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On the spectrum of a nontypical eigenvalue problem

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Abstract. We study a nontypical eigenvalue problem in a bounded domain from the Euclidian space $\mathbb{R}^2$ subject to the homogeneous Dirichlet boundary condition. We show that the spectrum of the problem contains two distinct intervals separated by an interval where there are no other eigenvalues.

Keywords: nontypical eigenvalue problem, spectrum, Trudinger's inequality, Orlicz spaces.

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1 Introduction

1.1 The statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain with smooth boundary denoted by $\partial\Omega$. We consider the following problem

$$\begin{cases} -\Delta u(x) = \lambda h(u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a real parameter and $h : \mathbb{R} \to \mathbb{R}$ is the function given by

$$h(t) = \begin{cases} e^{2t} + t^p - 1, & t \geq 0, \\ e^t - 1, & t < 0, \end{cases} \tag{1.2}$$

with $p \in (0,1)$ a fixed real number. Note that this equation is not a typical eigenvalue problem since it has an inhomogeneous character (in the sense that if u is a nontrivial solution of the equation then tu fails to be its solution for all $t \in \mathbb{R}$). However, since in this paper we are interested in finding parameters $\lambda \in \mathbb{R}$ for which problem (1.1) has nontrivial solutions we will call it a *nontypical eigenvalue problem*. In this context, we will call such a parameter

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an *eigenvalue* of problem (1.1) and a corresponding nontrivial solution of the equation an *eigenfunction*. Moreover, we will refer to the set of all eigenvalues of problem (1.1) as being the *spectrum* of the problem. To be more precise, we will use the following definition in our subsequent analysis.

Definition 1.1. We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (1.1) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla \phi \, dx = \lambda \int_\Omega h(u)\phi \, dx, \qquad \forall\, \phi \in H_0^1(\Omega) \tag{1.3}$$

Function u from the above relation is called an *eigenfunction* associated to eigenvalue λ.

1.2 Background, motivation and main result

First, we recall that in the case when $h(t) = t$, for all $t \in \mathbb{R}$, problem (1.1) reduces to the celebrated eigenvalue problem of the Laplace operator, i.e.

$$\begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega\,. \end{cases} \tag{1.4}$$

It is well-known that the spectrum of problem (1.4) consists in an increasing and unbounded sequence of positive real numbers (see, e.g. [11, Theorem 8.2.1.]). Moreover, each eigenvalue has a variational characterisation given by *Poincaré's principle* (see, e.g. [11, Proposition 8.2.2]). In particular, we just recall that the first eigenvalue of problem (1.4) is obtained by minimizing the Rayleigh quotient associated to the problem

$$\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega u^2 \, dx}\,. \tag{1.5}$$

Furthermore, each eigenfunction corresponding to λ_1 has constant sign in Ω.

Next, we consider the case when $h(t) = |t|^{q-2}t$, for all $t \in \mathbb{R}$, where $q \in (1,\infty) \setminus \{2\}$ is a given real number. Then problem (1.1) becomes

$$\begin{cases} -\Delta u(x) = \lambda |u(x)|^{q-2} u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega\,. \end{cases} \tag{1.6}$$

For problem (1.6) the spectrum is continuous and consists exactly in the interval $(0,\infty)$ (see, e.g., [10, Théorème 27.3, p. 119] or [5, Theorem 1]).

On the other hand, in the case when the function h involved in problem (1.1) is of the form

$$h(t) = \begin{cases} f(t), & t \geq 0, \\ t, & t < 0, \end{cases}$$

with f satisfying the properties

(I) there exists a positive constant $C \in (0,1)$ such that $|f(t)| \leq Ct$ for any $t \geq 0$;

(II) there exists $t_0 > 0$ such that $\int_0^{t_0} f(s)\, ds > 0$;

(III) $\lim_{t\to\infty} \frac{f(t)}{t} = 0$;

it was proved in [6, Theorem 1] that the spectrum of problem (1.1) contains, on the one hand, the isolated eigenvalue λ_1 given by relation (1.5) and, on the other hand, a continuous part, consisting in an interval (μ_1, ∞) with $\mu_1 > \lambda_1$.

