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Uniqueness of the trivial solution of some inequalities with fractional Laplacian

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Abstract. We obtain sufficient conditions for the uniqueness of the trivial solution for some classes of nonlinear partial differential inequalities containing the fractional powers of the Laplace operator.

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1 Introduction

In the present paper we obtain sufficient conditions for the uniqueness of the trivial solution for some new classes of nonlinear inequalities and systems with fractional powers of the Laplacian by using a modification of the test function method developed in [7,8].

However, this method cannot be used directly, since it was developed for other types of differential operators, in particular, for integer powers of the Laplacian. But it is known [1] that the solution sets for many problems containing operators of such types are relatively small. For instance, harmonic functions cannot approximate a function with interior maxima or minima, functions of a single variable with null second derivatives are necessarily affine linear, and so on, which facilitates choosing additional nonlinear terms that yield non-existence of solutions at all. In contrast, for fractional differential operators many new solutions can arise. Their set can even become locally dense in $C(\mathbb{R}^n)$, as in the case of s-harmonic functions (u such that $(-\Delta)^s u = 0$), see [2], also in the case of higher order operators, see [1,6]. Thus, in order to obtain non-existence results one has to exclude the existence of this larger solution set. Therefore non-existence results in the fractional setting are always a delicate matter, which requires a substantial modification of the known techniques, and were obtained up to now only in some special cases. Namely, this problem was considered in [2] for systems of equations with fractional powers of the Laplacian, and by the authors of the present paper in [5,9] for some respective inequalities and their systems.

The rest of the paper consists of four sections. In §2 we obtain some auxiliary estimates for the fractional Laplacian. Further, we prove uniqueness theorems: in §3, for some elliptic

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inequalities, in §4, for systems of such inequalities, and in §5, for respective parabolic problems.

2 Auxiliary estimates

Let $s \in \mathbb{R}_+$, $[s] = \sup\{z \in \mathbb{Z} : z \le s\}$, $\{s\} = s - [s]$. We define the operator $(-\Delta)^s$ by the formula

$$(-\Delta)^s u(x) \overset{\text{def}}{=} c_{n,s} \cdot (-\Delta)^{[s]} \left(\text{p.v.} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x-y|^{n+2\{s\}}}\, dy \right), \tag{2.1}$$

where

$$c_{n,s} \overset{\text{def}}{=} \frac{2^{\{s\}} \Gamma\left(\frac{n+\{s\}}{2}\right)}{\pi^{n/2} \left|\Gamma\left(-\frac{\{s\}}{2}\right)\right|}$$

for all functions such that the right-hand side of (2.1) makes sense at least in the distributional setting.

Remark 2.1. Note that this definition implies

$$(-\Delta)^s = (-\Delta)^{[s]} \cdot (-\Delta)^{\{s\}}. \tag{2.2}$$

For $u \in H^{2s}_{\text{loc}}(\mathbb{R}^n)$, this order can be reversed (see [3]).

We will use definition (2.1) for the proof of the following Lemmas 2.2 and 2.4.

Lemma 2.2. *Let $s \in \mathbb{R}_+$, $q > p > 0$ and $\alpha, \beta \in \mathbb{R}$. Consider a function $\varphi_1 : \mathbb{R}^n \to \mathbb{R}$ defined as*

$$\varphi_1(x) \overset{\text{def}}{=} \begin{cases} 1 & (|x| \le 1), \\ (2 - |x|)^\lambda & (1 < |x| < 2), \\ 0 & (|x| \ge 2) \end{cases} \tag{2.3}$$

with $\lambda > \max\left(2[s] + 1, \frac{2sq}{q-p} - n\right)$. Then one has

$$0 < \int_{\mathbb{R}^n} |(-\Delta)^s \varphi_1|^{\frac{q}{q-p}} (1 + |x|)^{\frac{\alpha q - \beta p}{q-p}} \varphi_1^{-\frac{p}{q-p}}\, dx < \infty. \tag{2.4}$$

Remark 2.3. In the Mitidieri–Pohozaev approach such estimates were established by direct calculation of the iterated Laplacian of the test functions that were given explicitly. This does not work for the fractional Laplacian, so we need to establish some additional estimates.

