{if } p_+ = 2; \\ \sqrt{\lambda^2+y_0^2} = \left(\frac{\gamma}{|y_0|}\right)^{\frac{1}{p_+-2}}\cdot\left(\frac{\lambda}{\varkappa}\right)^{\frac{p_+-1}{p_+-2}}, & \text{if } p_+ \ne 2. \end{cases} \tag{2.6}$$

Case $p_+ = 2.$

The inequality (2.5) is true if $p(x) \equiv 2$. Now, let $1 < p(x) \le p_+ = 2, \quad \forall x \in \Gamma_0^d$. From ($\lambda$) we have

$$|y_0| = \gamma(1+\mu) = \frac{\gamma\lambda}{\varkappa} \implies \frac{\varkappa}{\lambda}\sqrt{\lambda^2+y_0^2} \ge \frac{\varkappa}{\lambda}|y_0| = \gamma \ge 1$$
$$(p(x)-p_+)\ln\left(\frac{\varkappa}{\lambda}\sqrt{\lambda^2+y_0^2}\right) \le 0 \implies (2.5) \text{ is true.}$$

Case $p_+ > 2.$

From (λ) and (2.6) it follows that

$$|y_0| \le \frac{\lambda}{\varkappa}\cdot\gamma^{\frac{1}{p_+-1}} \quad \text{and} \quad \sqrt{\lambda^2+y_0^2} \ge \frac{\lambda}{\varkappa}\cdot\gamma^{\frac{1}{p_+-1}} \implies$$
$$(p(x)-p_+)\ln\left(\frac{\varkappa}{\lambda}\sqrt{\lambda^2+y_0^2}\right) \le \frac{p(x)-p_+}{p_+-1}\ln\gamma \le 0 \implies (2.5) \text{ is true.}$$

Case $p_+ < 2.$

From (λ) and (2.6) we obtain that

$$|y_0| = \gamma\left(\frac{\lambda}{\varkappa}\right)^{p_+-1}\left(\sqrt{\lambda^2+y_0^2}\right)^{2-p_+} \ge \gamma\left(\frac{\lambda}{\varkappa}\right)^{p_+-1}|y_0|^{2-p_+} \quad |y_0| \ge \frac{\lambda}{\varkappa}\cdot\gamma^{\frac{1}{p_+-1}} \implies$$
$$\text{and} \quad \sqrt{\lambda^2+y_0^2} = \gamma^{\frac{1}{p_+-2}}\left(\frac{\lambda}{\varkappa}\right)^{\frac{p_+-1}{p_+-2}}|y_0|^{\frac{1}{2-p_+}} \ge \frac{\lambda}{\varkappa}\cdot\gamma^{\frac{1}{p_+-1}} \implies$$
$$(p(x)-p_+)\ln\left(\frac{\varkappa}{\lambda}\sqrt{\lambda^2+y_0^2}\right) \le \frac{p(x)-p_+}{p_+-1}\ln\gamma \le 0 \implies (2.5) \text{ is true.}$$

□

3 Comparison principle

In $G_0^d \subset G$ we consider the second order quasi-linear degenerate operator Q of the form

$$Q(u,\eta) \equiv \int\limits_{G_0^d} \Big\langle \mathcal{A}_i(x,u_x)\eta_{x_i} + b(x,u,u_x)\eta(x)\Big\rangle dx + \int\limits_{\Gamma_0^d} \frac{\gamma(\omega)}{r^{p(x)-1}} u|u|^{p(x)-2}\eta(x)ds$$
$$- \int\limits_{\Omega_d} \mathcal{A}_i(x,u_x)\cos(r,x_i)\eta(x)d\Omega_d; \quad \gamma(\omega) \ge \gamma_0 > 0 \tag{3.1}$$

for $u(x) \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$ and for all non-negative $\eta(x)$ belonging to $\mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$ under the following assumptions:

functions $\mathcal{A}_i(x,\xi), b(x,u,\xi)$ are Carathéodory, continuously differentiable with respect to the u,ξ variables in $\mathfrak{M} = \overline{G} \times \mathbb{R} \times \mathbb{R}^n$ and satisfy in $\mathfrak{M}$ the following inequalities:

(i) $\frac{\partial \mathcal{A}_i(x,\xi)}{\partial \xi_j}\zeta_i\zeta_j \ge \varkappa_p |\xi|^{p(x)-2}\zeta^2,\ \forall \zeta \in \mathbb{R}^n \setminus \{0\};\ \varkappa_p > 0;$

(ii) $\sqrt{\sum\limits_{i=1}^n \left|\frac{\partial b(x,u,\xi)}{\partial \xi_i}\right|^2} \le b_1|u|^{-1}|\xi|^{p(x)-1}; \quad \frac{\partial b(x,u,\xi)}{\partial u} \ge b_2|u|^{-2}|\xi|^{p(x)}; \quad b_1 \ge 0,\ b_2 \ge 0;$

(iii) $p(x) \ge p_- > 1.$

Proposition 3.1. *Let Q satisfy assumptions (i)–(iii) and functions $u, w \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$ satisfy the inequality*

$$Q(u,\eta) \le Q(w,\eta) \tag{3.2}$$

for all non-negative $\eta \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$. Assume also that the inequality

$$u(x) \le w(x) \text{ on } \Omega_d \tag{3.3}$$

holds. Then $u(x) \le w(x)$ in G_0^d.