Finally, we consider the case when $h(t) = e^t$, for all $t \in \mathbb{R}$. Then problem (1.1) reads as follows

$$\begin{cases} -\Delta u(x) = \lambda e^{u(x)}, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.7}$$

Problems of type (1.7) have been extensively studied in the literature (see, e.g. [2] or [3] and the reference therein). For instance in [2, Theorem 1.3 & Theorem 5.8] it was proved that there exist two positive constants μ_1 and μ_2 (with $\mu_1 < \mu_2$) such that each $\lambda \in (0, \mu_1)$ is an eigenvalue of problem (1.7) while any $\lambda \in (\mu_2, \infty)$ can not be an eigenvalue of problem (1.7).

Motivated by the above results, in this paper we study equation (1.1) when function h involved in its formulation is given by relation (1.2). We reveal a new situation which can occur in the description of the spectrum of this problem, namely the fact that it contains two separate intervals. More precisely, we prove the following result.

Theorem 1.2. *Assume function h from problem (1.1) is given by relation (1.2) and λ_1 is given by relation (1.5). Then there exist two positive real numbers $\lambda_\star$ and $\lambda^\star$ with $\lambda_\star < \lambda^\star$ such that each $\lambda \in (0, \lambda_\star) \cup (\lambda^\star, \infty)$ is an eigenvalue of problem (1.1). Moreover, any $\lambda \in (\frac{\lambda_1}{2}, \lambda_1)$ is not an eigenvalue of problem (1.1).*

2 Proof of the main result

In order to prove Theorem 1.2 we start by recalling a series of known results that will be essential in the analysis of problem (1.1).

2.1 Auxiliary results

Given an N-function $\Phi : \mathbb{R} \to \mathbb{R}^+$ (i.e., Φ is even, convex; $\Phi(t) = 0$ iff $t = 0$; $\lim_{t\to 0} t^{-1}\Phi(t) = 0$ and $\lim_{t\to\infty} t^{-1}\Phi(t) = \infty$, see [1, Chapter 8] for more details) we can define the Orlicz space

$$L^\Phi(\Omega) := \left\{ u : \Omega \to \mathbb{R} : \ u \text{ is measurable and } \int_\Omega \Phi(|u(x)|)\, dx < \infty \right\}.$$

We point out a few examples of N-functions: $\Phi(t) = |t|^q$, with $q \in (1, \infty)$, or $\Phi(t) = e^{t^2} - 1$, or $\Phi(t) = \frac{e^{2|t|}-1}{2}$. Moreover, in the case when $\Phi(t) = |t|^q$, with $q \in (1, \infty)$, the Orlicz space $L^\Phi(\Omega)$ is, actually, the classical Lebesgue space $L^q(\Omega)$.

A well-known result (see, e.g. [8, pp. 221–222]) asserts that the Sobolev space $H_0^1(\Omega)$ is continuously embedded in the Orlicz space $L^{\Phi_0}(\Omega)$, where $\Phi_0(t) := e^{t^2} - 1$, for all $t \in \mathbb{R}$. This result is a consequence of *Trudinger's inequality* (see [9] or [4, Theorem 7.15]) which ensures that there exist two positive constants c_1 and c_2 (independent of Ω) such that

$$\int_\Omega \left(e^{\frac{|u(x)|^2}{c_1 \|\nabla u\|^2_{L^2(\Omega)}}} - 1 \right) dx \le c_2 |\Omega|, \qquad \forall\, u \in H_0^1(\Omega).$$

