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An existence result for fractional differential equations of neutral type with infinite delay

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Abstract

In this paper, the existence of mild solutions for the fractional differential equations of neutral type with infinite delay is obtained under the conditions in respect of the Kuratowski's measure of noncompactness. As an application, the existence of mild solution for some integrodifferential equation is obtained.

keywords: fractional differential equation, neutral differential equation, mild solution, infinite delay, measure of noncompactness

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1 Introduction

The main purpose of this paper is to prove existence of the mild solution for fractional differential equations of neutral type with infinite delay in Banach space X

$$\begin{cases} \dfrac{d^q}{dt^q}(x(t)-h(t,\,x_t)) = A(x(t)-h(t,\,x_t)) + f(t,\,x(t),\,x_t), & t\in[0,\,T],\\ x(t)=\phi(t)\in\mathcal{P}, & t\in(-\infty,0], \end{cases} \tag{1.1}$$

where $T>0$, $0<q<1, \mathcal{P}$ is an admissible phase space that will be defined later. The fractional derivative is understood here in the Caputo sense. A is a generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X, then there exists $M \geq 1$ such that $\|S(t)\| \leq M$. $h : [0, T] \times \mathcal{P} \to X$, $f : [0, T] \times X \times \mathcal{P} \to X$, and $x_t : (-\infty, 0] \to X$ defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in (-\infty, 0]$, ϕ belongs to $\mathcal{P}$ and $\phi(0) = 0$.

The fractional differential equations have been of much interest to many researchers due to its applications in various fields, such as Physics, Chemistry, Engineering, Economy, Aerodynamics, etc(cf., e.g. [2, 5, 6, 14, 15, 17] and the references therein). Moreover, the Cauchy problem for various delay equations in Banach spaces has been receiving more and more attention during the past decades(cf., e.g. [7, 11, 12, 15] and the references therein).

Neutral differential equations with infinite delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last few years(cf., e.g. [2, 9, 10] and the references therein). To the author's knowledge, few papers can be found in the literature for the solvability of the fractional order functional differential equations of neutral type with infinite delay.

In this paper, we study the solvability of Eq. (1.1) and obtain the existence result of Eq. (1.1) by using the Kuratowski's measures of noncompactness. Moreover, an example is presented to show an application of the abstract result.

2 Preliminaries

Throughout this paper, we set $J := [0, T]$ and denote by X a Banach space, by $L(X)$ the Banach space of all linear and bounded operators on X, and $C(J, X)$ the space of all X-valued continuous functions on J.

The following definition about phase space is due to Hale and Kato([7]).

Definition 2.1. *A linear space $\mathcal{P}$ consisting of functions from $\mathbf{R}^-$ into X with semi-norm $\|\cdot\|_{\mathcal{P}}$ is called an admissible phase space if $\mathcal{P}$ has the following properties.*

(1) If $x : (-\infty, T] \to X$ is continuous on J and $x_0 \in \mathcal{P}$, then $x_t \in \mathcal{P}$ and x_t is continuous in $t \in J$, and

$$\|x(t)\| \leq C\|x_t\|_{\mathcal{P}}, \tag{2.1}$$

where $C \geq 0$ is a constant.

(2) *There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$ such that*

$$\|x_t\|_{\mathcal{P}} \leq C_1(t) \sup_{s\in[0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{P}} \tag{2.2}$$

for $t \in [0, T]$ and x as in (1).

(3) *The space $\mathcal{P}$ is complete.*

Remark 2.2. *Equation (2.1) in (1) is equivalent to $\|\phi(0)\| \leq C\|\phi\|_{\mathcal{P}}$, for all $\phi \in \mathcal{P}$.*

Next, we recall the definition of Kuratowski's measure of noncompactness.

Definition 2.3. *Let B be a bounded subset of a semi-normed linear space Y. The Kuratowski's measure of noncompactness of B is defined as*

$$\alpha(B) = \inf\{d > 0 : B \text{ has a finite cover by sets of diameter} \leq d\}.$$

This measure of noncompactness satisfies some important properties([3]).