Proof. Let us define $z = u - w$ and $u^\tau = \tau u + (1-\tau)w,\ \tau \in [0,1]$. Then we have

$$0 \ge Q(u,\eta) - Q(w,\eta)$$
$$= \int\limits_{G_0^d} \left\langle \eta_{x_i} z_{x_j} \int_0^1 \frac{\partial \mathcal{A}_i(x,u_x^\tau)}{\partial u^\tau_{x_j}} d\tau + \eta z_{x_i} \int_0^1 \frac{\partial b(x,u^\tau,u_x^\tau)}{\partial u^\tau_{x_i}} d\tau + \eta z \int_0^1 \frac{\partial b(x,u^\tau,u_x^\tau)}{\partial u^\tau} d\tau \right\rangle dx$$
$$- \int\limits_{\Omega_d} \left(\int_0^1 \frac{\partial \mathcal{A}_i(x,u_x^\tau)}{\partial u^\tau_{x_j}} d\tau \right) \cos(r,x_i)\cdot z_{x_j}\eta(x)d\Omega_d$$
$$+ \int\limits_{\Gamma_0^d} \frac{\gamma(\omega)}{r^{p(x)-1}} \left(\int_0^1 \frac{\partial (u^\tau|u^\tau|^{p(x)-2})}{\partial u^\tau} d\tau \right) z(x)\eta(x)ds \tag{3.4}$$

for all non-negative $\eta \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$.

Now, we introduce the sets

$$(G_0^d)^+ := \{x \in G_0^d \mid \; u(x) > w(x)\} \subset G_0^d,$$
$$(\Gamma_0^d)^+ := \{x \in \Gamma_0^d \mid \; u(x) > w(x)\} \subset \Gamma_0^d$$

and assume that $(G_0^d)^+ \neq \emptyset$, $(\Gamma_0^d)^+ \neq \emptyset$. Let $k \geq 1$ be any **odd** number. We choose $\eta = \max\{(u-w)^k, 0\}$ as a test function in the integral inequality (3.4). We have

$$\int_0^1 \frac{\partial(u^\tau |u^\tau|^{p(x)-2})}{\partial u^\tau} d\tau = (p(x)-1) \int_0^1 |u^\tau|^{p(x)-2} d\tau > 0.$$

Then, by assumptions *(i)–(iii)* and $\eta\big|_{\Omega_d} = 0$, we obtain from (3.4) that

$$\int\limits_{(G_0^d)^+} \left\{ k\varkappa_p z^{k-1} \left(\int_0^1 |\nabla u^\tau|^{p(x)-2} d\tau \right) |\nabla z|^2 + b_2 z^{k+1} \left(\int_0^1 |u^\tau|^{-2} |\nabla u^\tau|^{p(x)} d\tau \right) \right\} dx$$
$$\leq b_1 \cdot \int\limits_{(G_0^d)^+} z^k \left(\int_0^1 |u^\tau|^{-1} |\nabla u^\tau|^{p(x)-1} d\tau \right) |\nabla z| dx. \tag{3.5}$$

By the Cauchy inequality,

$$b_1 z^k |\nabla z| |u^\tau|^{-1} |\nabla u^\tau|^{p(x)-1} = \left(|u^\tau|^{-1} z^{\frac{k+1}{2}} |\nabla u^\tau|^{\frac{p(x)}{2}} \right) \cdot \left(b_1 z^{\frac{k-1}{2}} |\nabla z| |\nabla u^\tau|^{\frac{p(x)}{2}-1} \right)$$
$$\leq \frac{\varepsilon}{2} |u^\tau|^{-2} z^{k+1} |\nabla u^\tau|^{p(x)} + \frac{b_1^2}{2\varepsilon} z^{k-1} |\nabla z|^2 |\nabla u^\tau|^{p(x)-2}, \quad \forall \varepsilon > 0.$$

Hence, taking $\varepsilon = 2b_2$, we obtain from (3.5) the inequality

$$\left(k\varkappa_p - \frac{b_1^2}{4b_2} \right) \int\limits_{(G_0^d)^+} z^{k-1} |\nabla z|^2 \left(\int_0^1 |\nabla u^\tau|^{p(x)-2} d\tau \right) dx \leq 0. \tag{3.6}$$