Proof. of Lemma 2.2. It suffices to consider $x \in \mathbb{R}^n$ such that $\frac{3}{2} < |x| < 2$, since otherwise the integrand is obviously regular and bounded.

We start with the case $[s] = 0$ using (2.1) with notation $f(x, y) = \frac{\varphi_1(x) - \varphi_1(y)}{|x-y|^{n+2s}}$, where $s = \{s\}$:

$$|(-\Delta)^s \varphi_1)(x)| = c_{n,s} \left| \int_{\mathbb{R}^n} f(x, y)\, dy \right| = c_{n,s} \left| \sum_{i=1}^{2} \int_{D_i(x)} f(x, y)\, dy \right|, \tag{2.5}$$

where

$$D_1(x) \overset{\text{def}}{=} \{y \in \mathbb{R}^n : |x - y| \ge (2 - |x|)/2\},$$
$$D_2(x) \overset{\text{def}}{=} \{y \in \mathbb{R}^n : |x - y| < (2 - |x|)/2\}$$

(here and below the singular integrals are understood in the sense of the Cauchy principal value).

For any $\varepsilon \in (0, 2s)$, since we have $|x-y| \geq (2-|x|)/2$ in $D_1(x)$, we get

$$\begin{aligned}\int_{D_1(x)} f(x,y)\,dy &= \int_{D_1(x)} \frac{\varphi_1(x)-\varphi_1(y)}{|x-y|^{n+2s}}\,dy \\ &\leq \varphi_1(x) \int_{D_1(x)} \frac{dy}{|x-y|^{n+2s}} \\ &\leq \varphi_1(x) \cdot \left(\frac{2-|x|}{2}\right)^{\varepsilon-2s} \int_{D_1(x)} \frac{dy}{|x-y|^{n+\varepsilon}} \leq c_1(2-|x|)^{\lambda+\varepsilon-2s}\end{aligned} \tag{2.6}$$

with some constant $c_1 > 0$.

On the other hand, the Lagrange Mean Value Theorem implies that

$$\begin{aligned}\int_{D_2(x)} f(x,y)\,dy &= \int_{D_2(x)} \frac{\varphi_1(x)-\varphi_1(y)}{|x-y|^{n+2s}}\,dy \\ &= \frac{1}{2}\int_{\tilde{D}_2(x)} \frac{2\varphi_1(x)-\varphi_1(x+z)+\varphi_1(x-z)}{|z|^{n+2s}}\,dz \\ &\leq c_2 \cdot \max_{z\in\tilde{D}_2(x)} |((2-|x+z|)^\lambda)''| \int_{\tilde{D}_2(x)} \frac{|z|^2}{|z|^{n+2s}\,dy} \\ &= c_3 \cdot \max_{z\in\tilde{D}_2(x)} (2-|x+z|)^{\lambda-2}|z|^{2-\varepsilon-2s} \cdot \int_{\tilde{D}_2(x)} \frac{dz}{|z|^{n-\varepsilon}},\end{aligned}$$

where $\tilde{D}_2(x) = \{z \in \mathbb{R}^n : |z| < (2-|x|)/2\}$, with constants c_2, $c_3 > 0$ and arbitrary small $\varepsilon > 0$.

For $z \in \tilde{D}_2(x)$ we have

$$2-|x+z| = 2-|x|+|x|-|x+z| \leq (2-|x|)+|z| \leq \frac{3}{2}(2-|x|).$$

Hence

$$\int_{D_2(x)} f(x,y)\,dy \leq c_4(2-|x|)^{\lambda-\varepsilon-2s} \tag{2.7}$$

for any $\varepsilon > 0$ and some constant $c_4 > 0$.

