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A state-dependent delay equation with chaotic solutions

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Abstract. We exhibit a scalar-valued state-dependent delay differential equation

$$x'(t) = f(x(t - d(x_t)))$$

that has a chaotic solution. This equation has continuous (semi-strictly) monotonic negative feedback, and the quantity $t - d(x_t)$ is strictly increasing along solutions.

Keywords: differential delay equation, state-dependent delay, chaotic solution.

2010 Mathematics Subject Classification: 34K23.

1 Introduction

In this paper we consider the following differential delay equation with continuous monotonic negative feedback and state-dependent delay:

$$x'(t) = f(x(t - d(x_t))). \tag{1.1}$$

Here "negative feedback" means that $uf(u) < 0$ for all $u \neq 0$.

We shall write $C = C([-1, 0], \mathbb{R})$ for the space of continuous real-valued functions defined on $[-1, 0]$, equipped with the sup norm. In the usual way, if $I \subseteq \mathbb{R}$ is any interval containing $[t - 1, t]$ and $x : I \to \mathbb{R}$ is continuous, we write x_t for the point in C defined by

$$x_t(s) = x(t + s), \qquad -1 \leq s \leq 0,$$

and refer to x_t as a *segment* of x. We shall take as the phase space for Equation (1.1) an appropriate subset $X \subseteq C$ on which existence and uniqueness of solutions holds, and on which a continuous solution semiflow is defined. In particular, the delay functional d will be defined on X and will assume values between 0 and 1. By a *solution* of Equation (1.1) we mean either a continuous function $x : [-1, \infty) \to \mathbb{R}$ such that $x_t \in X$ for all $t \geq 0$ and (1.1) holds for all $t > 0$, or a continuous function $x : \mathbb{R} \to \mathbb{R}$ such that $x_t \in X$ for all $t \in \mathbb{R}$ and (1.1) holds for all $t \in \mathbb{R}$. In either case, we say that x is the *continuation of x_0 as a solution of Equation* (1.1).

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We are interested in the possibility of irregular or chaotic solutions of Equation (1.1) when Equation (1.1) has monotonic negative feedback. Our primary motivation is the impossibility of chaos in the "corresponding" constant-delay equation

$$x'(t) = f(x(t-1)). \tag{1.2}$$

Suppose that $x : [-1, \infty) \to \mathbb{R}$ is a bounded solution of Equation (1.2), and write $\omega(x_0) \subseteq C$ for the ω-limit set of x_0. A version of the Poincaré–Bendixson theorem, due to Mallet-Paret and Sell [9] (and that actually has much wider applicability than we describe here), describes the structure of $\omega(x_0)$ in the case that f is smooth and strictly monotonic with negative feedback. On the "dynamical" side, Mallet-Paret and Sell's Poincaré–Bendixson theorem states that $\omega(x_0)$ must be either a single non-constant periodic orbit, or must consist of the equilibrium point $\{0\}$ and (perhaps) solutions homoclinic to $\{0\}$ (though solutions homoclinic to $\{0\}$ seem unlikely to exist, and results in [8] imply that they cannot exist if $\{0\}$ is hyperbolic). On the "geometric" side, the theorem states that the map $\Pi : \omega(x_0) \to \mathbb{R}^2$ given by the formula

$$\omega(x_0) \ni \varphi \mapsto \Pi(\varphi) = (\varphi(0), \varphi(-1)) \in \mathbb{R}^2$$

is injective. This injectivity has the consequence that, if $z_0 < z_1 < z_2$ are three successive zeros of a nontrivial periodic solution p, then p has minimal period $z_2 - z_0$ – loosely speaking, p has "one oscillation about zero per minimal period."