Lemma 2.4. ([3]) *Let A and B be bounded subsets of X. Then*

(1) $\alpha(A) \leq \alpha(B)$ *if $A \subseteq B$.*

(2) $\alpha(A) = \alpha(\overline{A})$, *where $\overline{A}$ denotes the closure of A.*

(3) $\alpha(A) = 0$ *if and only if A is precompact.*

(4) $\alpha(\lambda A) = |\lambda|\alpha(A)$, $\lambda \in \mathbf{R}$.

(5) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(6) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, *where $A + B = \{x + y : x \in A, y \in B\}$.*

(7) $\alpha(A + a) = \alpha(A)$ *for any $a \in X$.*

(8) $\alpha(\overline{conv}A) = \alpha(A)$, *where $\overline{conv}A$ is the closed convex hull of A.*

For $H \subset C(J, X)$ and $t \in J$, we define

$$\int_0^t H(s)ds = \left\{ \int_0^t u(s)ds : u \in H \right\},$$

where $H(s) = \{u(s) \in X : u \in H\}$.

The following lemmas will be needed.

Lemma 2.5. ([3]) *If $H \subset C(J, X)$ is a bounded, equicontinuous set, then*

$$\alpha(H) = \sup_{t \in J} \alpha(H(t)).$$

Lemma 2.6. ([8]) *If $\{u_n\}_{n=1}^\infty \subset L^1(J, X)$ and there exists an $m \in L^1(J, \mathbf{R}^+)$ such that $\|u_n(t)\| \le m(t)$, a.e. $t \in J$, then $\alpha(\{u_n(t)\}_{n=1}^\infty)$ is integrable and*

$$\alpha\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^\infty\right) \le 2\int_0^t \alpha(\{u_n(s)\}_{n=1}^\infty)ds.$$

Lemma 2.7. ([4], P125) *If U is a bounded set of X, then for any $\varepsilon > 0$, there exists $\{u_n\}_{n=1}^\infty \subset U$, such that $\alpha(U) \le 2\alpha(\{u_n\}_{n=1}^\infty) + \varepsilon$.*

The following result will be used later.

Lemma 2.8. ([1, 16]) *Let D be a bounded, closed and convex subset of a Banach space X such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv} N(V) \text{ or } V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Let Ω be set defined by

$$\Omega = \{x : (-\infty, T] \to X \text{ such that } x|_{(-\infty, 0]} \in \mathcal{P} \text{ and } x|_J \in C(J, X)\}.$$

Following [5, 6, 17], we introduce the definition of mild solution of Eq. (1.1).

Definition 2.9. *A function $x \in \Omega$ satisfying the equation*

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ -Q(t)h(0, \phi) + h(t, x_t) + \displaystyle\int_0^t R(t-s) f(s, x(s), x_s)ds, & t \in J, \end{cases}$$

is called a mild solution of Eq. (1.1), *where*

$$\begin{aligned} Q(t) &= \int_0^\infty \xi_q(\sigma) S(t^q\sigma) d\sigma, \\ R(t) &= q\int_0^\infty \sigma t^{q-1} \xi_q(\sigma) S(t^q\sigma) d\sigma \end{aligned}$$

and ξ_q *is a probability density function defined on* $(0, \infty)$ *such that*

$$\xi_q(\sigma) = \frac{1}{q}\sigma^{-1-\frac{1}{q}}\varpi_q(\sigma^{-\frac{1}{q}}) \geq 0,$$

where

$$\varpi_q(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \sigma \in (0, \infty).$$

Remark 2.10. *According to [13], direct calculation gives that*

$$\|R(t)\| \leq \frac{M}{\Gamma(q)}t^{q-1}, \quad t > 0.$$

3 Main Results

We will require the following assumptions.

(H1) $f : J \times X \times \mathcal{P} \to X$ satisfies $f(\cdot, v, w) : J \to X$ is measurable for all $(v, w) \in X \times \mathcal{P}$ and $f(t, \cdot, \cdot) : X \times \mathcal{P} \to X$ is continuous for a.e. $t \in J$, and there exist two positive functions $\mu_i(\cdot) \in L^p(J, \mathbf{R}^+)(p > \frac{1}{q} > 1,\ i = 1, 2)$ such that

$$\|f(t, v, w)\| \leq \mu_1(t)\|v\| + \mu_2(t)\|w\|_{\mathcal{P}}, \quad (t, v, w) \in J \times X \times \mathcal{P}.$$

(H2) For any bounded sets $D_1 \subset X$, $D_2 \subset \mathcal{P}$ and $0 \leq s \leq t \leq T$, there exist two integrable functions β_1, β_2 such that

$$\alpha\left(R(t-s)f(s, D_1, D_2)\right) \leq \beta_1(t, s)\alpha(D_1) + \beta_2(t, s)\sup_{-\infty<\theta\leq 0}\alpha(D_2(\theta)),$$

where $\sup_{t\in J}\int_0^t \beta_i(t, s)ds := \beta_i < \infty (i = 1, 2)$.