Actually, the above inequality can be improved (see, e.g. [7]), since there exists a constant $C_0 > 0$, which is independent of Ω, such that

$$\int_\Omega \left(e^{\frac{|u(x)|^2}{\|\nabla u\|^2_{L^2(\Omega)}}} - 1 \right) dx \leq C_0 \frac{\displaystyle\int_\Omega u(x)^2\, dx}{\displaystyle\int_\Omega |\nabla u(x)|^2\, dx} \leq C_0 \frac{|\Omega|}{\pi}, \qquad \forall\, u \in H_0^1(\Omega) \setminus \{0\}. \tag{2.1}$$

Finally, note that for any N-function Ψ that satisfies the property: $\lim_{t\to\infty} \frac{\Psi(kt)}{\Phi_0(t)} = 0$, $\forall\, k > 0$ we know that $H_0^1(\Omega)$ is compactly embedded in $L^\Psi(\Omega)$ (see, [8, pp. 221–222]). Setting $\Psi_0(t) := \frac{e^{2|t|}-1}{2}$, for all $t \in \mathbb{R}$, we observe that

$$\lim_{t\to\infty} \frac{\Psi_0(kt)}{\Phi_0(t)} = \lim_{t\to\infty} \frac{e^{2kt}-1}{2(e^{t^2}-1)} = 0,$$

and consequently $H_0^1(\Omega)$ is compactly embedded in $L^{\Psi_0}(\Omega)$. Similarly, it can be checked that $H_0^1(\Omega)$ is compactly embedded in each Lebesgue space $L^q(\Omega)$, with $q \in (1, \infty)$.

2.2 Proof of Theorem 1.2

First, we note that the embedding of $H_0^1(\Omega)$ into the Orlicz spaces $L^{\Phi_0}(\Omega)$, $L^{\Psi_0}(\Omega)$ and $L^{p+1}(\Omega)$ guarantees the fact that the integrals involved in Definition 1.1 are well-defined and, thus, problem (1.1) is well-posed.

Next, in order to go further, it is convenient to observe that problem (1.1) can be reformulated as follows

$$\begin{cases} -\Delta u(x) = \lambda[(e^{2u_+(x)} + u_+^p(x) - 1) + (e^{-u_-(x)} - 1)], & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

where $u_\pm(x) := \max\{\pm u(x), 0\}$ for all $x \in \Omega$. Recalling that for each $u \in H_0^1(\Omega)$ we have that $u(x) = u_+(x) - u_-(x)$ and $|u(x)| = u_+(x) + u_-(x)$, for all $x \in \Omega$, and furthermore, $u_\pm \in H_0^1(\Omega)$ and

$$\nabla u_+(x) = \begin{cases} 0, & \text{if } [u(x) \leq 0], \\ \nabla u(x), & \text{if } [u(x) > 0] \end{cases} \quad \text{and} \quad \nabla u_-(x) = \begin{cases} 0, & \text{if } [u(x) \geq 0], \\ \nabla u(x), & \text{if } [u(x) < 0] \end{cases}$$

for all $x \in \Omega$ (see, e.g. [4, Lemma 7.6]), we can also rewrite relation (1.3) in the following manner

$$\begin{aligned} &\int_\Omega \nabla u_+ \nabla\phi dx - \int_\Omega \nabla u_- \nabla\phi dx \\ &\qquad = \lambda \int_\Omega \left(e^{2u_+} + u_+^p - 1\right)\phi dx + \lambda \int_\Omega \left(e^{-u_-} - 1\right)\phi dx, \qquad \forall\, \phi \in H_0^1(\Omega). \end{aligned} \tag{2.3}$$

In other words, $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if and only if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that relation (2.3) holds true.

The proof of Theorem 1.2 will be a simple consequence of the conclusions of Propositions 2.1, 2.3 and 2.5 below.