It is known already that the state-dependent delay in Equation (1.1) allows for substantial changes in the dynamics of solutions. In [4], for example, an instance of Equation (1.1) is devised for which f is smooth and strictly decreasing and that has a periodic solution p for which the above-described map Π is not injective on $\{p_t\} = \omega(p_0)$. The papers [7], [13], [14], and [15] describe a state-dependent delay equation

$$x'(t) = -\alpha x(t - d(x_t)), \qquad \alpha > 0$$

with negative feedback and for which $\{0\}$ is hyperbolic, but that nevertheless has a solution homoclinic to $\{0\}$ and chaotic solutions near this homoclinic solution.

In the present paper, we wish to restrict attention to cases of Equation (1.1) where the delay functional is, in some (subjective) sense, not too dissimilar to the constant delay case. In particular, we are interested in cases where the delay $d(x_t)$ is strictly positive but bounded, and where the delayed time $t - d(x_t)$ is strictly increasing with respect to t for any solution x. We shall abbreviate this latter condition as (DTI). More specifically, condition (DTI) is defined as follows, where $x : [-1, \infty) \to \mathbb{R}$ is a solution of Equation (1.1):

$$t + \epsilon - d(x_{t+\epsilon}) > t - d(x_t) \quad \text{for any } t \geq 0 \text{ and any } \epsilon > 0. \tag{DTI}$$

Condition (DTI) (or close analogs) has been used by many authors in the study of state-dependent delay equations. On the conceptual side, (DTI) means that states influence the feedback response in the expected temporal order (though for a discussion of the delayed time $t - d(x_t)$ being *monotonic* but not necessarily increasing see [12]); on the analytical side, (DTI) facilitates the by-now-familiar organization of the phase space according to a non-increasing "oscillation speed." (In the above-mentioned papers [7], [13], [14], and [15], the "oscillation speed" of the solution homoclinic to $\{0\}$ increases, and condition (DTI) does not hold; condition (DTI) does hold for the equation considered in [4].)

Here, then, is the main theorem for the paper.

Theorem 1.1. *There is an instance of Equation (1.1) for which the following hold.*

(i) f *is continuous, non-increasing, and satisfies the negative feedback condition.*

(ii) A continuous solution semiflow for Equation (1.1) is defined on a subset $X \subseteq C$ *of bounded functions with bounded Lipschitz constant.*

(iii) (DTI) holds for all solutions of Equation (1.1) with segments in X.

(iv) Given any $k \in \mathbb{N}$, *there is a periodic solution* p^k *of Equation (1.1) such that* $p^k_t \in X$ *for all* $t \in \mathbb{R}$ *and such that the interval* $[0, \rho]$ *contains precisely* $2k+1$ *zeros of* p^k, *where* ρ *is the minimal period of* p^k.

(v) There is a solution v *of Equation (1.1) such that* $v_t \in X$ *for all* $t \in \mathbb{R}$ *and such that, given any* $k \in \mathbb{N}$, *there is a sequence* $t_n \to \infty$ *such that* $v_{t_n} \to p^k_0$.

Point (iv) demonstrates the violation of the "geometric" part of Mallet-Paret and Sell's Poincaré–Bendixson theorem; point (v) demonstrates the violation of the "dynamic" part. Points (iv) and (v) together constitute what we mean by our particular instance of Equation (1.1) being "chaotic". As we proceed, however, we shall see that it is possible to specify the version of Equation (1.1) described in Theorem 1.1 so that the following is also true: there is a subset $M \subseteq X$, a return map $R : M \to M$ for Equation (1.1), an interval $I \subseteq [0, 1]$, and a map $h : \mathbb{R} \to I$ such that

- the restricted dynamical system $h^2 : I \to I$ is semiconjugate to $R : M \to M$; and
- there is a subinterval of I on which h^2 is chaotic (in the sense of Devaney).

This is another sense in which our example equation can be considered to be "chaotic."

Remark 1.2. Chaotic solutions in the constant-delay case (Equation (1.2)) are known to be possible in the case that f is non-monotonic with negative feedback (see, for example, [10] and [11], or [5] and [6]).

