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On asymptotic properties of solutions to third-order delay differential equations

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Abstract. The purpose of the paper is to show that the *canonical* operator L_3 given by

$$L_3(\cdot) = \left(r_2\left(r_1(\cdot)'\right)'\right)'$$

where the functions $r_i(t) \in \mathcal{C}([t_0,\infty),[0,\infty))$ satisfy

$$\int_{t_0}^{\infty} \frac{\mathrm{d}s}{r_i(s)} = \infty, \quad (i = 1,2),$$

can be written in a certain *strongly noncanonical* form

$$L_3(\cdot) \equiv b_3\left(b_2\left(b_1\left(b_0(\cdot)\right)'\right)'\right)',$$

such that the functions $b_i(t) \in \mathcal{C}([t_0,\infty),[0,\infty))$ satisfy

$$\int_{t_0}^{\infty} \frac{\mathrm{d}s}{b_i(s)} < \infty, \quad (i = 1,2).$$

We study some relations between canonical and strongly noncanonical operators, showing the advantage of this reverse approach based on the use of a noncanonical representation of L_3 in the study of oscillatory and asymptotic properties of third-order delay differential equations.

Keywords: linear differential equation, delay, third-order, noncanonical operators, oscillation.

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1 Introduction

This paper deals with asymptotic and oscillatory properties of solutions to linear third-order delay differential equations of the form

$$\left(r_2\left(r_1 y'\right)'\right)'(t) + q(t)y(\tau(t)) = 0, \quad t \geq t_0 > 0. \tag{E}$$

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Throughout, we assume that

(H_1) the functions r_1, r_2, $q \in \mathcal{C}([t_0,\infty),\mathbb{R})$ are positive;

(H_2) $\tau \in \mathcal{C}^1([t_0,\infty),\mathbb{R})$ is strictly increasing, $\tau(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \infty$.

For the brevity sake, we define the operators

$$L_0 y = y, \quad L_i y = r_i \left(L_{i-1} y\right)', \quad (i = 1,2), \quad L_3 y = (L_2 y)'.$$

Under a solution of equation (E), we mean a nontrivial function $y \in \mathcal{C}^1([T_y,\infty),\mathbb{R})$ with $T_y \ge t_0$, which has the property $L_1 y$, $L_2 y \in \mathcal{C}^1([T_y,\infty),\mathbb{R})$, and satisfies ($E$) on $[T_y,\infty)$. We only consider those solutions of (E) which exist on some half-line $[T_y,\infty)$ and satisfy the condition

$$\sup\{|y(t)| : T \le t < \infty\} > 0 \quad \text{for any } T \ge T_y.$$

As is customary, a solution y of (E) is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is said to be *nonoscillatory*. The equation itself is termed *oscillatory* if all its solutions oscillate.

From Trench theory [18], it is known that $L_3 y$ can be always written in an equivalent *canonical form*

$$L_3 y(t) \equiv a_3(t) \left(a_2 \left(a_1 \left(a_0 y\right)'\right)'\right)'(t)$$

such that the functions $a_i(t) \in \mathcal{C}([t_0,\infty),\mathbb{R})$, $i = 0,1,2,3$, are positive,

$$\int_{t_0}^{\infty} \frac{\mathrm{d}s}{a_i(s)} = \infty, \quad (i = 1,2)$$

and uniquely determined up to positive multiplicative constants with the product 1. The explicit forms of functions a_i generally depend on the convergence or divergence of certain integrals and may be calculated using the proof of Lemmas 1 and 2 in [18]. As a matter of fact, the investigation of asymptotic properties of canonical third-order differential equations, especially with regard to oscillation and nonoscillation, has became the subject of extensive research, see e.g. [1–9, 11, 12, 15, 17] and the references cited therein.