Proposition 2.1. *Any $\lambda \in \left(\frac{\lambda_1}{2}, \lambda_1\right)$ is not an eigenvalue for problem (1.1), where λ_1 is the first eigenvalue for the Laplace operator $-\Delta$ with homogeneous Dirichlet boundary condition given by relation (1.5).*

Proof. Let $\lambda > 0$ be an eigenvalue for problem (1.1) with its corresponding eigenfunction $u \in H_0^1(\Omega) \setminus \{0\}$. Note that since $u \neq 0$ then at least one of the functions u_+ and u_- is nontrivial in Ω. Taking $\phi = u_-$ in (2.3) we obtain

$$-\int_\Omega |\nabla u_-|^2 \, dx = \lambda \int_\Omega \left(e^{-u_-} - 1\right) \, dx,$$

which, in view of the fact that $1 - e^{-y} \leq y$ for all $y \geq 0$, yields

$$\lambda_1 \int_\Omega u_-^2 \, dx \leq \int_\Omega |\nabla u_-|^2 \, dx = \lambda \int_\Omega \left(1 - e^{-u_-}\right) u_- \, dx \leq \lambda \int_\Omega u_-^2 \, dx \,.$$

Thus, if $u_- \neq 0$ then $\displaystyle\int_\Omega u_-^2 \, dx > 0$ and the above facts imply

$$\lambda \geq \lambda_1. \tag{2.4}$$

Otherwise, if $u_- \equiv 0$ and $\lambda > 0$ is an eigenvalue for problem (1.1) then $u_+ \neq 0$ and relation (2.3) reads as follows

$$\int_\Omega \nabla u_+ \nabla \phi \, dx = \lambda \int_\Omega \left(e^{2u_+} + u_+^p - 1\right) \phi \, dx, \qquad \forall \, \phi \in H_0^1(\Omega). \tag{2.5}$$

Let $e_1 \in H_0^1(\Omega) \setminus \{0\}$, $e_1(x) > 0$, $\forall \, x \in \Omega$, be an eigenfunction associated to the eigenvalue λ_1 defined in relation (1.5), i.e.

$$\int_\Omega \nabla e_1 \nabla \phi \, dx = \lambda_1 \int_\Omega e_1 \phi \, dx, \qquad \forall \, \phi \in H_0^1(\Omega). \tag{2.6}$$

Testing with $\phi = u_+$ in (2.6) and $\phi = e_1$ in (2.5) we find

$$\lambda_1 \int_\Omega u_+ e_1 \, dx = \int_\Omega \nabla e_1 \nabla u_+ \, dx = \lambda \int_\Omega \left(e^{2u_+} + u_+^p - 1\right) e_1 \, dx \geq 2\lambda \int_\Omega u_+ e_1 \, dx \,,$$

since $e^{2y} + y^p - 1 \geq e^{2y} - 1 \geq 2y$, $\forall \, y \geq 0$. Taking into account that $\int_\Omega u_+ e_1 dx > 0$ we deduce that

$$\frac{\lambda_1}{2} \geq \lambda. \tag{2.7}$$

Collecting the above pieces of information we find out that if $\lambda > 0$ is an eigenvalue for problem (1.1) then either relation (2.4) holds true or relation (2.7) holds true. In conclusion, any $\lambda \in \left(\frac{\lambda_1}{2}, \lambda_1\right)$ cannot be an eigenvalue for problem (1.1).

The proof of Proposition 2.1 is complete. □

Next, we consider the problem

$$\begin{cases} -\Delta u(x) = \lambda (e^{-u_-(x)} - 1), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega \,. \end{cases} \tag{2.8}$$

Definition 2.2. We say that λ is an *eigenvalue* for problem (2.8) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla \phi \, dx = \lambda \int_\Omega \left(e^{-u_-} - 1\right) \phi \, dx, \qquad \forall \, \phi \in H_0^1(\Omega).$$

Testing in the above relation with $\phi = u_+$ we find

$$\int_\Omega |\nabla u_+|^2 \, dx = \lambda \int_\Omega \left(e^{-u_-} - 1\right) u_+ = 0\,,$$

which implies $\|u_+\|_{H_0^1(\Omega)} = 0$, or $u_+ \equiv 0$. Consequently, problem (2.8) possesses only nonpositive eigenfunctions. Thus, it is enough to analyse the problem

$$\begin{cases} -\Delta u_-(x) = \lambda \left(1 - e^{-u_-(x)}\right), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega\,. \end{cases} \tag{2.9}$$