Choosing the odd number $k \geq \max(1; \frac{b_1^2}{2b_2 \varkappa_p})$, in view of $z(x) \equiv 0$ on $\partial(G_0^d)^+$, we get from (3.6) that $z(x) \equiv 0$ in $(G_0^d)^+$. We got a contradiction to our definition of the set $(G_0^d)^+$, this completes the proof. □

Remark 3.2. For the $p(x)$-Laplacian assumption (i) is satisfied with

$$\varkappa_p = \begin{cases} 1, & \text{if } \; p(x) \geq 2; \\ p_- - 1, & \text{if } \; 1 < p_- \leq p(x) < 2. \end{cases}$$

3.1 Barrier function and eigenvalue problem (*OEVP*)

We shall study the **barrier** function $w(r, \omega) \not\equiv 0$ as a solution of the auxiliary problem:

$$\begin{cases} -\triangle_{p_+} w = \mu w^{-1} |\nabla w|^{p_+}, & x \in G_0^d, \\ |\nabla w|^{p_+-2} \frac{\partial w}{\partial \vec{n}} + \frac{\gamma}{|x|^{p_+-1}} w |w|^{p_+-2} = 0, & x \in \Gamma_0^d, \\ 0 < d \leq d_0. \end{cases} \tag{BFP}$$

By direct calculations, we derive a solution of this problem in the form

$$w = w(r,\omega) = r^{\varkappa}\psi^{\varkappa/\lambda}(\omega), \qquad \varkappa = \frac{p_+ - 1}{p_+ - 1 + \mu}\lambda, \tag{BF}$$

where $(\lambda, \psi(\omega))$ is the solution of the eigenvalue problem (*OEVP*). For this function we calculate with regard to $y(\omega) = \frac{\psi'(\omega)}{\psi(\omega)}$:

$$\begin{gathered}\frac{\partial w}{\partial r} = \varkappa r^{\varkappa-1}\psi^{\varkappa/\lambda}(\omega); \quad \frac{\partial w}{\partial \omega} = \frac{\varkappa}{\lambda} r^{\varkappa}\psi^{\frac{\varkappa}{\lambda}-1}(\omega)\psi'(\omega);\\ |\nabla w| = \frac{\varkappa}{\lambda} r^{\varkappa-1}\psi^{\frac{\varkappa}{\lambda}-1}(\omega)\sqrt{\lambda^2\psi^2(\omega)+\psi'^2(\omega)} = \frac{\varkappa}{\lambda} r^{\varkappa-1}\psi^{\frac{\varkappa}{\lambda}}(\omega)\sqrt{\lambda^2+y^2(\omega)}.\end{gathered} \tag{3.7}$$

Proposition 3.3. $w \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$.

Proof. From (*BF*) and (2.2) it follows that $w \in L_\infty(G_0^d)$. Next,

$$\int\limits_{G_0^d} r^{-p(x)} w^{p(x)} dx = \int\limits_{G_0^d} r^{(\varkappa-1)p(x)} \psi^{\frac{\varkappa}{\lambda}p(x)}(\omega) dx.$$

By $r \le d \ll 1$ and assumption **(i)**, we have

$$\begin{aligned} &r^{(\varkappa-1)p(x)} \le r^{(\varkappa-1)p_-}, \quad \text{if } \varkappa \ge 1;\\ &r^{(\varkappa-1)p(x)} \le r^{(\varkappa-1)p_+}, \quad \text{if } \varkappa \le 1;\\ &\psi^{\frac{\varkappa}{\lambda}p(x)}(\omega) \le \psi_0^{p_+} \quad \text{in virtue of (2.2) and } \varkappa \le \lambda. \end{aligned}$$

Hence it follows that

$$\int\limits_{G_0^d} r^{-p(x)} w^{p(x)} dx \le \psi_0^{p_+} \operatorname{meas}\Omega \cdot \begin{cases} \frac{d^{(\varkappa-1)p_-+n}}{(\varkappa-1)p_-+n}, & \text{if } \varkappa \ge 1;\\ \frac{d^{(\varkappa-1)p_++n}}{(\varkappa-1)p_++n}, & \text{if } \varkappa \le 1. \end{cases} \tag{3.8}$$

From (3.7) with regard to (2.4) we obtain that

$$\begin{aligned} \int\limits_{G_0^d} w^{-1}|\nabla w|^{p(x)} dx &= \int\limits_{G_0^d} \left(\frac{\varkappa}{\lambda}\right)^{p(x)} r^{(p(x)-1)\varkappa-p(x)} \psi^{(p(x)-1)\frac{\varkappa}{\lambda}-p(x)}(\omega) \left(\lambda^2\psi^2(\omega)+\psi'^2(\omega)\right)^{p(x)/2} dx\\ &\le \psi_0^{p_+-1} \int\limits_{G_0^d} r^{(p(x)-1)\varkappa-p(x)} \left(\lambda^2+y^2(\omega)\right)^{p(x)/2} dx\\ &\le \psi_0^{p_+-1} \int\limits_{G_0^d} r^{(p(x)-1)\varkappa-p(x)} \left(\lambda^2+z_0^2\right)^{p(x)/2} dx. \end{aligned}$$

