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SOME PROPERTIES OF THE DULAC FUNCTIONS SET

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Abstract. In order to rule out the existence of periodic orbits in the plane for a given system of differential equations, we discuss the feature of the set of Dulac functions, establishing some of its properties as well as some results for special cases where this set of functions is not empty. We give some examples to illustrate applications of these results.

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1. Introduction

Many problems of the qualitative theory of differential equations in the plane refer to the existence of periodic orbits, for example in mechanical or electrical engineering, biological models and many others. However, until now we can not answer in general whether, given an arbitrary system of differential equations, it has periodic orbits or not.

There are some criteria that allow us to rule out the existence of periodic orbits in the plane such as Poincaré-Bendixson, the index theory and special systems such as the system gradient, among others, see ([1],[9],[8] and [5]).

A classical criterion to discard the existence of periodic orbits (or limiting the number of these) in a given region is the Bendixson-Dulac theorem.

For convenience, we recall the last criterion, see ([8] pag. 262, [9] pag. 202-203).

Theorem 1. (Bendixson-Dulac criterion) *Let $f_1(x_1,x_2)$, $f_2(x_1,x_2)$ and $h(x_1,x_2)$ be functions C^1 in a simply connected domain $D \subset \mathbb{R}^2$ such that $\frac{\partial(f_1 h)}{\partial x_1} + \frac{\partial(f_2 h)}{\partial x_2}$ does not change sign in D and vanishes at most on a set of measure zero. Then the system*

$$\begin{cases} \dot{x}_1 &= f_1(x_1,x_2), \\ \dot{x}_2 &= f_2(x_1,x_2), \quad (x_1,\ x_2) \in D, \end{cases} \tag{1}$$

does not have periodic orbits in D.

According to this criterion, to rule out the existence of periodic orbits of the system (1) in a simply connected region D, we need to find a function $h(x_1, x_2)$ that satisfies the conditions of the theorem of Bendixson-Dulac. Such function h is called a *Dulac function*.

Usually it is not easy to determine such a function, however it is possible to propose some candidates of the form $h = 1, x_1^s, x_2^s, e^{ax_1+bx_2}, x_1^s x_2^t, s, t \in \mathbb{Q}, a, b \in \mathbb{R}$, among others. In the particular case $h = 1$ this theorem is known as Bendixson's criterion.

There are some papers constructing the function of Dulac for special systems, for example see [3], [2], [6] and [7], also see [4] and [10] for more general situations.

In this paper we will introduce and study the set $\mathcal{H}_D^+(F)$ of Dulac functions for a region D and the vector field $F = (f_1, f_2)$ defined by system (1), showing some characteristics that allow us to say whether the set $\mathcal{H}_D^+(F)$ is different from the empty set.

2. Properties of the Dulac functions

Consider the vector field $F(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, then the system (1) can be rewritten in the form

$$\dot{x} = F(x), \qquad x = (x_1, x_2) \in D, \tag{2}$$

now let $C^0(D, \mathbb{R})$ be the set of continuous functions and define the set

$\mathcal{F}_D = \{f \in C^0(D, \mathbb{R}) : f$ doesn't change sign and vanishes only on a measure zero set$\}$.

Also for the simply connected region D, we introduce the sets

$$\mathcal{H}_D^+(F) = \{h \in C^1(D, \mathbb{R}) : k := \frac{\partial(hf_1)}{\partial x_1} + \frac{\partial(hf_2)}{\partial x_2} \geq 0, \;\; k \in \mathcal{F}_D\}$$

and

$$\mathcal{H}_D^-(F) = \{h \in C^1(D, \mathbb{R}) : k := \frac{\partial(hf_1)}{\partial x_1} + \frac{\partial(hf_2)}{\partial x_2} \leq 0, \;\; k \in \mathcal{F}_D\}.$$

A Dulac function in the system (1) of the Bendixson-Dulac theorem is an element in the set

$$\mathcal{H}_D(F) := \mathcal{H}_D^+(F) \cup \mathcal{H}_D^-(F).$$

This set has the following properties that are listed below in the next result.