Taking into account the definition of an eigenvalue for problem (1.1) (see relation (2.3)) we observe that if $\lambda > 0$ is an eigenvalue of problem (2.8) then it is an eigenvalue of problem (1.1). Now, let us define $h_1 : [0, \infty) \to \mathbb{R}$ by $h_1(t) = 1 - e^{-t}$ for all $t \geq 0$. It is easy to check that h_1 satisfies the following properties

- $|h_1(t)| \leq t$, $\forall\, t \geq 0$;
- $\lim_{t\to\infty} \int_0^t h_1(s) ds = \lim_{t\to\infty} \int_0^t (1 - e^{-s}) ds = \lim_{t\to\infty} t + e^{-t} - 1 = +\infty$. It follows that there exists $t_0 > 0$ such that $\int_0^{t_0} h_1(s) ds > 0$;
- $\lim_{t\to\infty} \frac{h_1(t)}{t} = \lim_{t\to\infty} \frac{1-e^{-t}}{t} = 0$.

In other words, conditions $(H1) - (H3)$ from [6, page 320] are fulfilled with $h(x,t) = h_1(t)$. Similar arguments as those used in the proofs of [6, Lemmas 4 & 5] can be considered in order to show that following result.

Proposition 2.3. *There exists $\lambda^* > 0$ such that every $\lambda \in (\lambda^*, \infty)$ is an eigenvalue for problem* (2.8) *having the corresponding eigenfunction nonpositive. Consequently, such a λ is an eigenvalue of problem* (1.1).

Finally, we consider the problem

$$\begin{cases} -\Delta u(x) = \lambda(e^{2u_+(x)} + u_+(x)^p - 1), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega\,. \end{cases} \tag{2.10}$$

Definition 2.4. We say that λ is an *eigenvalue* for problem (2.10) if there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla \phi \, dx = \lambda \int_\Omega \left(e^{2u_+} + u_+^p - 1\right) \phi \, dx, \ \forall\, \phi \in H_0^1(\Omega)\,.$$

Testing in the above relation with $\phi = u_-$ we obtain

$$-\int_\Omega |\nabla u_-|^2 \, dx = \lambda \int_\Omega \left(e^{2u_+} + u_+^p - 1\right) u_- = 0\,.$$

We infer that $\|u_-\|_{H_0^1(\Omega)} = 0$ which implies that $u_- \equiv 0$. Thus, problem (2.10) possesses only nonnegative eigenfunctions. Taking into account the definition of an eigenvalue for problem (1.1) (see relation (2.3)) we deduce that an eigenvalue of problem (2.10) is in fact an eigenvalue for problem (1.1).

In order to go further we introduce the Euler–Lagrange functional associated to problem (2.10), i.e. $J_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \lambda \left(\frac{1}{2} \int_\Omega \left(e^{2u_+} - 1\right) dx + \frac{1}{p+1} \int_\Omega u_+^{p+1} dx - \int_\Omega u_+ \, dx \right).$$

Standard arguments assure that $J_\lambda \in C^1(H_0^1(\Omega);\mathbb{R})$ and its derivative is given by

$$\langle J'_\lambda(u),\phi\rangle = \int_\Omega \nabla u \nabla \phi \, dx - \lambda \int_\Omega \left(e^{2u_+} + u_+^p - 1\right) \phi \, dx, \qquad \forall\, u,\phi \in H_0^1(\Omega)\,.$$

We note that the weak solutions of problem (2.10) are exactly the critical points of the functional J_λ. In view of Definition 2.4, λ is an eigenvalue of problem (2.10) if and only if the functional J_λ has a nontrivial and nonnegative critical point.