The purpose of the paper is to show the reverse, i.e. that the canonical operator L_3 can be written in a certain *strongly noncanonical* form

$$L_3 y(t) \equiv b_3(t) \left(b_2 \left(b_1 \left(b_0 y\right)'\right)'\right)'(t), \tag{1.1}$$

such that the functions $b_i(t) \in \mathcal{C}([t_0,\infty),\mathbb{R})$, $i = 0,1,2,3$, are positive and

$$\int_{t_0}^{\infty} \frac{\mathrm{d}s}{b_i(s)} < \infty, \quad (i = 1,2).$$

Consequently, we study some relations between canonical and strongly noncanonical operators and corresponding classes of nonoscillatory solutions of studied equations, showing the advantage and usefulness of this reverse approach based on the use of a noncanonical representation of L_3 in the study of oscillatory and asymptotic properties of solutions of third-order delay differential equations.

2 Noncanonical representation

Define the functions

$$R_i(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{r_i(s)}, \quad (i = 1, 2), \quad R_{12}(t) = \int_{t_0}^{t} \frac{R_2(s)}{r_1(s)}\mathrm{d}s, \quad R_{21}(t) = \int_{t_0}^{t} \frac{R_1(s)}{r_2(s)}\mathrm{d}s.$$

In the sequel, we will assume that L_3 is in canonical form, that is,

$$\text{(H}_3\text{)} \qquad R_i(\infty) = \infty, \quad i = 1, 2.$$

The following result is a modification of the well known Kiguradze lemma [13, Lemma 1.1] based on (H_3).

Lemma 2.1. *Assume* (H_1)−(H_3). *The set of all nonoscillatory solutions y of* (E) *can be divided into the following two classes*

$$N_0 = \{y(t) : (\exists T \geq t_0)(\forall t \geq T)\, (y(t)L_1y(t) < 0,\ y(t)L_2y(t) > 0)\}$$
$$N_2 = \{y(t) : (\exists T \geq t_0)(\forall t \geq T)\, (y(t)L_1y(t) > 0,\ y(t)L_2y(t) > 0)\}$$

Theorem 2.2. *Assume* (H_1)−(H_3). *Then L_3 has a certain strongly noncanonical form* (1.1), *where*

$$b_0 = \frac{1}{R_{12}}, \quad b_1 = \frac{r_1 R_{12}^2}{R_{21}}, \quad b_2 = \frac{r_2 R_{21}^2}{R_{12}}, \quad b_3 = \frac{1}{R_{21}}.$$

Proof. By some computations, we have

$$\frac{r_2 R_{21}^2}{R_{12}} \left(\frac{r_1 R_{12}^2}{R_{21}} \left(\frac{y}{R_{12}} \right)' \right)' = L_2 y R_{21} - L_1 y R_1 - y\frac{R_{21}}{R_{12}} + y\frac{R_1 R_2}{R_{12}}. \tag{2.1}$$

Integrating the equality

$$(R_1 R_2)' = \frac{R_2}{r_1} + \frac{R_1}{r_2}$$

from t_0 to t, we obtain

$$R_1 R_2 = R_{12} + R_{21}. \tag{2.2}$$

Using (2.2) in (2.1), we get

$$\frac{r_2 R_{21}^2}{R_{12}} \left(\frac{r_1 R_{12}^2}{R_{21}} \left(\frac{y}{R_{12}} \right)' \right)' = L_2 y R_{21} - L_1 y R_1 + y.$$

Therefore,

$$\widetilde{L}_3 y = \frac{1}{R_{21}} \left(\frac{r_2 R_{21}^2}{R_{12}} \left(\frac{r_1 R_{12}^2}{R_{21}} \left(\frac{y}{R_{12}} \right)' \right)' \right)' = L_3 y. \tag{2.3}$$

It remains to show that $\widetilde{L}_3$ is strongly noncanonical, that is,

$$\int_{t_0}^{\infty} \frac{R_{21}(t)}{r_1(t) R_{12}^2(t)} \mathrm{d}t = \int_{t_0}^{\infty} \frac{R_{12}(t)}{r_2(t) R_{21}^2(t)} \mathrm{d}t < \infty. \tag{2.4}$$