Since $\sqrt{\lambda^2+z_0^2} = const(n, \lambda, \omega_0, p_+)$ and $p(x) \in [p_-, p_+]$, we have

$$\left(\lambda^2+y^2(\omega)\right)^{p(x)/2} \le C_1 = const(n, \lambda, \omega_0, p_+, p_-).$$

From the above inequality we obtain that

$$\begin{aligned} \int\limits_{G_0^d} w^{-1}|\nabla w|^{p(x)} dx &\le C_1\psi_0^{p_+-1} \int\limits_{G_0^d} r^{(p(x)-1)\varkappa-p(x)} dx\\ &= C_1\psi_0^{p_+-1} \int\limits_{G_0^d} r^{(\varkappa-1)(p(x)-p_+)} \cdot r^{(\varkappa-1)p_+-\varkappa} dx. \end{aligned} \tag{3.9}$$

Now, by assumptions **(i)–(ii)** and $r \ll 1$, we derive

$$r^{(\varkappa-1)(p(x)-p_+)} \le \begin{cases} r^{(1-\varkappa)Lr}, & \text{if } \varkappa > 1, \\ 1, & \text{if } \varkappa \le 1. \end{cases}$$

Using the well known inequality

$$r^{\alpha}|\ln r| \le \frac{1}{\alpha e}, \quad \forall \alpha > 0,\ 0 < r < 1, \tag{3.10}$$

where e is the Euler number, we establish for $\varkappa > 1$ the inequality

$$r^{(1-\varkappa)Lr} \le e^{\frac{L(\varkappa-1)}{e}}, \quad 0 < r < 1.$$

Thus, from (3.9) it follows that

$$\int\limits_{G_0^d} w^{-1}|\nabla w|^{p(x)}dx \le C_1 \psi_0^{p_+-1} e^{\frac{L(\varkappa-1)}{e}} \operatorname{meas}\Omega \cdot \frac{d^{\varkappa(p_+-1)+n-p_+}}{\varkappa(p_+-1)+n-p_+}. \qquad \square$$

4 The proof of the main Theorem 1.3.

Let $A > 1$, and let $w(r,\omega)$ be the barrier function defined above. By the definition of the operator Q in (II), we have

$$\begin{aligned} Q(Aw,\eta) \equiv & \int\limits_{G_0^d} \left\langle A^{p(x)-1}|\nabla w|^{p(x)-2} w_{x_i}\eta_{x_i} + b\,(Aw, A\nabla w)\,\eta(x) \right\rangle dx \\ & + \gamma \int\limits_{\Gamma_0^d} A^{p(x)-1} r^{1-p(x)} w^{p(x)-1}\eta(x)dS - \int\limits_{\Omega_d} A^{p(x)-1}|\nabla w|^{p(x)-2}\frac{\partial w}{\partial r}\eta(x)d\Omega_d \end{aligned} \tag{4.1}$$

for all $d \in (0, d_0)$ and all non-negative $\eta \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$. Integrating by parts, we obtain that

$$\begin{aligned} & \int\limits_{G_0^d} A^{p(x)-1}|\nabla w|^{p(x)-2} w_{x_i}\eta_{x_i} dx \\ & \quad = -\int\limits_{G_0^d} \frac{d}{dx_i}\left\langle A^{p(x)-1}|\nabla w|^{p(x)-2} w_{x_i}\right\rangle \eta(x)dx + \int\limits_{\Gamma_0^d} A^{p(x)-1}|\nabla w|^{p(x)-2}\frac{dw}{dn}\eta(x)dS \\ & \quad\quad + \int\limits_{\Omega_d} A^{p(x)-1}|\nabla w|^{p(x)-2}\frac{\partial w}{\partial r}\eta(x)d\Omega_d. \end{aligned}$$