Proposition 2.5. *There exists $\lambda_* > 0$ such that every $\lambda \in (0,\lambda_*)$ is an eigenvalue for problem (2.10) with the corresponding eigenfunction nonnegative. Consequently, such a λ is an eigenvalue of problem (1.1).*

Remark 2.6. Since $p \in (0,1)$, the Hilbert space $H_0^1(\Omega)$ is compactly embedded in the Lebesgue space $L^{p+1}(\Omega)$ with $p+1 \in (1,2)$ which implies that there exists a positive constant $\tilde{C}$ such that

$$\|u\|_{L^{p+1}(\Omega)} \le \tilde{C}\|u\|_{H_0^1(\Omega)}, \qquad \forall \;\; u \in H_0^1(\Omega). \tag{2.11}$$

In order to prove Proposition 2.5 it is useful to first establish two auxiliary results.

Lemma 2.7. *Define*

$$\lambda_\star := \frac{1}{8\left[C_0\dfrac{|\Omega|}{\pi} + (e^2+1)|\Omega| + \dfrac{\tilde{C}^{p+1}}{2^p(p+1)}\right]} > 0\,, \tag{2.12}$$

where $\tilde{C}$ is given by relation (2.11). Then, for every $\lambda \in (0,\lambda_\star)$ we have

$$J_\lambda(u) > \frac{1}{16}, \qquad \forall\; u \in H_0^1(\Omega) \text{ with } \|u\|_{H_0^1(\Omega)} = \frac{1}{2}.$$

Proof. By relation (2.1) we deduce that for each $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} = \frac{1}{2}$ we have

$$\int_\Omega \left(e^{4|u(x)|^2} - 1\right) dx = \int_\Omega \left(e^{\frac{|u(x)|^2}{\|\nabla u\|^2_{L^2(\Omega)}}} - 1\right) dx \le C_0\, \frac{|\Omega|}{\pi}.$$

Since $e^{2y} \le e^{2y^2} + e^2$ for all $y \ge 0$, we deduce that for all $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} = \frac{1}{2}$ the following estimates hold true

$$\int_\Omega e^{2u_+} dx \le \int_\Omega \left(e^{2u_+^2} + e^2\right) dx \le \int_\Omega \left(e^{2|u|^2} + e^2\right) dx \le C_0\,\frac{|\Omega|}{\pi} + (e^2+1)|\Omega|. \tag{2.13}$$

Taking into account inequalities (2.13) and (2.11), it follows that for all $u \in H_0^1(\Omega)$ with

$\|u\|_{H_0^1(\Omega)} = \frac{1}{2}$ and all $\lambda \in (0, \lambda_\star)$ (where $\lambda_\star$ is given by relation (2.12)) we get

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \frac{\lambda}{2}\int_\Omega \left(e^{2u_+} - 2u_+ - 1\right) dx - \frac{\lambda}{p+1}\int_\Omega u_+^{p+1}\, dx \\ &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \frac{\lambda}{2}\int_\Omega e^{2u_+} dx - \frac{\lambda}{p+1}\int_\Omega u_+^{p+1}\, dx \\ &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \frac{\lambda}{2}\left(C_0\frac{|\Omega|}{\pi} + (e^2+1)|\Omega|\right) - \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}\, dx \\ &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \lambda\left(C_0\frac{|\Omega|}{2\pi} + \frac{e^2+1}{2}|\Omega|\right) - \frac{\lambda}{p+1}\,\tilde{C}^{p+1}\|u\|^{p+1}_{H_0^1(\Omega)} \\ &\geq \frac{1}{8} - \lambda\left(C_0\frac{|\Omega|}{2\pi} + \frac{e^2+1}{2}|\Omega| + \frac{\tilde{C}^{p+1}}{2^{p+1}(p+1)}\right) \\ &\geq \frac{1}{8} - \lambda_\star\left(C_0\frac{|\Omega|}{2\pi} + \frac{e^2+1}{2}|\Omega| + \frac{\tilde{C}^{p+1}}{2^{p+1}(p+1)}\right) \\ &= \frac{1}{16}. \end{aligned}$$