By virtue of (2.2), we see that

$$\int_{t_0}^{\infty} \frac{R_{21}(t)}{r_1(t)R_{12}^2(t)}\mathrm{d}t = -\int_{t_0}^{\infty} \left[\left(\frac{1}{R_{12}(t)} \right)' R_1(t) + \frac{1}{r_1(t)R_{12}(t)} \right] \mathrm{d}t = \left. \frac{R_1(t)}{R_{12}(t)} \right|_{\infty}^{t_0}.$$

Using the l'Hospital rule, we have

$$\lim_{t\to\infty} \frac{R_1(t)}{R_{12}(t)} = \lim_{t\to\infty} \frac{1}{R_2(t)} = 0. \tag{2.5}$$

Hence,

$$\int_{t_0}^{\infty} \frac{R_{21}(t)}{r_1(t)R_{12}^2(t)}\mathrm{d}t < \infty.$$

Convergence of the second integral in (2.4) can be shown in the same way. The proof is complete. □

Corollary 2.3. *The equation* (E) *possesses a solution* y *if and only if the equation*

$$\left(b_2\left(b_1 x'\right)'\right)'(t) + q(t)R_{21}(t)R_{12}(\tau(t))x(\tau(t)) = 0. \tag{E'}$$

has a solution $x = y/R_{12}$.

Similarly as before, one can define the operators

$$\widetilde{L}_0 x = x = \frac{y}{R_{12}}, \quad \widetilde{L}_i x = b_i\left(\widetilde{L}_{i-1}x\right)', \quad (i = 1,2), \quad \widetilde{L}_3 x = \left(\widetilde{L}_2 x\right)',$$

where b_i, $i = 1,2$ are as in Theorem 2.2. Also, we set

$$\widetilde{q}(t) = q(t)R_{21}(t)R_{12}(\tau(t)).$$

Then (E') can be rewritten in the form

$$\widetilde{L}_3 x(t) + \widetilde{q}(t)x(\tau(t)) = 0.$$

Let us explore various asymptotic properties of (E') which will be useful in the next. The following obvious result gives the structure of possible nonoscillatory solutions of (E').

Lemma 2.4. *Assume* $(H_1)-(H_3)$. *The set of all nonoscillatory solutions* $x = y/R_{12}$ *of* (E') *can be divided into the following four classes*

$$\begin{aligned}
\widetilde{N}_0 &= \left\{x(t) : (\exists T \geq t_0)(\forall t \geq T)\left(x(t)\widetilde{L}_1 x(t) < 0,\ x(t)\widetilde{L}_2 x(t) > 0\right)\right\},\\
\widetilde{N}_a &= \left\{x(t) : (\exists T \geq t_0)(\forall t \geq T)\left(x(t)\widetilde{L}_1 x(t) > 0,\ x(t)\widetilde{L}_2 x(t) < 0\right)\right\},\\
\widetilde{N}_b &= \left\{x(t) : (\exists T \geq t_0)(\forall t \geq T)\left(x(t)\widetilde{L}_1 x(t) > 0,\ x(t)\widetilde{L}_2 x(t) > 0\right)\right\},\\
\widetilde{N}_* &= \left\{x(t) : (\exists T \geq t_0)(\forall t \geq T)\left(x(t)\widetilde{L}_1 x(t) < 0,\ x(t)\widetilde{L}_2 x(t) < 0\right)\right\}.
\end{aligned}$$

Lemma 2.5. *Assume* $(H_1)-(H_3)$. *If*

$$\int_{t_0}^{\infty} \frac{1}{b_2(t)} \int_{t_0}^{t} \widetilde{q}(s)\mathrm{d}s\,\mathrm{d}t = \infty, \tag{2.6}$$

then $\widetilde{N}_a = \widetilde{N}_b = \emptyset$ *for* (E') .