Hence and (4.1), with regard to problem (*BFP*), it follows that

$$Q(Aw,\eta) = J_{G_0^d} + J_{\Gamma_0^d}, \tag{4.2}$$

where

$$J_{G_0^d} \equiv \int\limits_{G_0^d} \left\langle \mu A^{p(x)-1} w^{-1} |\nabla w|^{p(x)} - A^{p(x)-1} |\nabla w|^{p_+-2} w_{x_i} \frac{d|\nabla w|^{p(x)-p_+}}{dx_i} \right.$$

$$\left. - \frac{\partial A^{p(x)-1}}{\partial x_i} w_{x_i} |\nabla w|^{p(x)-2} + b\,(Aw, A\nabla w) \right\rangle \eta(x) dx; \tag{4.3}$$

$$J_{\Gamma_0^d} \equiv \gamma \int\limits_{\Gamma_0^d} \left(\frac{Aw}{r} \right)^{p(x)-1} \left\langle 1 - \left(\frac{r|\nabla w|}{w} \right)^{p(x)-p_+} \right\rangle \eta(x) dS.$$

At first, we assert that $J_{\Gamma_0^d} \geq 0$. Indeed, by (3.7),

$$\left. \left(\frac{r|\nabla w|}{w} \right) \right|_{\Gamma_0^d} = \frac{\varkappa}{\lambda} \sqrt{\lambda^2 + y_0^2}$$

and desired inequality follows from Proposition 2.1. Thus, from (4.2) it follows that

$$Q(Aw, \eta) \geq J_{G_0^d}. \tag{4.4}$$

Further, we proceed to the estimating of integral $J_{G_0^d}$. Setting $W(x) = |\nabla w|^{p(x)-p_+}$, we calculate

$$\ln W(x) = (p(x) - p_+) \ln |\nabla w|, \quad \Longrightarrow \quad \frac{1}{W(x)} \cdot \frac{\partial W}{\partial x_i} = \frac{\partial p}{\partial x_i} \ln |\nabla w| + \frac{p(x) - p_+}{|\nabla w|} \cdot \frac{d|\nabla w|}{dx_i} \quad \Longrightarrow$$

$$\frac{d}{dx_i} \left(|\nabla w|^{p(x)-p_+} \right) = |\nabla w|^{p(x)-p_+} \left\langle \frac{\partial p}{\partial x_i} \ln |\nabla w| + \frac{p(x) - p_+}{|\nabla w|} \cdot \frac{d|\nabla w|}{dx_i} \right\rangle.$$

Similarly,

$$\frac{d}{dx_i} \left(A^{p(x)-1} \right) = A^{p(x)-1} \frac{\partial p}{\partial x_i} \ln A.$$

By (4.3), we obtain that

$$J_{G_0^d} \geq \int\limits_{G_0^d} \left\{ A^{p(x)-1} |\nabla w|^{p(x)-2} \left\langle \mu w^{-1} |\nabla w|^2 - (\nabla p \cdot \nabla w)(\ln A + \ln |\nabla w|) \right. \right.$$

$$\left. \left. - \frac{p(x) - p_+}{|\nabla w|} \cdot w_{x_i} \frac{d|\nabla w|}{dx_i} \right\rangle + b\,(Aw, A\nabla w) \right\} \eta(x) dx. \tag{4.5}$$

Passing to polar coordinates, we calculate

$$w_{x_i} \frac{d|\nabla w|}{dx_i} = \frac{\partial w}{\partial r} \cdot \frac{\partial |\nabla w|}{\partial r} + \frac{1}{r^2} \frac{\partial w}{\partial \omega} \cdot \frac{\partial |\nabla w|}{\partial \omega}.$$

Now, by (4.4)–(4.5) with regard to assumption **(iv)**, we obtain that

$$Q(Aw, \eta) \geq \int\limits_{G_0^d} A^{p(x)-1} \left\{ |\nabla w|^{p(x)-2} \left\langle \sigma w^{-1} |\nabla w|^2 - (\nabla p \cdot \nabla w)(\ln A + \ln |\nabla w|) \right. \right.$$

$$\left. \left. - \frac{p(x) - p_+}{|\nabla w|} \cdot \left(\frac{\partial w}{\partial r} \cdot \frac{\partial |\nabla w|}{\partial r} + \frac{1}{r^2} \frac{\partial w}{\partial \omega} \cdot \frac{\partial |\nabla w|}{\partial \omega} \right) \right\rangle - b_0 w^{p(x)-1} \right\} \eta(x) dx, \tag{4.6}$$

with

$$\sigma = \begin{cases} \mu - \delta, & \text{if } \mu > 0; \\ \nu, & \text{if } \mu = 0. \end{cases}$$

Taking into account (3.7), the Lipschitz condition of $p(x)$ **(ii)** and $\frac{\psi'(\omega)}{\psi(\omega)} = y(\omega)$, we directly calculate:

1)

$$\begin{aligned} |(\nabla p \cdot \nabla w)(\ln A + \ln |\nabla w|)| &\le |\nabla p| \cdot |\nabla w| \cdot (\ln A + |\ln |\nabla w||) \\ &\le L|\nabla w| \cdot (\ln A + |\ln |\nabla w||); \end{aligned}$$

in virtue of (3.7), (2.2), (2.4), we derive:

$$\begin{aligned} |\ln |\nabla w|| &\le \left|\ln \frac{\varkappa}{\lambda}\right| + |\varkappa - 1| \cdot |\ln r| + \frac{\varkappa}{\lambda}\psi^{\frac{\varkappa}{\lambda}-1} \ln \psi + \frac{1}{2}|\ln(\lambda^2 + y^2(\omega))| \\ &\le \ln \frac{\lambda}{\varkappa} + \ln \psi_0 + \frac{1}{2}|\ln(\lambda^2 + z_0^2)| + |\varkappa - 1| \cdot |\ln r| \\ &= \ln C_1(n, p_+, \lambda, \omega_0) + |\varkappa - 1| \cdot |\ln r|; \end{aligned}$$

(note that $C_1 \ge 1$: indeed, by virtue of (*BF*) and (2.2) $\frac{\lambda}{\varkappa}\psi_0 \ge 1$; from (13) [4] and (2.5) it follows that $\sqrt{\lambda^2 + z_0^2} \ge \sqrt{\lambda^2 + y_0^2} \ge \frac{\lambda}{\varkappa} \ge 1 \implies C_1 = \frac{\lambda}{\varkappa}\psi_0\sqrt{\lambda^2 + z_0^2} \ge 1$); using inequality (3.10) with $\alpha = \frac{1}{2}$, we get that

$$\begin{aligned} |\nabla w| \cdot |\ln |\nabla w|| &\le \frac{\varkappa}{\lambda} r^{\varkappa-1}\psi^{\frac{\varkappa}{\lambda}}(\omega)\sqrt{\lambda^2 + y^2(\omega)} \cdot (\ln C_1 + |\varkappa - 1| \cdot \ln r) \\ &= \frac{\varkappa}{\lambda} r^{\varkappa-2}\psi^{\frac{\varkappa}{\lambda}}(\omega)\sqrt{\lambda^2 + y^2(\omega)} \cdot \left(r \ln C_1 + (|\varkappa - 1|\sqrt{r}) \cdot \sqrt{r} \ln r\right) \\ &\le \frac{\varkappa}{\lambda} r^{\varkappa-2}\psi^{\frac{\varkappa}{\lambda}}(\omega)\sqrt{\lambda^2 + y^2(\omega)} \cdot (r \ln C_1 + |\varkappa - 1|\sqrt{r}); \end{aligned}$$

from which we obtain that

$$|(\nabla p \cdot \nabla w)(\ln A + \ln |\nabla w|)| \le L\frac{\varkappa}{\lambda} r^{\varkappa-2}\psi^{\frac{\varkappa}{\lambda}}(\omega)\sqrt{\lambda^2 + y^2(\omega)} \cdot (r \ln(AC_1) + |\varkappa - 1|\sqrt{r}).$$

2)

$$\begin{aligned} &\frac{\partial |\nabla w|}{\partial r} = \frac{\varkappa - 1}{r}|\nabla w|; \\ &\frac{\partial |\nabla w|}{\partial \omega} = |\nabla w| \left(\frac{\varkappa}{\lambda} + \frac{y'(\omega)}{\lambda^2 + y^2(\omega)}\right) \cdot y(\omega) = |\nabla w| \left(\frac{p_+ - 1}{p_+ - 1 + \mu} + \frac{y'(\omega)}{\lambda^2 + y^2(\omega)}\right) \cdot y(\omega), \end{aligned}$$

and by (*BF*) it follows that

$$\begin{aligned} &\frac{p_+ - p(x)}{|\nabla w|} \cdot \left(\frac{\partial w}{\partial r} \cdot \frac{\partial |\nabla w|}{\partial r} + \frac{1}{r^2}\frac{\partial w}{\partial \omega} \cdot \frac{\partial |\nabla w|}{\partial \omega}\right) \\ &\qquad = \varkappa(p_+ - p(x))\frac{w}{r^2}\left\langle \varkappa - 1 + \frac{y^2}{\lambda}\left(\frac{p_+ - 1}{p_+ - 1 + \mu} + \frac{y'(\omega)}{\lambda^2 + y^2(\omega)}\right)\right\rangle; \end{aligned}$$

using (2.3)–(2.4), we establish that

$$\begin{aligned} \frac{p_+ - 1}{p_+ - 1 + \mu} + \frac{y'(\omega)}{\lambda^2 + y^2(\omega)} &\ge -\lambda\frac{(n - p_+)}{(p_+ - 1)y^2 + \lambda^2} - \mu\frac{p_+ - 1}{p_+ - 1 + \mu} \cdot \frac{y^2 + \lambda^2}{(p_+ - 1)y^2 + \lambda^2} \\ &\ge -\lambda\frac{(n - p_+)}{(p_+ - 1)y^2 + \lambda^2} - \mu\frac{y^2 + \lambda^2}{(p_+ - 1)y^2 + \lambda^2}; \end{aligned}$$