(H3) There exists a constant $L > 0$ such that

$$\|h(t_1, \varphi) - h(t_2, \widetilde{\varphi})\| \le L(|t_1 - t_2| + \|\varphi - \widetilde{\varphi}\|_{\mathcal{P}}), \ \ t_1,\, t_2 \in J, \ \varphi,\, \widetilde{\varphi} \in \mathcal{P}.$$

(H4) There exists $M^* \in (0, 1)$ such that

$$LC_1^* + \frac{MT_{p,\,q}M_{p,\,q}}{\Gamma(q)}(\|\mu_1\|_{L^p(J,\mathbf{R}^+)} + C_1^*\|\mu_2\|_{L^p(J,\mathbf{R}^+)}) < M^*, \tag{3.1}$$

where $T_{p,\,q} := T^{q-\frac{1}{p}}$, $M_{p,\,q} := \left(\frac{p-1}{pq-1}\right)^{\frac{p-1}{p}}$, $C_1^* = \sup\limits_{0\le\eta\le T} C_1(\eta)$.

Let us consider the operator $\Phi : \Omega \to \Omega$ defined by

$$(\Phi x)(t) = \begin{cases} \phi(t), & t \in (-\infty,\, 0], \\ -Q(t)h(0,\, \phi) + h(t,\, x_t) + \displaystyle\int_0^t R(t-s)f(s,\, x(s),\, x_s)ds, & t \in J. \end{cases}$$

It is easy to see that Φ is well-defined.

Let $y(\cdot) : (-\infty,\, T] \to X$ be the function defined by

$$y(t) = \begin{cases} \phi(t), & t \in (-\infty,\, 0], \\ 0, & t \in J. \end{cases}$$

Let $x(t) = y(t) + z(t)$, $t \in (-\infty,\, T]$.

It is clear to see that z satisfies $z_0 = 0$ and

$$z(t) = -Q(t)h(0,\, \phi) + h(t,\, y_t + z_t) + \int_0^t R(t-s)f(s,\, y(s) + z(s),\, y_s + z_s)ds, \qquad t \in J$$

if and only if x satisfies

$$x(t) = -Q(t)h(0,\, \phi) + h(t,\, x_t) + \int_0^t R(t-s)f(s,\, x(s),\, x_s)ds, \qquad t \in J$$

and $x(t) = \phi(t), \ t \in (-\infty,\, 0]$.

Let $Z_0 = \{z \in \Omega : z_0 = 0\}$. For any $z \in Z_0$,

$$\|z\|_{Z_0} = \sup_{t\in J} \|z(t)\| + \|z_0\|_{\mathcal{P}} = \sup_{t\in J} \|z(t)\|.$$

Thus $(Z_0,\, \|\cdot\|_{Z_0})$ is a Banach space.

Define the operator $\widetilde{\Phi} : Z_0 \to Z_0$ by $(\widetilde{\Phi}z)(t) = 0$, $t \in (-\infty, 0]$ and

$$(\widetilde{\Phi}z)(t) = -Q(t)h(0, \phi) + h(t, y_t + z_t) + \int_0^t R(t-s)f(s, y(s)+z(s), y_s + z_s)ds, \quad t \in J.$$

Obviously, the operator Φ has a fixed point is equivalent to $\widetilde{\Phi}$ has one. Now we show that $\widetilde{\Phi}$ has a fixed point.

Before going further we need the lemma as follows.