Combining (2.5) and (2.7), we obtain

$$|(-\Delta)^s\varphi_1(x)| \leq c_5(2-|x|)^{\lambda-\varepsilon-2s}, \tag{2.8}$$

which together with (2.3) implies

$$|(-\Delta)^s\varphi_1(x)|^{\frac{q}{q-p}}(1+|x|)^{\frac{\alpha q-\beta p}{q-p}}\varphi_1^{-\frac{p}{q-p}}(x) \leq c_6(2-|x|)^{\frac{(\lambda-\varepsilon-2s)q-\lambda p}{q-p}} = c_6(2-|x|)^{\lambda-\frac{(2s+\varepsilon)q}{q-p}} \tag{2.9}$$

with some constants $c_5, c_6 > 0$ independent of x. Hence, in case $[s] = 0$ (2.4) follows by assumption $\lambda > \frac{2sq}{q-p} - n$, if $\varepsilon > 0$ is sufficiently small.

For $[s] > 0$, we use the identity (2.2) and the representation of the radial Laplacian

$$\Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r}\cdot\frac{\partial v}{\partial r}. \tag{2.10}$$

It is easy to see that for $1 < |x| = r < 2$ (2.2) and (2.10) imply

$$|(-\Delta)^s\varphi_1(x)| \leq c\sum_{k=1}^{2[s]}\left|\frac{\partial^k(-\Delta^{\{s\}}\varphi_1)(x)}{\partial r^k}\right| \tag{2.11}$$

with some $c > 0$. This holds, both for $0 \le r \le 1$ and for $r > 2$, since in these cases both parts of the inequality are zero.

Differentiating the integral in the definition (2.1) up to order $2[s]$ and repeating the previous arguments for the respective derivatives (note that they can be exchanged with $(-\Delta)^s$ by Remark 2.1), we obtain

$$\left| \frac{\partial^k((-\Delta^{\{s\}}\varphi_1)(x)}{\partial r^k} \right| \le c_5(2-|x|)^{\lambda-\varepsilon-2\{s\}-k} \qquad (k=1,\dots,2[s];\ r=|x|)),$$

which together with (2.10) implies (2.9) and hence (2.4) for arbitrary $s \in \mathbb{R}_+$. □

Lemma 2.4. *Let $s \in \mathbb{R}_+$, $q > p > 0$ and α, $\beta \in \mathbb{R}$. For the family of functions $\varphi_R(x) = \varphi_1\left(\frac{x}{R}\right)$, where $R > 0$, one has*

$$\int_{\mathbb{R}^n} |(-\Delta)^s \varphi_R|^{\frac{q}{q-p}} (1+|x|)^{\frac{\alpha q-\beta p}{q-p}} \varphi_R^{-\frac{p}{q-p}}\, dx \le c R^{n+\frac{(\alpha-2s)q-\beta p}{q-p}} \tag{2.12}$$

for every $R > 0$ and some $c > 0$ independent of R.

Sketch of the proof. By (2.1) and a change of variables $\tilde{y} = \frac{y}{R}$, we have

$$(-\Delta)^s \varphi_R(x) = R^{-2s}(-\Delta)^s \varphi_1\left(\frac{x}{R}\right). \tag{2.13}$$

Substituting (2.13) into the left-hand side of (2.12) and applying Lemma 2.2, we obtain the claim. □

3 Single elliptic inequalities

Now consider the nonlinear elliptic inequality

$$(-\Delta)^s(|x|^\alpha |u|^{p-1}u) \ge c|u|^q(1+|x|)^\beta \qquad (x \in \mathbb{R}^n), \tag{3.1}$$

where $s > 0$, $c > 0$, $q > p > 0$ and α are real numbers.