Remark 1.3. While Theorem 1.1 illustrates how variable delay can complicate solution behavior, we emphasize that the feedback function f in Theorem 1.1 is only nonincreasing, rather than strictly decreasing; accordingly, Theorem 1.1, by itself, does not quite illustrate that the Poincaré-Bendixson theorem fails merely by the introduction of state-dependent delay.

The paper is organized as follows. In Section 2 we give a very simple existence and uniqueness result that is adequate for our purposes. In Section 3 we define our particular equation of interest. In Section 4 we make the explicit estimates that we need, and prove Theorem 1.1.

2 Existence and uniqueness

Throughout, if A and B are metric spaces with metrics d_A and d_B, respectively, and $H : A \to B$ is any function, we write $\ell(H)$ for the global Lipschitz constant for H, provided that it exists:

$$\ell(H) = \sup_{a_1 \neq a_2} \frac{d_B(H(a_1), H(a_2))}{d_A(a_1, a_2)}.$$

As stated above, we write $C = C([-1,0],\mathbb{R})$, equipped with the sup norm. Throughout we write

$$X = \{\varphi \in C \ : \ \|\varphi\| \le 1, \ \ell(\varphi) \le 1\}.$$

By the Ascoli–Arzelà Theorem, X is a compact subset of C.

We shall assume throughout that $f : \mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses:

$$\begin{cases} f \text{ is nonincreasing, } f \text{ is Lipschitz (i.e. } \ell(f) < \infty\text{), and } uf(u) < 0 \text{ for all } u \neq 0; \\ |f(u)| \le 1 \text{ for all } u \in \mathbb{R}. \end{cases} \tag{Hf}$$

We shall assume throughout that $d : X \to \mathbb{R}$ satisfies the following hypotheses:

$$\begin{cases} 0 < d_{\min} \le d(\varphi) \le d_{\max} \le 1 \text{ for all } \varphi \in X; \\ d \text{ is Lipschitz (i.e. } \ell(d) < \infty\text{);} \\ \text{There is some } \alpha \in (0,1) \text{ such that, if } \varphi \in X \text{ and } \psi \in X \text{ and} \\ \varphi(s) = \psi(s) \text{ for all } s \in [-1,-\alpha]\text{, then } d(\varphi) = d(\psi). \end{cases} \tag{Hd}$$

The assumptions on the range of d and that d is Lipschitz are familiar; the final assumption says that $d(\varphi)$ depends only on the restriction of φ to $[-1,-\alpha]$ and substantially simplifies our work both in this section and later.

Here is the existence and uniqueness theorem that we shall use. The ideas are, by now, standard.

Proposition 2.1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies Hypotheses (Hf) and that $d : X \to \mathbb{R}$ satisfies Hypotheses (Hd). Then Equation (1.1) has a uniquely defined continuous solution semiflow $F : \mathbb{R}_+ \times X \to X$.*

Proof. Choose and fix $\beta \in (0, \min(d_{\min}, \alpha))$, where $d_{\min}$ and α are as in (Hd).

Given $\varphi \in X$, let $\tilde{x} : [-1,\beta] \to \mathbb{R}$ be any extension of φ to $[-1,\beta]$ with $|\tilde{x}(s)| \le 1$ for all $s \in [-1,\beta]$ and $\ell(\tilde{x}) \le 1$ – otherwise put, $\tilde{x}_s \in X$ for all $s \in [0,\beta]$ (we may, in particular, take $\tilde{x}$ to be constant on $[0,\beta]$). Observe that the function

$$[0,\beta] \ni s \mapsto \tilde{x}(s - d(\tilde{x}_s))$$

is continuous and does not depend on the particular extension $\tilde{x}$ of φ. Define $x : [-1,\beta] \to \mathbb{R}$ by the formula

$$x_0 = \varphi; \ \ x(t) = x(0) + \int_0^t f(\tilde{x}(s - d(\tilde{x}_s)))\, ds, \qquad s \in [0,\beta].$$

Observe that, for $t \in (0,\beta)$,

$$|x'(t)| = |f(\tilde{x}(s - d(\tilde{x}_s)))| \le 1,$$

and so $\ell(x) \le 1$.