hence it follows that

$$\frac{y^2}{\lambda}\left(\frac{p_+-1}{p_+-1+\mu}+\frac{y'(\omega)}{\lambda^2+y^2(\omega)}\right)\geq-\frac{1}{\lambda}\cdot\frac{y^2}{(p_+-1)y^2+\lambda^2}\cdot\left\langle(n-p_+)\lambda+\mu(y^2+\lambda^2)\right\rangle$$
$$\geq-\frac{n-p_+}{p_+-1}-\frac{\mu(y^2+\lambda^2)}{\lambda(p_+-1)};$$

using (2.4) again, we derive

$$\frac{p_+-p(x)}{|\nabla w|}\cdot\left(\frac{\partial w}{\partial r}\cdot\frac{\partial|\nabla w|}{\partial r}+\frac{1}{r^2}\frac{\partial w}{\partial\omega}\cdot\frac{\partial|\nabla w|}{\partial\omega}\right)$$
$$\geq-\varkappa L\frac{w}{r}\left\langle|\varkappa-1|+\frac{n-p_+}{p_+-1}+\frac{\mu(y^2+\lambda^2)}{\lambda(p_+-1)}\right\rangle\geq-LC_0(n,p_+,\lambda,\omega_0,\nu)\frac{w}{r};$$

3) $|w|^{-1}|\nabla w|^2=\left(\frac{\varkappa}{\lambda}\right)^2r^{\varkappa-2}\psi^{\frac{\varkappa}{\lambda}}(\omega)\left(\lambda^2+y^2(\omega)\right).$

From **1)–3)** it follows that

$$(\mu-\delta)\frac{|\nabla w|^2}{w}-(\nabla p\cdot\nabla w)(\ln A+\ln|\nabla w|)-\frac{p(x)-p_+}{|\nabla w|}\cdot\left(\frac{\partial w}{\partial r}\cdot\frac{\partial|\nabla w|}{\partial r}+\frac{1}{r^2}\frac{\partial w}{\partial\omega}\cdot\frac{\partial|\nabla w|}{\partial\omega}\right)$$
$$\geq\left(\frac{\varkappa}{\lambda}\right)^2r^{\varkappa-2}\psi^{\frac{\varkappa}{\lambda}}(\omega)\left(\lambda^2+y^2(\omega)\right)$$
$$\times\left\langle(\mu-\delta)-\frac{Ld}{\varkappa}\ln A-\frac{Ld}{\varkappa}\ln C_1-\frac{L|\varkappa-1|}{\varkappa}\sqrt{d}-\frac{LC_0d}{\lambda^2}\right\rangle. \tag{4.7}$$

Now, we require the fulfilment of the following inequalities:

$$\begin{cases}\frac{L}{\varkappa}\ln A\cdot d & \leq\frac{1}{5}\sigma,\\ \frac{Ld}{\varkappa}\ln C_1d & \leq\frac{1}{5}\sigma,\\ \frac{L|\varkappa-1|}{\varkappa}\sqrt{d} & \leq\frac{1}{5}\sigma,\\ \frac{LC_0}{\lambda^2}d & \leq\frac{1}{5}\sigma,\\ b_0d & \leq\frac{\varkappa_0}{10}\sigma,\quad\text{where }\varkappa_0=\begin{cases}1, & \text{for }\varkappa\geq1,\\ \varkappa^{p_+}, & \text{for }\varkappa<1.\end{cases}\end{cases} \tag{4.8}$$

Using (*BF*), (3.7) and the last inequality of (4.8), we get

$$|\nabla w|^{p(x)-2}\Big\langle\sigma w^{-1}|\nabla w|^2-(\nabla p\cdot\nabla w)(\ln A+\ln|\nabla w|)$$
$$-\frac{p(x)-p_+}{|\nabla w|}\left(\frac{\partial w}{\partial r}\cdot\frac{\partial|\nabla w|}{\partial r}+\frac{1}{r^2}\frac{\partial w}{\partial\omega}\cdot\frac{\partial|\nabla w|}{\partial\omega}\right)\Big\rangle-b_0w^{p(x)-1}$$
$$\geq\frac{\sigma}{5}\left(\frac{\varkappa}{\lambda}\right)^{p(x)}r^{\varkappa(p(x)-1)-p(x)}\psi^{\frac{\varkappa}{\lambda}(p(x)-1)}(\omega)(\lambda^2+y^2(\omega))^{\frac{p(x)}{2}}-b_0r^{\varkappa(p(x)-1)}\psi^{\frac{\varkappa}{\lambda}(p(x)-1)}(\omega)$$
$$\geq r^{\varkappa(p(x)-1)-p(x)}\psi^{\frac{\varkappa}{\lambda}(p(x)-1)}(\omega)\left(\frac{\sigma}{5}\varkappa^{p(x)}-b_0r^{p(x)}\right)\geq\left(\frac{\sigma}{5}\varkappa_0-b_0d\right)r^{(\varkappa-1)p(x)-\varkappa}$$
$$\geq\frac{\sigma}{10}\varkappa_0r^{(\varkappa-1)p(x)-\varkappa}. \tag{4.9}$$