Lemma 1. *Let $F : D \to \mathbb{R}^2, C^1, D$ simply connected, then*

(a) *if $\mathcal{H}_D(F) \neq \emptyset$, then (1) has no periodic orbits entirely contained in D.*

(b) $\mathcal{H}_D^-(F) = -\mathcal{H}_D^+(F)$.

(c) $\mathcal{H}_D(F) \neq \emptyset$ *if and only if* $\mathcal{H}_D^+(F) \neq \emptyset$.

(d) *If $h_1,\ h_2 \in \mathcal{H}_D^+(F)$ and $\lambda_1, \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 > 0$, then $\lambda_1 h_1 + \lambda_2 h_2 \in \mathcal{H}_D^+(F)$.*

(e) *Let $D_1,\ D_2$ be simply connected sets such that $D_1 \subset D_2$, if $\mathcal{H}_{D_2}^+(F) \neq \emptyset$, then $\mathcal{H}_{D_1}^+(F) \neq \emptyset$, in particular $\mathcal{H}_{\mathbb{R}^2}^+(F) \subset \mathcal{H}_{D_2}^+(F) \subset \mathcal{H}_{D_1}^+(F)$.*

(f) *Let $D \subset \mathbb{R}^2$ simply connected. Suppose that for all $D_1 \subset D$ simply connected bounded, $\mathcal{H}_{D_1}^+(F) \neq \emptyset$, then there are no periodic orbits in D.*

Proof. items (a), (b) and (d) are direct from the definition.

(c) It follows from (b).

(e) If $h \in \mathcal{H}_{D_2}^+(F)$, then we can take $h_| \in C^1(D_1, R)$ such that $h_|$ is the restriction of h to the set D_1.

(f) Suppose there is a periodic orbit γ in D. Take D_1 as the region bounded by γ, then by hypothesis, there exists a function $h \in \mathcal{H}_{D_1}^+(F)$ and so, D_1 can not have periodic orbits. $\square$

Now we examine conditions that imply that the set $\mathcal{H}_D^+(F) \neq \emptyset$. Our results are established with the help of the techniques developed by the authors in [10], let us recall the following proposition

Theorem 2. ([10]). *If there exist $c \in \mathcal{F}_D$ such that h is a solution of the equation*

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h\left(c(x_1, x_2) - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right), \tag{3}$$

with $h \in \mathcal{F}_D$, then h is a Dulac function for (1) on D.

A first result of the existence of Dulac functions is as follows

Theorem 3. *Suppose there is $c \in \mathcal{F}_D$, such that*

$$\mu_i := \frac{1}{f_i}\left(c - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right) \textit{ depends only on } x_i, \textit{ for some } i \in \{1, 2\}$$

and is continuous, then the set $\mathcal{H}_D^+(F)$ is not empty.

Proof. We consider the case μ_1 depending only on x_1. We seek a Dulac function, using the theorem 2, so that the associated equation is

$$f_1\frac{\partial h}{\partial x_1}+f_2\frac{\partial h}{\partial x_2}=h\left(c(x_1,x_2)-\left(\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}\right)\right),$$

Assume that h depends only on x_1. Thus the previous equation reduces to

$$f_1\frac{\partial h}{\partial x_1}=h\left(c(x_1,x_2)-\left(\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}\right)\right),$$

which is rewritten as

$$\frac{\partial \log h}{\partial x_1}=\frac{1}{f_1}\left(c(x_1,x_2)-\left(\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}\right)\right)=\mu_1.$$

From our hypothesis $h=\exp\left(\int \mu_1 dx_1\right)$ is a solution and satisfies the conditions of theorem 2, therefore the system has a Dulac function. The proof is complete. □

Example 1. *Consider the system*

$$\begin{aligned}\dot{x_1} &= x_1x_2,\\ \dot{x_2} &= (x_1x_2)^2\cos x_1+2x_2^3+5x_2,\end{aligned}$$

calculating μ_1, we have

$$\mu_1=\frac{1}{f_1}\left[c(x_1,x_2)-\frac{\partial f_1}{\partial x_1}-\frac{\partial f_2}{\partial x_2}\right],$$

or replacing

$$\mu_1=\frac{1}{x_1x_2}\left[c(x_1,x_2)-x_2-[2(x_1^2x_2)\cos x_1+6x_2^2+5]\right]$$