Imagine (for example) that $x(t) > 1$ for some $t \in (0,\beta)$. By the mean value theorem there must be some $t_0 \in (0,t)$ with $x(t_0) > 1$ and $x'(t_0) > 0$. Since $\ell(x) \le 1$, however, we must have $x(s) > 0$ for all $s \in [t_0 - 1, t_0]$; in particular, $\tilde{x}(t_0 - d(\tilde{x}_{t_0})) = x(t_0 - d(\tilde{x}_{t_0})) > 0$ and so $x'(t_0) = f(\tilde{x}(t_0 - d(\tilde{x}_{t_0}))) < 0$, a contradiction. We conclude that $x(t) \le 1$ for all $t \in [0,\beta]$; the proof that $x(t) \ge -1$ for all $t \in [0,\beta]$ is similar. Thus we see that $x_t \in X$ for all $t \in [0,\beta]$, and so conclude that $d(x_t)$ is defined and, by (Hd), equal to $d(\tilde{x}_t)$ for all $t \in [0,\beta]$. It follows that x solves Equation (1.1) for $t \in (0, \beta)$. Moreover, if $y : [-1, \beta] \to \mathbb{R}$ is continuous, solves Equation (1.1) for $t \in (0, \beta)$, and satisfies $y_0 = \varphi$, from (Hd) we see that $y'(t) = x'(t)$ for all $t \in (0, \beta)$; thus x is the *unique* solution of Equation (1.1) on $[-1, \beta]$ with $x_0 = \varphi$. Continuing forward by steps establishes the existence of the solution semiflow $F : \mathbb{R}_+ \times X \to X$.

Suppose that $\varphi, \psi \in X$ have continuations x and y, respectively, as solutions of Equation (1.1). Write $\tilde{\varphi}$ and $\tilde{\psi}$ for extensions of φ and ψ, respectively, to $[-1, \infty)$ that are constant on $[0, \infty)$. Observe that for any $t \in [0, \beta]$, where β is as above, by (Hd) we have $d(x_t) = d(\tilde{\varphi}_t)$ and $d(y_t) = d(\tilde{\psi}_t)$; we also of course have $\|\tilde{\varphi}_t - \tilde{\psi}_t\| \leq \|\varphi - \psi\|$. Thus for $t \in [0, \beta]$ we have

$$|d(x_t) - d(y_t)| = |d(\tilde{\varphi}_t) - d(\tilde{\psi}_t)| \leq \ell(d)\|\varphi - \psi\|,$$

and so (using the fact that $\ell(y) \leq 1$) we have, for $t \in [0, \beta]$, that

$$\begin{aligned}
|x'(t) - y'(t)| &= |f(x(t - d(x_t))) - f(y(t - d(y_t)))| \\
&\leq \ell(f)\Big[|x(t - d(x_t)) - y(t - d(x_t))| + |y(t - d(x_t)) - y(t - d(y_t))|\Big] \\
&\leq \ell(f)\Big[\|\varphi - \psi\| + \ell(d)\|\varphi - \psi\|\Big] = \ell(f)(1 + \ell(d))\|\varphi - \psi\|.
\end{aligned}$$

Thus, for $t \in [0, \beta]$ we have

$$|x(t) - y(t)| \leq [1 + \beta\ell(f)(1 + \ell(d))]\,\|\varphi - \psi\| =: m\|\varphi - \psi\|.$$

Therefore, given any $t, \bar{t} \geq 0$, assuming $t \in [k\beta, (k+1)\beta)$ we have

$$|x(t) - y(\bar{t})| \leq |x(t) - y(t)| + |y(t) - y(\bar{t})| \leq m^{k+1}\|\varphi - \psi\| + |t - \bar{t}|.$$

The continuity of the solution semiflow follows. □

Remark 2.2. The hypotheses of Proposition 2.1 are not sufficient for (DTI) to hold. It is not hard to see that certain additional assumptions on d (for example, that $\ell(d) < 1$) are enough to guarantee (DTI); below, our verification that (DTI) holds for our particular equation of interest is somewhat more involved.