We define the class $L^\alpha_{p,\mathrm{loc}}(\mathbb{R}^n)$ as that of all functions u such that for each compact set $\Omega \subset \mathbb{R}^n$ one has $\int_\Omega |x|^\alpha |u|^p\, dx < \infty$.

Definition 3.1. A weak solution of inequality (3.1) is a function $u \in L_{q,\mathrm{loc}}(\mathbb{R}^n) \cap L^\alpha_{p,\mathrm{loc}}(\mathbb{R}^n)$ such that for any nonnegative function $\varphi \in C_0^{2[s]+1}(\mathbb{R}^n)$ there holds the inequality

$$\int_{\mathbb{R}^n} |x|^\alpha |u|^{p-1} u(-\Delta)^s \varphi\, dx \ge c \int_{\mathbb{R}^n} |u|^q (1+|x|)^\beta \varphi\, dx. \tag{3.2}$$

We will prove the following theorem.

Theorem 3.2. *Inequality (3.1) has no nontrivial (i.e., distinct from zero a.e.) weak solutions for $n+\alpha-2s > 0$ and*

$$p < q \le \frac{(n+\beta)p}{n+\alpha-2s}. \tag{3.3}$$

Proof. We make use of the test function $\varphi_R(x) = \varphi_1\left(\frac{x}{R}\right)$ defined in Lemma 2.4.

Substituting $\varphi(x) = \varphi_R(x)$ into (3.1) and applying the Hölder inequality, we get

$$\begin{aligned} c\int_{\mathbb{R}^n} &|u|^q(1+|x|)^\beta \varphi_R\,dx \\ &\le \int_{\mathbb{R}^n} |u|^{p-1}u|x|^\alpha(-\Delta)^s\varphi_R\,dx \\ &\le \int_{\mathbb{R}^n} |u|^p|x|^\alpha|(-\Delta)^s\varphi_R|\,dx \\ &\le \left(\int_{\mathbb{R}^n}|u|^q(1+|x|)^\beta\varphi_R\,dx\right)^{\frac{p}{q}}\left(\int_{\operatorname{supp}|(-\Delta)^s\varphi_R|}|(-\Delta)^s\varphi_R|^{\frac{q}{q-p}}(1+|x|)^{\frac{\alpha q-\beta p}{q-p}}\varphi_R^{-\frac{p}{q-p}}\,dx\right)^{\frac{q-p}{q}}. \end{aligned} \tag{3.4}$$

Hence,

$$\int_{\mathbb{R}^n}|u|^q(1+|x|)^\beta\varphi_R\,dx \le c\int_{\mathbb{R}^n}|(-\Delta)^s\varphi_R|^{\frac{q}{q-p}}(1+|x|)^{\frac{\alpha q-\beta p}{q-p}}\varphi_R^{-\frac{p}{q-p}}\,dx. \tag{3.5}$$

From (3.5) by Lemma 2.4 we obtain

$$\int_{\mathbb{R}^n}|u|^q(1+|x|)^\beta\varphi_R\,dx \le cR^{n+\frac{(\alpha-2s)q-\beta p}{q-p}}.$$

Taking $R\to\infty$, in case of strict inequality in (3.3) we come to a contradiction, which proves the claim. In case of equality, we have

$$\int_{\mathbb{R}^n}|u|^q(1+|x|)^\beta\,dx < \infty,$$

whence

$$\int_{\operatorname{supp}|(-\Delta)^s\varphi_R|}|u|^q(1+|x|)^\beta\varphi_R\,dx \to 0 \quad \text{as } R\to\infty$$

and by (3.4)

$$\int_{\mathbb{R}^n}|u|^q(1+|x|)^\beta\,dx = 0,$$

which completes the proof in this case as well. □

Remark 3.3. From the results of [7] it follows that at least for $\alpha = 0$ and integer s the upper bound given for uniqueness of the trivial solution in (3.3) is optimal. Its optimality for $\alpha \neq 0$ and/or non-integer s is an open problem.