Proof. To show the nonexistence of solutions from classes $\widetilde{N}_a$ and $\widetilde{N}_b$, we proceed as the in proof of cases (3) and (4), respectively, in [10, Theorem 1]. □

In the sequel, we consider the following auxiliary functions

$$\pi_1(t) = \int_t^\infty \frac{1}{b_1(s)}\,\mathrm{d}s = \frac{R_1(t)}{R_{12}(t)}, \qquad \pi_2(t) = \int_t^\infty \frac{1}{b_2(s)}\,\mathrm{d}s = \frac{R_2(t)}{R_{21}(t)},$$
$$\pi(t) = \int_t^\infty \frac{1}{b_1(s)}\pi_2(s)\,\mathrm{d}s = \frac{1}{R_{12}(t)}.$$

Lemma 2.6. *Assume* $(H_1)-(H_3)$*. If*

$$\int_{t_0}^\infty \widetilde{q}(s)\pi_1(\tau(s))\,\mathrm{d}s = \infty, \tag{2.7}$$

then every nonoscillatory solution $x(t) \in \widetilde{N}_0$ *of* (E') *satisfies*

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \widetilde{L}_1 x(t) = 0.$$

Proof. Let $x(t)$ be a positive solution of (E') such that $x(t) \in \widetilde{N}_0$ eventually, say for $t \ge t_1$, where $t_1 \in [t_0, \infty)$ is large enough. Assume on the contrary that $\lim_{t\to\infty} x(t) = \ell > 0$. An integration of (E') yields

$$\widetilde{L}_2 x(t_1) \ge \int_{t_1}^\infty \widetilde{q}(s)x(\tau(s))\,\mathrm{d}s \ge \ell \int_{t_1}^\infty \widetilde{q}(s)\,\mathrm{d}s. \tag{2.8}$$

On the other of hand, since $\lim_{t\to\infty} \pi_1(t) = 0$, (2.7) implies that $\int_{t_1}^\infty \widetilde{q}(s)\,\mathrm{d}s = \infty$. In view of (2.8), this, however, contradicts the fact that $L_2 x$ is decreasing and we conclude that $x(t) \to 0$ as $t \to \infty$.

Now assume that $\lim_{t\to\infty} \widetilde{L}_1 x(t) = -\ell < 0$. Then $-\widetilde{L}_1 x(t) \ge \ell$ eventually, and so

$$x(t) \ge \ell \int_t^\infty \frac{1}{b_1(s)}\,\mathrm{d}s = \ell\pi_1(t). \tag{2.9}$$

Integrating (E') from t_1 to ∞ and using (2.7) and (2.9) in the resulting inequality yield

$$\widetilde{L}_2 x(t_1) \ge \int_{t_1}^\infty \widetilde{q}(s)x(\tau(s))\,\mathrm{d}s \ge \ell \int_{t_1}^\infty \widetilde{q}(s)\pi_1(\tau(s))\,\mathrm{d}s \to \infty \quad \text{as} \quad t \to \infty.$$

A contradiction and the proof is complete. □

The next result is crucial in establishing important relations between solutions of (E) and those of the corresponding strongly noncanonical equation (E').

Lemma 2.7. *Let* $(H_1)-(H_3)$, (2.6) *and* (2.7) *hold. Assume that* $x(t)$ *is a positive solution of* (E')*. If* $x(t) \in \widetilde{N}_0$*, then*

$$(xR_{12})'(t) \le 0. \tag{2.10}$$

.

If $x(t) \in \widetilde{N}_*$*, then*

$$(xR_{12})'(t) \ge 0. \tag{2.11}$$

Proof. At first assume that $x(t) \in \widetilde{N}_0$. By Lemma 2.6 and the monotonicity of $\widetilde{L}_2 x$, we see that

$$-\widetilde{L}_1 x(t) = \int_t^\infty \frac{1}{b_2(s)} \widetilde{L}_2 x(s)\, \mathrm{d}s \le \widetilde{L}_2 x(t) \pi_2(t).$$