3 A particular instance of Equation (1.1)

We begin by specifying our feedback function f as follows: where $\eta > 0$,

$$f(u) = \begin{cases} 1, \; u \leq -\eta; \\ -u/\eta, \; u \in [-\eta, \eta]; \\ -1, \; u \geq \eta. \end{cases}$$

(We shall impose additional conditions on η later.) The function f is pictured in Figure 3.1. Note that f is nonincreasing, that f has negative feedback and Lipschitz constant η^{-1}, and that $|f(u)| \leq 1$ for all $u \in \mathbb{R}$ – that is, that f satisfies Hypotheses (Hf). We henceforth take f and η as described above.

We now turn to the definition of our delay functional d. We shall need the following lemma.

Lemma 3.1. *Let $0 < a < b \leq 1$ be given. There is a Lipschitz map $g : \mathbb{R} \to [a, b]$ that satisfies $\ell(g) \leq 2$ and that has the following properties:*

(i) Given any $k \in \mathbb{N}$, the discrete dynamical system $g^2 : [a, b] \to [a, b]$ has a periodic point $\tilde{q}^k$ that has minimal period k.

(ii) There is a point $\tilde{q} \in [a, b]$ such that, given any $k \in \mathbb{N}$, there is a strictly increasing sequence n_j of natural numbers such that $g^{2n_j}(\tilde{q}) \to \tilde{q}^k$ as $j \to \infty$.

Proof. First let $G : \mathbb{R} \to [0, 1]$ be any map with $\ell(G) \leq 2$ and for which there is a subinterval $S \subseteq [0, 1]$ such that

- $G^2(S) \subseteq S$;
- For each $k \in \mathbb{N}$, the subinterval S contains a periodic point of minimal period k of the restricted discrete dynamical system $G^2 : [0, 1] \to [0, 1]$;
- The restricted discrete dynamical system $G^2 : [0, 1] \to [0, 1]$ has an orbit that is dense in S.

Examples of such maps G are well-known; see Remark 3.3 below.

Now given $0 < a < b \leq 1$, we define $g : \mathbb{R} \to [a, b]$ by the formula

$$g(x) = a + (b - a)G\left(\frac{x - a}{b - a}\right).$$

Observe that $\ell(g) = \ell(G)$.

The restricted discrete dynamical system $g : [a, b] \to [a, b]$ is conjugate to the restricted discrete dynamical system $G : [0, 1] \to [0, 1]$ (via the conjugacy $H(x) = (b - a)x + a$), and so likewise $g^2 : [a, b] \to [a, b]$ is conjugate to $G^2 : [0, 1] \to [0, 1]$. Thus g^2, analogously to G^2, has a forward-invariant interval $\tilde{S} \subseteq [a, b]$ that contains periodic points of every possible minimal period and a dense orbit. Any point in this dense orbit satisfies point (ii) of the lemma. □

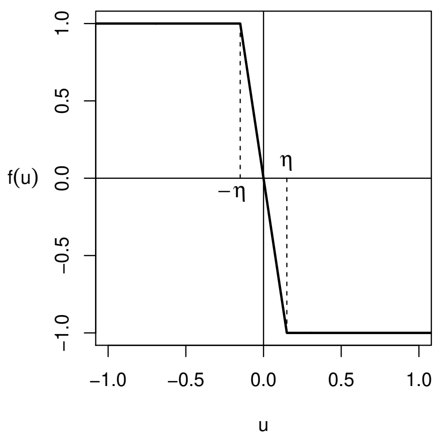

Figure 3.1: The feedback function f.

We shall henceforth write g, a and b as in Lemma 3.1.

As we proceed, we shall also need the function described in the following lemma, which is slightly different from g.