4 Systems of elliptic inequalities

Here we consider a system of nonlinear elliptic inequalities

$$\begin{cases} (-\Delta)^{s_1}(|x|^{\alpha_1}|u|^{p_1-1}u) \ge c_1|v|^{q_1}(1+|x|)^{\beta_1} & (x\in\mathbb{R}^n), \\ (-\Delta)^{s_2}(|x|^{\alpha_2}|v|^{p_2-1}v) \ge c_2|u|^{q_2}(1+|x|)^{\beta_2} & (x\in\mathbb{R}^n), \end{cases} \tag{4.1}$$

where $s_1 > 1$, $s_2 > 1$, $q_1 > p_2 > 0$, $q_2 > p_1 > 0$, α_1, α_2, β_1 and β_2 are real numbers.

Definition 4.1. A weak solution of system of inequalities (4.1) is a pair of functions $(u,v) \in (L_{q_2,\mathrm{loc}}(\mathbb{R}^n)\cap L^{\alpha_1}_{p_1,\mathrm{loc}}(\mathbb{R}^n)) \times (L_{q_1,\mathrm{loc}}(\mathbb{R}^n)\cap L^{\alpha_2}_{p_2,\mathrm{loc}}(\mathbb{R}^n))$ such that for any nonnegative function $\varphi \in C_0^{2\max([s_1],[s_2])+1}(\mathbb{R}^n)$ there hold the inequalities

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |x|^{\alpha_1}|u|^{p_1-1}u(-\Delta)^{s_1}\varphi\,dx \geq c_1\int_{\mathbb{R}^n}|v|^{q_1}(1+|x|)^{\beta_1}\varphi\,dx,\\
&\int_{\mathbb{R}^n} |x|^{\alpha_2}|v|^{p_2-1}v(-\Delta)^{s_2}\varphi\,dx \geq c_1\int_{\mathbb{R}^n}|u|^{q_2}(1+|x|)^{\beta_2}\varphi\,dx.
\end{aligned}
\tag{4.2}
$$

We will prove the following theorem.

Theorem 4.2. *System (4.1) has no nontrivial (i.e., distinct from a pair of zero constants a.e.) weak solutions for*

$$
n+\frac{\max\{(\alpha_1-2s_1)q_1q_2+p_1[q_1(\alpha_2-2s_2-\beta_2)-\beta_1p_2],(\alpha_2-2s_2)q_1q_2+p_2[q_2(\alpha_1-2s_1-\beta_1)-\beta_2p_1]\}}{q_1q_2-p_1p_2}\leq 0. \tag{4.3}
$$

Proof. Introduce a test function $\varphi_R(x)$ as in the proof of the previous theorems. Similarly to (3.4), we get

$$
\begin{aligned}
c_1\int_{\mathbb{R}^n}|v|^{q_1}(1+|x|)^{\beta_1}\varphi_R\,dx &\leq \int_{\mathbb{R}^n}|u|^{p_1}|x|^{\alpha_1}|(-\Delta)^{s_1}\varphi_R|\,dx\\
&\leq \left(\int_{\mathbb{R}^n}|u|^{q_2}(1+|x|)^{\beta_2}\varphi_R\,dx\right)^{\frac{p_1}{q_2}}\left(\int_{\operatorname{supp}|(-\Delta)^{s_1}\varphi_R|}|(-\Delta)^{s_1}\varphi_R|^{\frac{q_2}{q_2-p_1}}(1+|x|)^{\frac{\alpha_1q_2-\beta_2p_1}{q_2-p_1}}\varphi_R^{-\frac{p_1}{q_2-p_1}}\,dx\right)^{\frac{q_2-p_1}{q_2}},\\
c_2\int_{\mathbb{R}^n}|u|^{q_2}(1+|x|)^{\beta_2}\varphi_R\,dx &\leq \int_{\mathbb{R}^n}|v|^{p_2}|x|^{\alpha_2}|(-\Delta)^{s_2}\varphi_R|\,dx\\
&\leq \left(\int_{\mathbb{R}^n}|v|^{q_1}(1+|x|)^{\beta_1}\varphi_R\,dx\right)^{\frac{p_2}{q_1}}\left(\int_{\operatorname{supp}|(-\Delta)^{s_2}\varphi_R|}|(-\Delta)^{s_2}\varphi_R|^{\frac{q_1}{q_1-p_2}}(1+|x|)^{\frac{\alpha_2q_1-\beta_1p_2}{q_1-p_2}}\varphi_R^{-\frac{p_2}{q_1-p_2}}\,dx\right)^{\frac{q_1-p_2}{q_1}}.
\end{aligned}
$$