From (4.6), (4.9) it follows that

$$Q(Aw,\eta)\geq\frac{\sigma}{10}\varkappa_0\int\limits_{G_0^d}A^{p(x)-1}r^{(\varkappa-1)p(x)-\varkappa}\eta(x)dx.$$

Since $p(x) \geq p_- > 1$ and $A > 1$, we have $A^{p(x)-1} \geq A^{p_- -1}$. Therefore, taking into consideration assumption **(iii)**, the last inequality takes the form

$$\begin{aligned} Q(Aw, \eta) &\geq \frac{\sigma}{10} \varkappa_0 A^{p_- -1} \int_{G_0^d} r^{(\varkappa-1)p(x)-\varkappa} \eta(x) dx \geq \frac{\sigma}{10} \varkappa_0 A^{p_- -1} \int_{G_0^d} r^{\beta(x)} \eta(x) dx \\ &\geq \int_{G_0^d} f_0 r^{\beta(x)} \eta(x) dx \geq \int_{G_0^d} |f(x)| \eta(x) dx \geq \int_{G_0^d} f(x) \eta(x) dx \\ &= Q(u, \eta), \text{ by } (II), \end{aligned}$$

for all non-negative $\eta \in \mathfrak{N}^{1,p(x)}_{-1,\infty}(G_0^d)$, if $A > 1$ satisfies

$$A \geq \left(\frac{10 f_0}{\sigma \varkappa_0} \right)^{\frac{1}{p_- -1}}, \tag{4.10}$$

Further, we show that $u(x) \leq Aw(x)$ on Ω_d. By (*BF*) and (2.2),

$$w(x)|_{\Omega_d} = d^{\varkappa} \psi^{\varkappa/\lambda}(\omega) \geq d^{\varkappa}.$$

In virtue of $|u(x)| \leq M_0$, $\forall x \in G_0^d$, we can choose A such that

$$A \geq \frac{M_0}{d^{\varkappa}} \tag{4.11}$$

and therefore

$$Aw(x)|_{\Omega_d} \geq Ad^{\varkappa} \geq M_0 \geq u(x)|_{\Omega_d}.$$

Thus, if we choose a small $d > 0$ according to (4.8) and a large $A > 1$ according to (4.10)–(4.11)

$$A \geq \max \left\{ \frac{M_0}{d^{\varkappa}}, \left(\frac{10 f_0}{\sigma \varkappa_0} \right)^{\frac{1}{p_- -1}} \right\},$$

then we come to the Comparison Principle

$$Q(u, \eta) \leq Q(Aw, \eta) \quad \text{in } G_0^d; \quad u(x) \leq Aw(x) \quad \text{on } \Omega_d.$$

For this purpose, we need to check the consistency for the system of two inequalities

$$\begin{cases} \frac{L}{\varkappa} \ln A \cdot d & \leq \frac{\sigma}{5}, \\ A & \geq \frac{M_0}{d^{\varkappa}}. \end{cases}$$

Indeed, from this system it follows that

$$\ln \left(M_0 d^{-\varkappa} \right) \leq \ln A \leq \frac{1}{d} \cdot \frac{\varkappa \sigma}{5L} \quad \Longrightarrow \quad d \cdot \ln \left(M_0 d^{-\varkappa} \right) \leq \frac{\varkappa \sigma}{5L}.$$

Now, in virtue of

$$\lim_{d \to +0} d \cdot \ln \left(M_0 d^{-\varkappa} \right) = \lim_{d \to +0} \frac{\ln M_0 - \varkappa \ln d}{\frac{1}{d}} = \varkappa \lim_{d \to +0} d \ln d = -\varkappa \lim_{d \to +0} d = 0,$$

consequently, there is $\widetilde{d} > 0$ such that for $d \in (0, \widetilde{d})$ the desired inequality is true.

Thus, the Comparison Principle implies that

$$u(x) \leq Aw(x) \quad \text{in } G_0^{\widetilde{d}}.$$

Similarly, we derive the estimate $u(x) \geq -Aw(x)$ in $G_0^{\widetilde{d}}$ if we replace $u(x)$ with $-u(x)$. By this and (2.2), we get the required estimate

$$|u(x)| \leq Aw(x) \leq C|x|^{\varkappa}, \quad \text{in } G_0^{\widetilde{d}}.$$

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