Hence,

$$\left(\frac{\widetilde{L}_1 x}{\pi_2}\right)'(t) = \frac{\widetilde{L}_2 x(t)\pi_2(t) + \widetilde{L}_1 x(t)}{\pi_2^2(t) b_2(t)} \le 0,$$

which implies that $\widetilde{L}_1 x(t)/\pi_2(t)$ is decreasing. Therefore,

$$x(t) = \int_t^\infty \frac{-\widetilde{L}_1 x(s)}{\pi_2(s)} \frac{1}{b_1(s)}\, \pi_2(s)\, \mathrm{d}s \le \frac{-\widetilde{L}_1 x(t)}{\pi_2(t)}\, \pi(t),$$

and we conclude that

$$(x R_{12})'(t) = \left(\frac{x}{\pi}\right)'(t) = \frac{\widetilde{L}_1 x(t)\pi(t) + \pi_2(t) x(t)}{\pi^2(t) b_1(t)} \le 0.$$

Now we assume that $x(t) \in \widetilde{N}_*$. By virtue of the fact that $-\widetilde{L}_1 x$ is increasing, we have

$$x(t) = x(\infty) - \int_t^\infty \frac{1}{b_1(s)} \widetilde{L}_1 x(s)\, \mathrm{d}s \ge -\widetilde{L}_1 x(t) \pi_1(t).$$

Thus,

$$\left(\frac{x}{\pi_1}\right)'(t) = \frac{\widetilde{L}_1 x(t) + x(t)}{\pi_1^2(t) b_1(t)} \ge 0,$$

that is,

$$\left(\frac{R_{12}}{R_1} x\right)'(t) \ge 0.$$

Hence,

$$0 \le \left(\frac{R_{12}}{R_1} x\right)'(t) = (R_{12} x)'(t) \frac{1}{R_1(t)} - x(t) R_{12}(t) \frac{1}{r_1(t) R_1^2(t)}.$$

Consequently, $(R_{12} x)'(t) \ge 0$ and the proof is complete. □

In view of Lemma 2.5, the essential classes for (E') are $\widetilde{N}_0$ and $\widetilde{N}_*$. In the next main result, they will be shown, under weak assumptions, to be equivalent to classes N_0 and N_2 of (E), respectively.

Theorem 2.8. *Let* (H_1)–(H_3), (2.6) *and* (2.7) *hold. Assume that* $y(t)$ *and* $x(t) = y(t)/R_{12}(t)$ *are corresponding nonoscillatory solutions of* (E) *and* (E'), *respectively. Then*

$$y(t) \in N_0 \quad \textit{if and only if} \quad x(t) \in \widetilde{N}_0,$$
$$y(t) \in N_2 \quad \textit{if and only if} \quad x(t) \in \widetilde{N}_*,$$

Proof. Assume that $y(t) \in N_0$. Then $y'(t) < 0$, and consequently $(R_{12} x)'(t) < 0$. By Lemma 2.7, $x(t) \notin \widetilde{N}_*$ and so $x(t) \in \widetilde{N}_0$.

On the other hand, if we assume that $y(t) \in N_2$, then $y'(t) > 0$, and consequently $(R_{12} x)'(t) > 0$. By Lemma 2.7, $x(t) \in \widetilde{N}_*$. □

3 Applications

In this section, we provide some oscillation criteria for (E) in two ways: using directly (E) and also using a strongly noncanonical corresponding equation (E'). Subsequently, we test the strength of these results on Euler type equations, showing the advantage of making use the strongly noncanonical equation (E').

As usual, all functional inequalities considered in this paper are supposed to be satisfied for all t large enough.

Theorem 3.1. *Assume* $(H_1)-(H_3)$*. If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s)R_{12}(\tau(s))\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{3.1}$$

then any nonoscillatory solution y of (E) belongs to the class N_0.

Proof. Let $y(t)$ be a nonoscillatory solution of (E). By Lemma 2.1, either $y \in N_0$ or $y \in N_2$. Assume on the contrary that $y \in N_2$. Without loss of generality, we may take $t_1 \geq t_0$ such that

$$y(t) > 0, \quad L_iy(t) > 0, \quad i = 1,2, \quad L_3y(t) < 0 \quad \text{for } t \geq t_1.$$