The proof of Lemma 2.7 is complete. □

Lemma 2.8. *Fix $\lambda \in (0, \lambda_*)$ where λ_* is given by relation (2.12). There exists $t_1 > 0$ sufficiently small such that $J_\lambda(t_1 e_1) < 0$, where e_1 is a positive eigenfunction associated to λ_1 following relation (1.5).*

Proof. Taking into account that $e^y - y - 1 \geq 0$, for all $y \geq 0$, we deduce that for any $t \in (0,1)$ we have

$$\begin{aligned} J_\lambda(te_1) &= \frac{1}{2}\int_\Omega |\nabla(te_1)|^2\, dx - \frac{\lambda}{2}\int_\Omega \left(e^{2te_1} - 2te_1 - 1\right) dx - \frac{\lambda}{p+1}\int_\Omega (te_1)^{p+1} dx \\ &\leq \frac{t^2}{2}\int_\Omega |\nabla e_1|^2\, dx - \frac{\lambda t^{p+1}}{p+1}\int_\Omega e_1^{p+1} dx\,. \end{aligned}$$

Therefore

$$J(te_1) < 0\,,$$

for all $t \in (0, \delta^{1/(1-p)})$ with δ a given real number satisfying

$$0 < \delta < \frac{2\lambda \displaystyle\int_\Omega e_1^{p+1} dx}{(p+1)\|e_1\|^2_{H_0^1(\Omega)}}\,.$$

The proof of Lemma 2.8 is complete. □

Proof of Proposition 2.5. Let λ_* be defined as in (2.12) and fix $\lambda \in (0, \lambda_*)$. Define by $B_{1/2}(0)$ the ball centred at the origin and of radius $\frac{1}{2}$ from $H_0^1(\Omega)$ and denote by $\partial B_{1/2}(0)$ its boundary. By Lemma 2.7 it follows that

$$\inf_{\partial B_{1/2}(0)} J_\lambda > 0\,. \tag{2.14}$$

On the other hand, by Lemma 2.8 there exists $t_1 > 0$ sufficiently small such that $J_\lambda(t_1 e_1) < 0$, where e_1 is a positive eigenfunction associated to λ_1 from relation (1.5). Moreover, taking into account relations (2.11) and (2.13) we deduce that

$$J_\lambda(u) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \lambda\left(C_0\frac{|\Omega|}{2\pi} + \frac{e^2+1}{2}|\Omega|\right) - \frac{\lambda}{p+1}\,\tilde{C}^{p+1}\|u\|^{p+1}_{H_0^1(\Omega)}$$

for any $u \in B_{1/2}(0)$. It follows that

$$-\infty < c := \inf_{\overline{B_{1/2}(0)}} J_\lambda < 0.$$

We consider $0 < \epsilon < \inf_{\partial B_{1/2}(0)} J_\lambda - \inf_{B_{1/2}(0)} J_\lambda$. Applying Ekeland's variational principle to the functional $J_\lambda : \overline{B_{1/2}(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{B_{1/2}(0)}$ such that

$$\begin{cases} J_\lambda(u_\epsilon) < \inf_{\overline{B_{1/2}(0)}} J_\lambda + \epsilon, \\ J_\lambda(u_\epsilon) < J_\lambda(v) + \epsilon \|v - u_\epsilon\|_{H_0^1(\Omega)}, \ \ v \neq u_\epsilon \,. \end{cases}$$