Lemma 3.1. *Let $C_2^* = \sup_{0\le\eta\le T} C_2(\eta)$, for $z \in Z_0$, we have*

$$\|y_t + z_t\|_{\mathcal{P}} \le C_2^*\|\phi\|_{\mathcal{P}} + C_1^* \sup_{0\le\tau\le t} \|z(\tau)\|. \tag{3.2}$$

Proof. Noting (2.2), we have

$$\begin{aligned}
& \|y_t + z_t\|_{\mathcal{P}} \le \|y_t\|_{\mathcal{P}} + \|z_t\|_{\mathcal{P}} \\
\le\ & C_1(t) \sup_{0\le\tau\le t} \|y(\tau)\| + C_2(t)\|y_0\|_{\mathcal{P}} + C_1(t) \sup_{0\le\tau\le t} \|z(\tau)\| + C_2(t)\|z_0\|_{\mathcal{P}} \\
=\ & C_2(t)\|\phi\|_{\mathcal{P}} + C_1(t) \sup_{0\le\tau\le t} \|z(\tau)\| \\
\le\ & C_2^*\|\phi\|_{\mathcal{P}} + C_1^* \sup_{0\le\tau\le t} \|z(\tau)\|.
\end{aligned}$$

□

For some $r > 0$, we set $B_r = \{z \in Z_0 : \|z\|_{Z_0} \le r\}$.

Now, from (3.2), for $z \in B_r$, we can see

$$\|y_t + z_t\|_{\mathcal{P}} \le C_2^*\|\phi\|_{\mathcal{P}} + C_1^* r := r^*. \tag{3.3}$$

In view of (H1) and (H3), we have

$$\begin{aligned}
\|f(t, y(t)+z(t), y_t + z_t)\| &\le \mu_1(t)\|y(t)+z(t)\| + \mu_2(t)\|y_t + z_t\|_{\mathcal{P}} \\
&\le \mu_1(t)r + \mu_2(t)r^*,
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\|h(t, y_t + z_t)\| &\le \|h(t, y_t + z_t) - h(t, 0)\| + \|h(t, 0)\| \\
&\le L\|y_t + z_t\|_{\mathcal{P}} + M_1 \\
&\le Lr^* + M_1,
\end{aligned} \tag{3.5}$$

where $M_1 = \sup_{t\in J} \|h(t, 0)\|$.

Proposition 3.2. *The operator $\widetilde{\Phi}$ maps B_r into itself.*

Proof. Suppose contrary that for each positive number r there exist a function $z^r(\cdot) \in B_r$ and some $t \in J$ such that $\|(\widetilde{\Phi} z^r)(t)\| > r$. Then from (3.4) and (3.5), we obtain

$$\begin{aligned} r &< \|(\widetilde{\Phi} z^r)(t)\| \\ &\leq \|-Q(t)h(0,\,\phi)\| + \|h(t,\,y_t + z_t^r)\| + \int_0^t \|R(t-s)f(s,\,y(s)+z^r(s),\,y_s+z_s^r)\|ds \\ &\leq LM\|\phi\|_{\mathcal{P}} + MM_1 + Lr^* + M_1 + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\mu_1(s)r + \mu_2(s)r^*]\,ds \\ &= M_2 + \frac{Mr}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mu_1(s)ds + \frac{Mr^*}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mu_2(s)ds, \end{aligned}$$

where $M_2 = LM\|\phi\|_{\mathcal{P}} + MM_1 + Lr^* + M_1$.

Noting that the Hölder inequality, we have

$$\int_0^t (t-s)^{q-1}\mu_i(s)ds \;\leq\; M_{p,\,q}\,\|\mu_i\|_{L^p(J,\,\mathbf{R}^+)}\,t^{\frac{pq-1}{p}} \leq T_{p,\,q}M_{p,\,q}\|\mu_i\|_{L^p(J,\,\mathbf{R}^+)},\; i=1,2.$$

Then

$$r < M_2 + \frac{MrT_{p,\,q}M_{p,\,q}}{\Gamma(q)}\|\mu_1\|_{L^p(J,\,\mathbf{R}^+)} + \frac{Mr^*T_{p,\,q}M_{p,\,q}}{\Gamma(q)}\|\mu_2\|_{L^p(J,\,\mathbf{R}^+)}. \tag{3.6}$$

Dividing both sides of (3.6) by r, and taking $r \to \infty$, we have

$$LC_1^* + \frac{MT_{p,\,q}M_{p,\,q}}{\Gamma(q)}(\|\mu_1\|_{L^p(J,\,\mathbf{R}^+)} + C_1^*\|\mu_2\|_{L^p(J,\,\mathbf{R}^+)}) \geq 1.$$