Next, we claim that (3.1) implies

$$\lim_{t\to\infty} L_2y(t) = 0. \tag{3.2}$$

Assume not, i.e. $\lim_{t\to\infty} L_2y(t) = \ell > 0$. Then $L_2y(t) \geq \ell$ eventually, say for $t_* \geq t_1$ and so $y(t) \geq \ell R_{12}(t)$. Using this inequality in (E) and integrating the resulting inequality from t_* to t, we see that

$$L_2y(t) \geq \int_{t_*}^{t} q(s)R_{12}(\tau(s))\mathrm{d}s \to \infty \quad \text{as} \quad t \to \infty. \tag{3.3}$$

Since

$$\int_{t_0}^{\infty} q(s)R_{12}(\tau(s))\mathrm{d}s = \infty$$

is necessary for the validity of (3.1), condition (3.3) clearly contradicts the fact that L_2y is decreasing. Thus, (3.2) holds. On the other hand, it follows from the monotonicity of $L_2y(t)$ that

$$\begin{aligned}
L_1y(t) &= L_1y(t_1) + \int_{t_1}^{t} \frac{1}{r_2(s)} L_2y(s)\mathrm{d}s \\
&\geq L_1y(t_1) + L_2y(t) \int_{t_1}^{t} \frac{\mathrm{d}s}{r_2(s)} \\
&= L_1y(t_1) + L_2y(t)R_2(t) - L_2y(t) \int_{t_0}^{t_1} \frac{\mathrm{d}s}{r_2(s)} \\
&\geq L_2y(t)R_2(t)
\end{aligned}$$

for $t \geq t_2$, where $t_2 > t_1$ is large enough. Dividing both sides of the latter inequality by $r_1(t)$ and integrating the resulting inequality from t_2 to t, we get

$$\begin{aligned}
y(t) &= y(t_2) + \int_{t_2}^{t} \frac{R_2(s)}{r_1(s)} L_2y(s)\mathrm{d}s \\
&\geq y(t_2) + L_2y(t) \int_{t_2}^{t} \frac{R_2(s)}{r_1(s)}\mathrm{d}s \\
&= y(t_2) + L_2y(t)R_{12}(t) - L_2y(t) \int_{t_0}^{t_2} \frac{R_2(s)}{r_1(s)}\mathrm{d}s \\
&\geq L_2y(t)R_{12}(t)
\end{aligned}$$

for $t \geq t_3$, where $t_3 > t_2$ is large enough. From (E), we see that $z(t) = L_2 y(t)$ is a positive solution of the first-order delay differential inequality

$$z'(t) + q(t)R_{12}(\tau(t))z(\tau(t)) \leq 0.$$

However, by [14, Theorem 2.1.1], condition (3.1) ensures that the above inequality does not possess a positive solution, which is a contradiction. The proof is complete. □

Remark 3.2. Theorem 3.1 given for canonical equation (E) improves [16, Theorem 6.2.2] in the sense that (3.1) does not depend on the value of the initial constant appearing in R_{12}.

The next result provides an alternative criterion for Theorem 3.1, based on the use of corresponding strongly noncanonical equation (E').

Theorem 3.3. *Assume* (H_1)–(H_3) *and* (2.7). *If*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \frac{1}{b_1(v)} \int_{t_1}^{v} \frac{1}{b_2(u)} \int_{t_1}^{u} \tilde{q}(s)\mathrm{d}s\mathrm{d}u\,\mathrm{d}v > \frac{1}{\mathrm{e}} \tag{3.4}$$

for any $t_1 \geq t_0$, *where* b_i, $i = 1, 2$ *are as in Theorem* 2.2, *then any nonoscillatory solution* y *of* (E) *belongs to the class* N_0.

Proof. Let $y(t)$ be a nonoscillatory solution of (E). By Lemma 2.1, either $y \in N_0$ or $y \in N_2$. Assume on the contrary that $y \in N_2$.