Estimating the second factors in the right-hand sides of the obtained inequalities by Lemma 2.4 similarly to (2.4), we get

$$
\int_{\mathbb{R}^n}|v|^{q_1}(1+|x|)^{\beta_1}\varphi_R\,dx \leq cR^{n+\alpha_1-2s_1-\frac{(n+\beta_2)p_1}{q_2}}\left(\int_{\mathbb{R}^n}|u|^{q_2}(1+|x|)^{\beta_2}\varphi_R\,dx\right)^{\frac{p_1}{q_2}}, \tag{4.4}
$$

$$
\int_{\mathbb{R}^n}|u|^{q_2}(1+|x|)^{\beta_2}\varphi_R\,dx \leq cR^{n+\alpha_2-2s_2-\frac{(n+\beta_1)p_2}{q_1}}\left(\int_{\mathbb{R}^n}|v|^{q_1}(1+|x|)^{\beta_1}\varphi_R\,dx\right)^{\frac{p_2}{q_1}} \tag{4.5}
$$

and, substituting (4.5) into (4.4) and vice versa,

$$
\int_{\mathbb{R}^n}|v|^{q_1}(1+|x|)^{\alpha_1}\varphi_R\,dx \leq cR^{n+\frac{(\alpha_1-2s_1)q_1q_2+p_1[q_1(\alpha_2-2s_2-\beta_2)-\beta_1p_2]}{q_1q_2-p_1p_2}},
$$

$$
\int_{\mathbb{R}^n}|u|^{q_2}(1+|x|)^{\alpha_2}\varphi_R\,dx \leq cR^{n+\frac{(\alpha_2-2s_2)q_1q_2+p_2[q_2(\alpha_1-2s_1-\beta_1)-\beta_2p_1]}{q_1q_2-p_1p_2}}.
$$

Passing to the limit as $R\to\infty$, we complete the proof of the theorem similarly to the previous ones, including the critical case. □

Remark 4.3. From the results of [7] it follows that at least for $\alpha_1=\alpha_2=0$ and integer s_1, s_2 the upper bound given for uniqueness of the trivial solution in (4.3) is optimal. Its optimality for arbitrary α_1,α_2 and/or non-integer s_1,s_2 is an open problem.

5 Nonlinear parabolic inequalities

Now let $u_0 \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, $u_0(x) \geq 0$ a.e. in $\mathbb{R}^n$. We consider the nonlinear parabolic inequality

$$u_t + (-\Delta)^s(|x|^\alpha u) \geq c|u|^q(1+|x|)^\beta \qquad ((x,t) \in \mathbb{R}^n \times \mathbb{R}_+) \tag{5.1}$$

with the initial condition

$$u(x,0) = u_0(x) \qquad (x \in \mathbb{R}^n). \tag{5.2}$$

Definition 5.1. A weak global (in time) solution of inequality (5.1) is a function $u \in L_{q,\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}_+) \cap L^\alpha_{1,\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}_+)$ such that for any nonnegative function $\varphi \in C^{2[s]+1,1}(\mathbb{R}^n \times \mathbb{R}_+)$ with $\operatorname{supp}\varphi(\cdot,t) \subset\subset \mathbb{R}^n$ for each $t > 0$ there holds the inequality

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} |x|^\alpha u[(-\Delta)^s\varphi - \varphi_t]\,dx\,dt \geq c\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} |u|^q(1+|x|)^\beta\varphi\,dx + \int_{\mathbb{R}_n} u_0(x)\varphi(x,0)\,dx. \tag{5.3}$$

We prove the following theorem.