This contradicts (3.1). Hence for some positive number r, $\widetilde{\Phi}(B_r) \subset B_r$. □

Proposition 3.3. *The operator $\widetilde{\Phi}$ is a continuous mapping of B_r into itself.*

Proof. Let $\{z^k\}_{k\in\mathbf{N}}$ be a sequence of B_r such that $z^k \to z$ in B_r as $k \to \infty$. Since f satisfies (H1), for almost every $t \in J$, we get

$$f(t,\,y(t)+z^k(t),\,y_t+z_t^k) \to f(t,\,y(t)+z(t),\,y_t+z_t),\;\text{as } k \to \infty. \tag{3.7}$$

In view of (3.3) and (3.4), we obtain $\|y_t + z_t^k\|_{\mathcal{P}} \leq r^*$ and

$$\|f(t,\,y(t)+z^k(t),\,y_t+z_t^k) - f(t,\,y(t)+z(t),\,y_t+z_t)\| \leq 2\mu_1(t)r + 2\mu_2(t)r^*,$$

then by the Lebesgue Dominated Convergence Theorem we have

$$\begin{aligned}
&\|(\widetilde{\Phi}z^k)(t)-(\widetilde{\Phi}z)(t)\| \\
\leq\ & \|h(t,\, y_t+z_t^k)-h(t,\, y_t+z_t)\| \\
&+\int_0^t \|R(t-s)[f(s,\, y(s)+z^k(s),\, y_s+z_s^k)-f(s,\, y(s)+z(s),\, y_s+z_s)]\|ds \\
\leq\ & L\|z_t^k-z_t\|_{\mathcal{P}} \\
&+\frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,\, y(s)+z^k(s),\, y_s+z_s^k)-f(s,\, y(s)+z(s),\, y_s+z_s)\|ds \\
\to\ & 0, \qquad k\to\infty.
\end{aligned}$$

Therefore, we obtain that $\lim\limits_{k\to\infty}\|\widetilde{\Phi}z^k-\widetilde{\Phi}z\|_{Z_0}=0$. □

Proposition 3.4. *The operator* $\widetilde{\Phi}$ *transforms* B_r *into equicontinuous set.*

Proof. Let $0<t_2<t_1<T$ and $z\in B_r$, we can see

$$\|(\widetilde{\Phi}z)(t_1)-(\widetilde{\Phi}z)(t_2)\| \leq I_1+I_2+I_3+I_4,$$

where

$$\begin{aligned}
I_1 &= \|Q(t_1)-Q(t_2)\|\cdot\|h(0,\, \phi)\|, \\
I_2 &= \|h(t_1,\, y_{t_1}+z_{t_1})-h(t_2,\, y_{t_2}+z_{t_2})\|, \\
I_3 &= \left\|\int_0^{t_2}[R(t_1-s)-R(t_2-s)]f(s,\, y(s)+z(s),\, y_s+z_s)ds\right\| \\
&\leq q\left\|\int_0^{t_2}\int_0^{\infty}\sigma[(t_1-s)^{q-1}-(t_2-s)^{q-1}]\xi_q(\sigma)S((t_1-s)^q\sigma)f(s,\, y(s)+z(s),\, y_s+z_s)d\sigma ds\right\| \\
&+ q\int_0^{t_2}\int_0^{\infty}\sigma(t_2-s)^{q-1}\xi_q(\sigma)\|S((t_1-s)^q\sigma)-S((t_2-s)^q\sigma)\|\|f(s,\, y(s)+z(s),\, y_s+z_s)\|d\sigma ds \\
&\leq \frac{M}{\Gamma(q)}\int_0^{t_2}\left|(t_1-s)^{q-1}-(t_2-s)^{q-1}\right|\|f(s,\, y(s)+z(s),\, y_s+z_s)\|ds \qquad (3.8) \\
&+ q\int_0^{t_2}\int_0^{\infty}\sigma(t_2-s)^{q-1}\xi_q(\sigma)\|S((t_1-s)^q\sigma)-S((t_2-s)^q\sigma)\|\|f(s,\, y(s)+z(s),\, y_s+z_s)\|d\sigma ds, \\
I_4 &= \int_{t_2}^{t_1}\|R(t_1-s)\|\|f(s,\, y(s)+z(s),\, y_s+z_s)\|ds \\
&\leq \frac{M}{\Gamma(q)}\int_{t_2}^{t_1}(t_1-s)^{q-1}\|f(s,\, y(s)+z(s),\, y_s+z_s)\|ds.
\end{aligned}$$

It follows the continuity of $S(t)$ in the uniform operator topology for $t > 0$ that I_1 tends to 0, as $t_2 \to t_1$. The continuity of h ensures that I_2 tends to 0, as $t_2 \to t_1$.