Clearly, condition

$$\int_{t_1}^{\infty} \frac{1}{b_1(v)} \int_{t_1}^{v} \frac{1}{b_2(u)} \int_{t_1}^{u} \tilde{q}(s)\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v = \infty, \tag{3.5}$$

is necessary for the validity of (3.4), which in view of the fact that $\pi(t_1) < \infty$ implies (2.6). By Theorem 2.8, it suffices to show that (E') does not possess a solution $x \in \widetilde{N}_*$. Assume the contrary. Without loss of generality, we may take $t_1 \geq t_0$ such that

$$x(t) > 0, \quad \widetilde{L}_i x(t) < 0, \quad i = 1, 2, 3 \quad \text{for } t \geq t_1.$$

Proceeding the same as in the proof of case (1) of [10, Theorem 2], we arrive at contradiction with (3.4). The proof is complete. □

Theorem 3.4. *Let all assumptions of Theorem* 3.1 *hold. If, moreover,*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} q(s) \int_{\tau(s)}^{\tau(t)} \frac{1}{r_1(u)} \int_{u}^{\tau(t)} \frac{\mathrm{d}x}{r_2(x)}\mathrm{d}u\,\mathrm{d}s > 1, \tag{3.6}$$

then (E) *is oscillatory.*

Proof. Assume to the contrary that y is a nonoscillatory solution of (E). By Theorem 3.1, we have that $y(t) \in N_0$. Proceeding the same as in the proof of case (2) of [9, Theorem 2], we arrive at contradiction with (3.6). The proof is complete. □

Theorem 3.5. *Let all assumptions of Theorem* 3.3 *hold. If, moreover,*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \tilde{q}(s) \int_{\tau(s)}^{\tau(t)} \frac{1}{b_1(u)} \int_{u}^{\tau(t)} \frac{\mathrm{d}x}{b_2(x)}\mathrm{d}u\,\mathrm{d}s > 1, \tag{3.7}$$

then (E) *is oscillatory.*

Proof. By Theorem 2.8 and the proof of Theorem 3.3, it suffices to show that (E') does not possess a solution $x \in N_0$. Assume the contrary. Without loss of generality, we may take $t_1 \geq t_0$ such that

$$x(t) > 0, \quad \widetilde{L}_1 x(t) < 0, \quad \widetilde{L}_2 x(t) > 0, \quad \widetilde{L}_3 x(t) < 0 \quad \text{for } t \geq t_1.$$

Proceeding the same as in the proof of case (2) of [9, Theorem 2], we arrive at contradiction with (3.7). The proof is complete. □

Example 3.6. Consider the Euler equation

$$y'''(t) + \frac{q_0}{t^3} y(\lambda t) = 0, \quad \lambda \in (0,1). \tag{3.8}$$

By Theorem 2.2, the corresponding strongly noncanonical equation is

$$\left(t^2 \left(t^2 x'(t)\right)'\right)' + q_0 \lambda^2 t x(\lambda t) = 0, \quad \lambda \in (0,1). \tag{3.9}$$

Both Theorems 3.1 and 3.3 reduce to the same condition

$$\frac{\lambda^2 q_0}{2} \ln \frac{1}{\lambda} > \frac{1}{\mathrm{e}},$$

which ensures that $N_2 = \emptyset$ for (3.8). On the other hand, condition

$$q_0 \lambda^2 \left[\ln \frac{1}{\lambda} - 2\left(\frac{1}{\lambda} - 1\right) + \frac{1}{2}\left(\frac{1}{\lambda^2} - 1\right)\right] > 2 \tag{3.10}$$

from Theorem 3.4 or condition

$$q_0 \left[\ln \frac{1}{\lambda} - 2(1-\lambda) + \frac{1-\lambda^2}{2}\right] > 2 \tag{3.11}$$

from Theorem 3.5 implies that $N_0 = \emptyset$. One can verify that (3.11) always provides a stronger result than (3.10), which clearly justifies the use of strongly noncanonical equations (3.9) in investigating the asymptotic properties of (E). This surprising feature has been revealed when evaluating the integrals (3.6) and (3.7).

Remark 3.7. In general, the nonexistence of solutions of (E) belonging to the class N_0 is due to a delay argument only. The idea of improving the criteria eliminating such solutions by rewriting the equation into a strongly noncanonical form which we present in this paper deserves to be further studied.

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