and taking

$$c=x_2+5+6x_2^2>0, \qquad \forall\, x_1,x_2\in\mathbb{R}^2.$$

We have

$$\mu_1=2x_1\cos x_1$$

and therefore the set $\mathcal{H}^+_{\mathbb{R}^2}(F)$ is not empty. ◇

Example 2. *Let $g\in C^1(\mathbb{R},\mathbb{R})$ and the system*

$$\begin{aligned}\dot{x_1} &= 2x_1^3-5x_1^2x_2+g(x_2),\\ \dot{x_2} &= x_1x_2+x_1x_2^3,\end{aligned}$$

then

$$\mu_2 = \frac{1}{x_1 x_2(1+x_2^2)}\left[c(x_1,x_2) - (6x_1^2 - 10x_1x_2) - 3x_1x_2^2\right].$$

We can take $c \in \mathcal{F}_{\mathbb{R}^2}$ *as* $c(x_1,x_2) := 6x_1^2 \geq 0$, *we have*

$$\mu_2 = \frac{10-3x_2}{1+x_2^2},$$

that only depend on x_2*, then by theorem 3,* $\mathcal{H}^+_{\mathbb{R}^2}(F) \neq \emptyset$*.* ◇

Now we use theorem 3 to study some special systems, consider an equation as follows

$$\begin{cases} \dot{x}_1 = r_1(x_1)r_2(x_2), \\ \dot{x}_2 = s_1(x_1)s_2(x_2). \end{cases} \tag{4}$$

We establish the following

Proposition 1. *Let* $D_0 \subset D$ *be a compact, simply connected set. If* $r_1 \neq 0$, $r_2 \in \mathcal{F}_D$ *and* $s_1 s_2' \geq 0$ *then the set* $\mathcal{H}^+_{D_0}(F)$ *is not empty.*

Proof. From theorem 3, it is enough to see that we can choose $\mu_1(x_1)$ a continuous function such that

$$c := \mu_1 r_1 r_2 + \left(\frac{\partial r_1 r_2}{\partial x_1} + \frac{\partial s_1 s_2}{\partial x_2}\right) \in \mathcal{F}_{D_0}.$$

Without loss of generality, suppose $r_2 \geq 0$ and $r_1 > 0$ in D. We take $\mu_1(x_1) := nr_1(x_1)$ with $n \in \mathbb{N}$ such that $\mu_1 r_1 + r_1' > 0$ in D_0. This is possible because r_1' is continuous and D_0 compact. So we have $r_2(\mu_1 r_1 + r_1') \in \mathcal{F}_{D_0}$. Therefore

$$c = r_2(\mu_1 r_1 + r_1') + s_1 s_2' \geq r_2(\mu_1 r_1 + r_1'),$$

thus $c \in \mathcal{F}_{D_0}$, which we needed to prove. □

Example 3. *Let* $D_0 = \{(x_1,x_2) : \sqrt{x_1^2+x_2^2} \leq 1\} \subset \mathbb{R}^2$*, and consider the system*

$$\begin{aligned} \dot{x_1} &= -x_1x_2 + x_1 + 2x_2 - 2, \\ \dot{x_2} &= x_1^2(x_2 - \cos x_2), \end{aligned}$$

as $-x_1x_2 + x_1 + 2x_2 - 2 = (2-x_1)(x_2-1)$*, and* $(x_2 - \cos x_2)' = 1 + \sin x_2$*. It follows from the last proposition that there is some* $h \in \mathcal{H}^+_{D_0}(F)$*.*

Example 4. *Let D_0 be the region bounded by $0 \leq x_1 \leq a$ and $0 \leq x_2 \leq b$ in $\mathbb{R}^2$ and consider the system*

$$\begin{aligned} \dot{x_1} &= x_1^2 x_2^2 + a x_2^2, \\ \dot{x_2} &= e^{x_1}(b + x_2), \end{aligned}$$

now $r_1 = x_1^2 + a \neq 0$ and $r_2 = x_2^2 \in \mathcal{F}_D$ and $s_1 s_2' \geq 0$, therefore from proposition 1, $\mathcal{H}^+_{D_0}(F) \neq \emptyset$.