Noting (3.4) and using the assumption of $\mu_i(s)(i = 1, 2)$, we see that the first term on the right-hand side of (3.8) tends to 0 as $t_2 \to t_1$. The second term on the right-hand side of (3.8) tends to 0 as $t_2 \to t_1$ as a consequence of the continuity of $S(t)$ in the uniform operator topology for $t > 0$.

In view of the assumption of $\mu_i(s)(i = 1, 2)$ and (3.4) we see that $I_4 \to 0$, as $t_2 \to t_1$. □

Theorem 3.5. *Assume that (H1)-(H4) are satisfied, and if $L + 4(\beta_1 + \beta_2) < 1$, then there exists a mild solution of Eq. (1.1) on $(-\infty, T]$.*

Proof. Let V be any subset of B_r such that $V \subset \overline{conv}(\widetilde{\Phi}(V) \cup \{0\})$.

Set $(\widetilde{\Phi}_1 z)(t) = h(t, y_t + z_t)$,

$$(\widetilde{\Phi}_2 z)(t) = -Q(t)h(0, \phi) + \int_0^t R(t-s) f(s, y(s) + z(s), y_s + z_s) ds.$$

Noting that for $z, \widetilde{z} \in V$, we have

$$\|h(t, y_t + \widetilde{z}_t) - h(t, y_t + z_t)\| \leq L \|\widetilde{z}_t - z_t\|_{\mathcal{P}},$$

thus

$$\alpha(h(t, y_t + V_t)) \leq L\alpha(V_t) \leq L \sup_{-\infty<\theta\leq 0} \alpha(V(t+\theta)) = L \sup_{0\leq\tau\leq t} \alpha(V(\tau)) \leq L\alpha(V),$$

where $V_t = \{z_t : z \in V\}$. Therefore, $\alpha(\widetilde{\Phi}_1 V) = \sup_{t\in J} \alpha((\widetilde{\Phi}_1 V)(t)) \leq L\alpha(V)$.

Moreover, from Lemma 2.4-2.7 and (H2), we have

$$\begin{aligned}
\alpha(\widetilde{\Phi}_2 V) &\leq 2\alpha(\{\widetilde{\Phi}_2 v_n\}) + \varepsilon = 2 \sup_{t\in J} \alpha(\{\widetilde{\Phi}_2 v_n(t)\}) + \varepsilon \\
&= 2 \sup_{t\in J} \alpha\left(\left\{\int_0^t R(t-s) f(s, y(s) + v_n(s), y_s + v_{ns}) ds\right\}\right) + \varepsilon \\
&\leq 4 \sup_{t\in J} \int_0^t \alpha(\{R(t-s) f(s, y(s) + v_n(s), y_s + v_{ns})\}) ds + \varepsilon \\
&\leq 4 \sup_{t\in J} \int_0^t [\beta_1(t, s)\alpha(\{v_n(s)\}) + \beta_2(t, s) \sup_{-\infty<\theta\leq 0} \alpha(\{v_n(\theta + s)\})] ds + \varepsilon \\
&\leq 4 \sup_{t\in J} \int_0^t [\beta_1(t, s)\alpha(\{v_n\}) + \beta_2(t, s) \sup_{0\leq\tau\leq s} \alpha(\{v_n(\tau)\})] ds + \varepsilon \\
&\leq 4(\beta_1 + \beta_2)\alpha(\{v_n\}) + \varepsilon \leq 4(\beta_1 + \beta_2)\alpha(V) + \varepsilon.
\end{aligned}$$