Lemma 3.2. *Let a, b, and g be as Lemma 3.1, and let $c \in \mathbb{R}$. The function $h : \mathbb{R} \to [a-c, b-c]$ given by*

$$h(u) = g(u+c) - c \quad \text{for all } u \in [a-c, b-c]$$

has the following properties:

(i) Given any $k \in \mathbb{N}$, the discrete dynamical system $h^2 : [a-c, b-c] \to [a-c, b-c]$ has a periodic point q^k that has minimal period k.

(ii) There is a point $q \in [a-c, b-c]$ such that, given any $k \in \mathbb{N}$, there is a strictly increasing sequence n_j of natural numbers such that $h^{2n_j}(q) \to q^k$ as $j \to \infty$.

Proof. The lemma follows from the fact that h is conjugate to g. □

Remark 3.3. There are many maps $G : \mathbb{R} \to [0,1]$ satisfying the conditions in the proof of Lemma 3.1. For example, the so-called *Tent Map* $T : \mathbb{R} \to [0,1]$ is given by

$$T(u) = \begin{cases} 0, u \leq 0; \\ 2u, u \in [0, 1/2]; \\ 2 - 2u, u \in [1/2, 1]; \\ 0, u \geq 1. \end{cases}$$

T^2 is well-known to have periodic points of all minimal periods and an orbit that is dense in $[0,1]$; moreover, the restricted map $T^2 : [0,1] \to [0,1]$ is chaotic in the sense of Devaney (see, for example, [1] or [3] for further discussion of this idea).

Here we introduce another class of such maps, primarily because it is these maps we use in our numerical examples below. Let $\kappa \in (0,1)$, and consider the function $G_\kappa : \mathbb{R} \to [0,1]$ given by

$$G_\kappa(u) = \begin{cases} 0, u \leq 0; \\ \frac{u}{\kappa},\ u \in [0, \kappa]; \\ 1 - (u-\kappa)(1+\kappa),\ u \in [\kappa, 1]; \\ \kappa^2,\ u \geq 1. \end{cases}$$

Observe that G_κ has slope $1/\kappa$ on $[0, \kappa]$ and slope $-(1+\kappa)$ on $[\kappa, 1]$. Therefore

$$\ell(G_\kappa) = \max\left\{\frac{1}{\kappa},\ 1+\kappa\right\}$$

and $\ell(G_\kappa) \leq 2$ as long as $\kappa \in [1/2, 1]$. (The minimum value for $\ell(G_\kappa)$ is attained at $\kappa = \frac{-1+\sqrt{5}}{2}$, where

$$\ell(G_\kappa) = \frac{1}{\kappa} = 1 + \kappa = \frac{1+\sqrt{5}}{2}.)$$

Graphs of G_κ and G_κ^2 are shown in Figures 3.2 and 3.3, respectively.

Write

$$A_1 = [\kappa^2, \kappa], \qquad A_2 = \left[\kappa, \frac{1+\kappa^2}{1+\kappa}\right], \quad \text{and} \quad A_3 = \left[\frac{1+\kappa^2}{1+\kappa}, 1\right].$$

As is clear from the figure and readily verified with direct computation,

$$G_\kappa^2(A_1) = A_1 \cup A_2 \cup A_3, \qquad G_\kappa^2(A_2) = A_1 \cup A_2 \cup A_3, \quad \text{and} \quad G_\kappa^2(A_3) = A_2 \cup A_3.$$

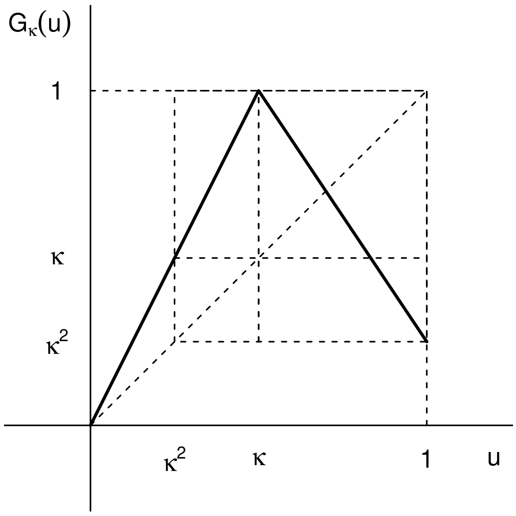

Figure 3.2: The function G_κ.