Another result that helps us to establish conditions for which the set $\mathcal{H}^+_D(F) \neq \emptyset$ is the next

Proposition 2. *Let $D \subseteq \mathbb{R}^2$, suppose that there exists a function $h : D \to \mathbb{R}$, C^1 which only vanishes on a set of measure zero such that*

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} > 0 \qquad \text{in } D, \tag{5}$$

then for any $D_1 \subset D$ simply connected compact, we have

$$\mathcal{H}^+_{D_1}(F) \neq \emptyset.$$

Proof. Note that $h^{2k} \in \mathcal{F}_D$ for all $k \in \mathbb{N}$. Let D_1 be a simply connected compact and take

$$0 < r_0 := min_{(x_1,x_2)\in D_1} \left\{ f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} \right\}$$

and take $m_0 > 0$ such that $|h \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) | \leq m_0$ in D_1. Now take $n := 2k + 1$ such that $n r_0 - m_0 > 0$ and consider

$$\begin{aligned} & f_1 \frac{\partial h^n}{\partial x_1} + f_2 \frac{\partial h^n}{\partial x_1} + h^n \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) = \\ & h^{n-1} \left[n \left(f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} \right) + h \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \right] \in \mathcal{F}_{D_1}, \end{aligned}$$

since it holds that

$$h^{2k} \left[n r_0 - m_0 \right] \in \mathcal{F}_{D_1}.$$

□

Example 5. *Consider*

$$\begin{aligned} \dot{x_1} &= -x_1^2 + x_1 x_2 + 1, \\ \dot{x_2} &= -x_1 x_2 + 2 x_1^2 \end{aligned}$$

and take $h(x_1,x_2)=x_1+x_2$*, then*

$$f_1\frac{\partial h}{\partial x_1}+f_2\frac{\partial h}{\partial x_2}=f_1+f_2=1+x_1^2>0,$$

therefore for any simply connected compact region $D\subset\mathbb{R}^2$, $\mathcal{H}_D^+(F)\neq\emptyset$. ◇

Example 6. *Consider*

$$\begin{aligned}\dot{x_1} &= -4x_1x_2^2+2x_2-1,\\ \dot{x_2} &= 2x_1^2x_2-2x_1x_2+1.\end{aligned}$$

Let $h=x_2^2-x_1$*, then*

$$(-4x_1x_2^2+2x_2-1)(-1)+(2x_1^2x_2-2x_1x_2+1)(2x_2)=1+4x_1^2x_2^2>0$$

and $\mathcal{H}_D^+(F)\neq\emptyset$ *for any simply connected compact region* $D\subset\mathbb{R}^2$. ◇

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References

[1] I. Bendixson, Sur les curbes définiés par des équations différentielles, *Acta Math.*, 24 (1901), p. 1-88.

[2] L. A. Cherkas, Dulac functions for polynomial autonomous systems on a plane, *Diff. eqs.*, 33, (1997), 692-701.

[3] M. Chamberland, A. Cima, A. Gasull, F. Mañosas, Characterizing Asymptotic Stability with Dulac Functions, *Discrete Contin. Dynam. Systems*, 17(1), (2007), 59-76.

[4] E. Saéz, I. Szánto, On the construction of certain Dulac function, *Transactions on Automatic Control*, vol. 33, (1988).

[5] M. Farkas, *Periodic Motions*, Applied Math. Sciences 104, Springer-Verlag, (1994).

[6] A. Gasull, H. Giacomini, Upper bounds for the number of limit cycles through linear differential equations, *Pacific J. Math.*, 226(2), (2006), 277-296.

[7] L. A. Cherkas, A. A Grin, K. R. Schneider, Dulac-Cherkas functions for generalized Liénard Systems, *E. J. of Qualitative Theory of Differential Equations*, No. 35, (2011), 1-23.

[8] Lawrence Perko, *Differential equations and dynamical systems*, Springer-Verlag, (2006).

[9] Strogatz, S. *Nonlinear dynamics and chaos*, Addison-Wesley, (1994).

[10] Osvaldo Osuna, Villaseñor-Aguilar G. On the Dulac functions, *Qualitative Theory of Dynamical Systems*, Vol. 10, No. 1, (2011), 43-49.

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