It follows from Lemma 2.4 that

$$\alpha(V) \ \ \leq \ \ \alpha(\widetilde{\Phi}V) \leq \alpha(\widetilde{\Phi}_1 V) + \alpha(\widetilde{\Phi}_2 V) \leq [L + 4(\beta_1 + \beta_2)]\alpha(V) + \varepsilon,$$

since ε is arbitrary, we can obtain

$$\alpha(V) \leq [L + 4(\beta_1 + \beta_2)]\alpha(V),$$

hence $\alpha(V) = 0$. Now, combining this with Proposition (3.2)-(3.3) and applying Lemma 2.8, we conclude that $\widetilde{\Phi}$ has a fixed point z^* in B_r. Let $x(t) = y(t)+z^*(t),\ t \in (-\infty, T]$, then $x(t)$ is a fixed point of the operator Φ which is a mild solution of Eq. (1.1). □

We make the following hypothesis:

(H4')There exists $M^* \in (0, 1)$ such that

$$\frac{MT_{p,q}M_{p,q}}{\Gamma(q)}(\|\mu_1\|_{L^p(J,\mathbf{R}^+)} + C_1^*\|\mu_2\|_{L^p(J,\mathbf{R}^+)}) < M^*.$$

From Theorem 3.5, we can see the following theorem.

Theorem 3.6. *Assume that (H1), (H2) and (H4') are satisfied, and if* $4(\beta_1+\beta_2) < 1$, *then there exists a mild solution of problem*

$$\begin{cases} \dfrac{d^q}{dt^q}x(t) = Ax(t) + f(t,\, x(t),\, x_t), & t \in [0,\, T], \\ x(t) = \phi(t), & t \in (-\infty, 0], \end{cases}$$

on $(-\infty,\, T]$.

4 Application

We consider the following integrodifferential model:

$$\begin{cases} \dfrac{\partial^q}{\partial t^q}\left[v(t,\,\xi) - t\displaystyle\int_{-\infty}^0 \frac{k_1(\theta)}{1+|v(t+\theta,\,\xi)|}d\theta\right] = \dfrac{\partial^2}{\partial \xi^2}\left[v(t,\,\xi) - t\displaystyle\int_{-\infty}^0 \frac{k_1(\theta)}{1+|v(t+\theta,\,\xi)|}d\theta\right] \\ \qquad\qquad + \dfrac{t^k}{k}\sin|v(t,\xi)| \cdot \displaystyle\int_0^t \cos v(s,\,\xi)ds + \int_{-\infty}^0 k_2(\theta)\sin(t^3|v(t+\theta,\,\xi)|)d\theta, \\ v(t,\,0) - t\displaystyle\int_{-\infty}^0 \frac{k_1(\theta)}{1+|v(t+\theta,\,0)|}d\theta = 0, \\ v(t,\,1) - t\displaystyle\int_{-\infty}^0 \frac{k_1(\theta)}{1+|v(t+\theta,\,1)|}d\theta = 0, \\ v(\theta,\,\xi) = v_0(\theta,\,\xi), \qquad -\infty < \theta \leq 0, \end{cases} \tag{4.1}$$

where $0 \le t \le 1$, $\xi \in [0,\, 1]$, $k \in \mathbf{N}$, $k_1,\, k_2 : (-\infty,\, 0] \to \mathbf{R}$, $v_0 : (-\infty, 0] \times [0,\, 1] \to \mathbf{R}$ are continuous functions, and $\int_{-\infty}^{0} |k_i(\theta)| d\theta < \infty (i = 1,\, 2)$.

Set $X = L^2([0,\, 1],\, \mathbf{R})$ and define A by

$$\begin{cases} D(A) = H^2(0,1) \cap H_0^1(0,1), \\ Au = u''. \end{cases}$$

Then A generates a compact, analytic semigroup $S(\cdot)$ of uniformly bounded linear operators, and $\|S(t)\| \le 1$.

Let the phase space $\mathcal{P}$ be $BUC(\mathbf{R}^-,\, X)$, the space of bounded uniformly continuous functions endowed with the following norm:

$$\|\varphi\|_{\mathcal{P}} = \sup_{-\infty < \theta \le 0} |\varphi(\theta)|, \quad \text{for all } \varphi \in \mathcal{P},$$

then we can see that $C_1(t) = 1$ in (2.2).