Since $J_\lambda(u_\epsilon) \leq \inf_{\overline{B_{1/2}(0)}} J_\lambda + \epsilon \leq \lim_{B_{1/2}(0)} J_\lambda + \epsilon < \inf_{\partial B_{1/2}(0)} J_\lambda$, we deduce that $u_\epsilon \in B_{1/2}(0)$. Now, we introduce $I_\lambda : \overline{B_{1/2}(0)} \to \mathbb{R}$ defined by $I_\lambda(v) = J_\lambda(v) + \epsilon \|v - u_\epsilon\|_{H_0^1(\Omega)}$. It is clear to see that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + tu) - I_\lambda(u_\epsilon)}{t} \geq 0$$

for small positive t and any $u \in B_1(0)$. The above relation yields

$$\frac{J_\lambda(u_\epsilon + tu) - J_\lambda(u_\epsilon)}{t} + \epsilon \|u\|_{H_0^1(\Omega)} \geq 0.$$

Letting $t \to 0$ we infer that $\langle J'_\lambda(u_\epsilon), u\rangle + \epsilon \|u\|_{H_0^1(\Omega)} \geq 0$ and this implies that $\|J'_\lambda(u_\epsilon)\| \leq \epsilon$. Thus, there exists a sequence $\{u_n\} \subset B_{1/2}(0)$ such that

$$J_\lambda(u_n) \to c \quad \text{and} \quad J'_\lambda(u_n) \to 0, \quad \text{as } n \to \infty. \tag{2.15}$$

It is clear that sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$ which implies that there exists $\overline{u} \in H_0^1(\Omega)$ such that, up to a subsequence, still denoted by $\{u_n\}$, $\{u_n\}$ converges weakly to $\overline{u}$ in $H_0^1(\Omega)$. It follows that

$$\|\overline{u}\|_{H_0^1(\Omega)} \leq \liminf_{n\to\infty} \|u_n\|_{H_0^1(\Omega)}$$

which implies that $\overline{u} \in B_{1/2}(0)$. By the compact embedding of $H_0^1(\Omega)$ in $L^{\Psi_0}(\Omega)$ and $L^{p+1}(\Omega)$, we deduce that $\{u_n\}$ converges strongly to $\overline{u}$ in $L^{\Psi_0}(\Omega)$ and $L^{p+1}(\Omega)$. It follows that

$$\lim_{n\to\infty} \langle J'(\overline{u}), u_n - \overline{u}) = 0\,,$$

or

$$\lim_{n\to\infty} \left[\int_\Omega \nabla \overline{u} \nabla (u_n - \overline{u}) \, dx - \lambda \int_\Omega \left(e^{2\overline{u}_+} + \overline{u}_+^p - 1\right) (u_n - \overline{u}) \, dx \right] = 0. \tag{2.16}$$

On the other hand, by relation (2.15) we conclude that

$$\lim_{n\to\infty} \langle J'_\lambda(u_n), u_n - \overline{u}\rangle = 0\,,$$

or

$$\lim_{n\to\infty} \left[\int_\Omega \nabla u_n \nabla (u_n - \overline{u}) \, dx - \lambda \int_\Omega \left(e^{2(u_n)_+} + (u_n)_+^p - 1\right) (u_n - \overline{u}) \, dx \right] = 0. \tag{2.17}$$

Subtracting (2.16) from (2.17) and using the above pieces of information we deduce that

$$\lim_{n\to\infty} \int_\Omega |\nabla (u_n - \overline{u})|^2 \, dx = 0.$$

Therefore, we obtain that $\{u_n\}$ converges strongly to $\overline{u}$ in $H^1_0(\Omega)$, and using (2.15) we deduce that

$$J_\lambda(\overline{u}) = c < 0 \quad \text{and} \quad J'_\lambda(\overline{u}) = 0\,.$$

We conclude that $\overline{u}$ is a nontrivial critical point of functional J_λ. Since $J_\lambda(v) \geq J_\lambda(|v|)$ for any $v \in H^1_0(\Omega)$, it follows that $\overline{u}$ is a nonnegative and nontrivial critical point of J_λ. Thus, any $\lambda \in (0, \lambda_\star)$ is an eigenvalue of problem (2.10). The proof of Proposition 2.5 is complete. □

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