Theorem 5.2. *Inequality (5.1) with initial condition (5.2) has no nontrivial weak global solutions for $\alpha < 2s$ and*

$$1 < q \leq 1 + \frac{2s-\alpha+\beta}{n}. \tag{5.4}$$

Proof. Introduce the test function $\varphi_{R,\theta}(x,t) = \varphi_1\left(\frac{x}{R}\right)\varphi_1\left(\frac{t}{R^\theta}\right)$, where φ_1 is defined as in Lemma 2.2, and the parameter $\theta > 0$ will be specified below. Substituting $\varphi(x,t) = \varphi_{R,\theta}(x,t)$ into (3.1) and using the Young inequality, we get

$$\begin{aligned} &c\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} |u|^q(1+|x|)^\beta\varphi_{R,\theta}\,dx\,dt \\ &\quad\leq \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} u\cdot\left[|(-\Delta)^s\varphi_{R,\theta}| + |x|^\alpha\cdot\left|\frac{\partial\varphi_{R,\theta}}{\partial t}\right|\right]dx\,dt \leq \frac{c}{2}\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} |u|^q(1+|x|)^\beta\varphi_{R,\theta}\,dx\,dt \\ &\quad\quad + d(c)\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\left[|(-\Delta)^s\varphi_{R,\theta}|^{\frac{q}{q-1}}(1+|x|)^{\frac{\alpha q-\beta}{q-1}} + \left|\frac{\partial\varphi_{R,\theta}}{\partial t}\right|^{\frac{q}{q-1}}(1+|x|)^{-\frac{\beta}{q-1}}\right]\varphi_{R,\theta}^{-\frac{1}{q-1}}\,dx\,dt, \end{aligned} \tag{5.5}$$

where $d(c) > 0$. Hence,

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} |u|^q(1+|x|)^\beta\varphi_{R,\theta}\,dx \leq \frac{2d(c)}{c}I, \tag{5.6}$$

where

$$I := \int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\left[|(-\Delta)^s\varphi_{R,\theta}|^{\frac{q}{q-1}} + \left|\frac{\partial\varphi_{R,\theta}}{\partial t}\right|^{\frac{q}{q-1}}\right](1+|x|)^{\frac{\alpha q-\beta}{q-1}}\varphi_{R,\theta}^{-\frac{1}{q-1}}\,dx.$$

From (5.6) and (2.4) due to the definition of $\varphi_{R,\theta}(x,t)$ we have

$$I \leq CR^{n+\theta-\frac{\beta}{q-1}}\left(R^{\frac{(\alpha-2s)q}{q-1}} + R^{-\frac{\theta q}{q-1}}\right)$$

with some $C > 0$. Choosing $\theta = 2s - \alpha$ and taking $R \to \infty$, in the case of a strict inequality in (5.4) we come to a contradiction, which proves the theorem. The case of equality is considered similarly to Theorem 3.2. □

Remark 5.3. Similar results can be obtained for the inequality

$$u_t + (-\Delta)^s(|x|^\alpha |u|^{p-1}u) \geq c|u|^q(1+|x|)^\beta \qquad ((x,t) \in \mathbb{R}^n \times \mathbb{R}_+) \tag{5.7}$$

with initial condition (5.2).

Remark 5.4. From the results of [7] it follows that at least for $\alpha = 0$ and integer s the upper bound given for uniqueness of the trivial solution in (5.4) is optimal. Its optimality for $\alpha \neq 0$ and/or non-integer s is an open problem.

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