For $t \in [0,\, 1]$, $\xi \in [0,\, 1]$ and $\varphi \in BUC(\mathbf{R}^-,\, X)$, we set

$$\begin{aligned} x(t)(\xi) &= v(t,\, \xi), \\ \phi(\theta)(\xi) &= v_0(\theta,\, \xi), \quad \theta \in (-\infty,\, 0], \\ h(t, \varphi)(\xi) &= t \int_{-\infty}^{0} \frac{k_1(\theta)}{1 + |\varphi(\theta)(\xi)|} d\theta, \\ f(t,\, x(t),\, \varphi)(\xi) &= \frac{t^k}{k} \sin |x(t)(\xi)| \cdot \int_0^t \cos x(s)(\xi) ds + \int_{-\infty}^{0} k_2(\theta) \sin(t^3 |\varphi(\theta)(\xi)|) d\theta. \end{aligned}$$

Then the above equation (4.1) can be written in the abstract form as Eq. (1.1).

Moreover, for $t \in [0,\, 1]$, we can see

$$\begin{aligned} \|f(t,\, x(t),\, \varphi)(\xi)\| &\le \frac{t^{k+1}}{k} \|x(t)\| + t^3 \|\varphi\|_{\mathcal{P}} \int_{-\infty}^{0} |k_2(\theta)| d\theta \\ &= \mu_1(t) \|x(t)\| + \mu_2(t) \|\varphi\|_{\mathcal{P}}, \end{aligned}$$

where $\mu_1(t) := \dfrac{t^{k+1}}{k}$, $\mu_2(t) := t^3 \displaystyle\int_{-\infty}^{0} |k_2(\theta)| d\theta$.

For $t_1,\, t_2 \in [0,\, 1]$, $\varphi, \widetilde{\varphi} \in \mathcal{P}$, we have

$$\begin{aligned}
\|h(t_1,\, \varphi) - h(t_2,\, \widetilde{\varphi})\| &\leq |t_1 - t_2| \int_{-\infty}^{0} \left\| \frac{k_1(\theta)}{1 + |\varphi(\theta)(\xi)|} \right\| d\theta \\
&\quad + t_2 \int_{-\infty}^{0} |k_1(\theta)| \left\| \frac{1}{1 + |\varphi(\theta)(\xi)|} - \frac{1}{1 + |\widetilde{\varphi}(\theta)(\xi)|} \right\| d\theta \\
&\leq |t_1 - t_2| \int_{-\infty}^{0} |k_1(\theta)| d\theta + \int_{-\infty}^{0} |k_1(\theta)| d\theta \cdot \|\varphi - \widetilde{\varphi}\|_{\mathcal{P}} \\
&= L(|t_1 - t_2| + \|\varphi - \widetilde{\varphi}\|_{\mathcal{P}}),
\end{aligned}$$

where $L = \int_{-\infty}^{0} |k_1(\theta)| d\theta$.

Suppose further that there exists a constant $M^* \in (0,\, 1)$ such that

$$L + \frac{M_{p,\,q}}{\Gamma(q)} (\|\mu_1\|_{L^p([0,\,1],\,\mathbf{R}^+)} + \|\mu_2\|_{L^p([0,\,1],\,\mathbf{R}^+)}) < M^*,$$

then (4.1) has a mild solution by Theorem 3.5.

For example, if we take

$$k_1(\theta) = k_2(\theta) = e^{k\theta},\ q = 0.5,\ p = 3,\ k = 3,$$

then $L = \frac{1}{3}$, $M_{p,q} = 4^{\frac{2}{3}}$, $\|\mu_1\|_{L^p([0,\,1],\,\mathbf{R}^+)} = \frac{1}{3}(\frac{1}{13})^{\frac{1}{3}}$, $\|\mu_2\|_{L^p([0,\,1],\,\mathbf{R}^+)} = \frac{1}{3}(\frac{1}{10})^{\frac{1}{3}}$, thus, we see

$$L + \frac{M_{p,\,q}}{\Gamma(q)} (\|\mu_1\|_{L^p([0,\,1],\,\mathbf{R}^+)} + \|\mu_2\|_{L^p([0,\,1],\,\mathbf{R}^+)}) = \frac{1}{3} + \frac{4^{\frac{2}{3}}}{3\sqrt{\pi}} \left((\frac{1}{13})^{\frac{1}{3}} + (\frac{1}{10})^{\frac{1}{3}} \right) < 0.8 < 1.$$

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