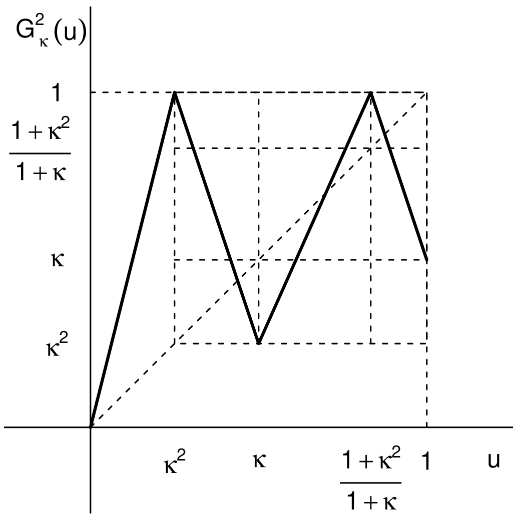

Figure 3.3: The function G_κ^2.

Using familiar ideas (see, for example, [2]) one can now show that the restricted dynamical system $G_\kappa^2 : A_1 \cup A_2 \cup A_3 \to A_1 \cup A_2 \cup A_3$ is semiconjugate to a chaotic subshift of finite type on three symbols, has periodic points of all minimal periods, and has a dense orbit (and is chaotic in the sense of Devaney).

We now define our delay functional d on X in several steps, showing that conditions (Hd) and (DTI) hold.

Choose $\alpha \in (0,1)$. First, given $\varphi \in X$, we write

$$\mu(\varphi) = \max_{s \in [-1,-\alpha]} |\varphi(s)|.$$

Lemma 3.4. *The functional $\mu : X \to \mathbb{R}$ satisfies $\ell(\mu) \leq 1$.*

Proof. Let $\varphi, \psi \in X$ be given. Suppose that

$$\max_{s\in[-1,-\alpha]} |\varphi(s)|$$

is attained at the point t_1 and that

$$\max_{s\in[-1,-\alpha]} |\psi(s)|$$

is attained at the point t_2. Then

$$\mu(\psi) \geq |\psi(t_1)| \geq |\varphi(t_1)| - \|\psi - \varphi\| = \mu(\varphi) - \|\psi - \varphi\|$$

and

$$\mu(\varphi) \geq |\varphi(t_2)| \geq |\psi(t_2)| - \|\psi - \varphi\| = \mu(\psi) - \|\psi - \varphi\|,$$

which yields

$$\|\psi - \varphi\| \geq \mu(\varphi) - \mu(\psi) \quad \text{and} \quad \|\psi - \varphi\| \geq \mu(\psi) - \mu(\varphi)$$

and so

$$|\mu(\psi) - \mu(\varphi)| \leq \|\psi - \varphi\|$$

– that is, $\ell(\mu) \leq 1$. ☐

With α and μ as above, we now define the functional $\nu : X \to \mathbb{R}$ by

$$\nu(\varphi) = \min\Big\{\mu(\varphi) - |\varphi(-1)|, \mu(\varphi) - |\varphi(-\alpha)|\Big\}.$$

Remark 3.5. Observe that $\nu(\varphi) = 0$ precisely when the maximum of $|\varphi(s)|_{s\in[-1,-\alpha]}$ is attained at either -1 or at $-\alpha$. One consequence of this observation is the following. If $x : [-1, \infty) \to \mathbb{R}$ is a function with $x_t \in X$ for all $t \geq 0$ and $\nu(x_t) \neq 0$ for all $t \in (t_0, t_1) \subseteq [0, \infty)$, then

$$\max\{|x_t(s)| : s \in [-1, -\alpha]\}$$

is never attained at $s = -1$ or at $s = -\alpha$ as t runs over (t_0, t_1). It follows that $t \mapsto \mu(x_t)$ is constant on (t_0, t_1) and therefore, by continuity, constant on $[t_0, t_1]$. Otherwise put, *if (t_0, t_1) is any interval where the function $t \mapsto \nu(x_t)$ is nonzero, the map $t \mapsto \mu(x_t)$ is constant on $[t_0, t_1]$.*

The following lemma is elementary and we omit the proof.

Lemma 3.6.

(1) Suppose that Y is a metric space and that $h_1 : Y \to \mathbb{R}_+$ and $h_2 : Y \to \mathbb{R}_+$ are two Lipschitz functions. Then the function defined by

$$h(y) = \min(h_1(y), h_2(y))$$

is Lipschitz with $\ell(h) \leq \max(\ell(h_1), \ell(h_2))$.

(2) Suppose that W, Y, and Z are metric spaces and that $h_1 : W \to Y$ and $h_2 : Y \to Z$ are Lipschitz functions. Then $h_2 \circ h_1$ is Lipschitz with

$$\ell(h_2 \circ h_1) \leq \ell(h_2) \times \ell(h_1).$$

(3) *Suppose that Y is a compact metric space and that $h_1 : Y \to \mathbb{R}_+$ and $h_2 : Y \to \mathbb{R}_+$ are two Lipschitz functions. Then the function $h : Y \to \mathbb{R}_+$ defined by*

$$h(y) = h_1(y) \times h_2(y)$$

is Lipschitz with

$$\ell(h) \leq \max_Y |h_1(y)|\ell(h_2) + \max_Y |h_2(y)|\ell(h_1).$$

Lemma 3.7. $\ell(\nu) \leq 2$.

Proof. Consider the map defined by

$$\nu_*(\varphi) = \mu(\varphi) - |\varphi(-1)|.$$

Given $\varphi, \psi \in X$, by Lemma 3.4 and the triangle inequality we have

$$\begin{aligned}|\nu_*(\varphi) - \nu_*(\psi)| &= \Big|\mu(\varphi) - \mu(\psi) + |\psi(-1)| - |\varphi(-1)|\Big| \leq \Big|\mu(\varphi) - \mu(\psi)\Big| + \Big||\psi(-1)| - |\varphi(-1)|\Big| \\ &\leq 2\|\varphi - \psi\|.\end{aligned}$$

Thus $\ell(\nu_*) \leq 2$. A similar argument shows that the map

$$\nu^*(\varphi) = \mu(\varphi) - |\varphi(-\alpha)|$$

also has Lipschitz constant no more than 2. The lemma now follows from the first part of Lemma 3.6. □

Given $\gamma_0 \in (0, 1)$ and $0 < c_1 < c_2 < \infty$, we now define $\gamma : (0, \infty) \to (0, \infty)$ as follows:

$$\gamma(u) = \begin{cases} \gamma_0, \ u \in [0, c_1]; \\ \gamma_0 + \frac{1-\gamma_0}{c_2-c_1}(u - c_1), \ u \in [c_1, c_2]; \\ 1, \ u \geq c_2. \end{cases}$$

That is, γ is constant on $[0, c_1]$; linear on $[c_1, c_2]$; and equal to 1 on $[c_2, \infty)$. Observe that $\ell(\gamma) = \frac{1-\gamma_0}{c_2-c_1}$.

Now let η be as in the definition of f at the beginning of Section 3. Let g be as in the statement of Lemma 3.1. Remember in particular that g is Lipschitz with $\ell(g) \leq 2$ and that g maps $\mathbb{R}$ into $[a, b] \subseteq (0, 1]$ (we shall specify the constants a and b later). We now define our delay functional $d : X \to \mathbb{R}$ as follows:

$$d(\varphi) = \gamma(\nu(\varphi))g(\mu(\varphi) + \eta/2). \tag{D}$$

We shall take d as defined in (D) henceforth.

Lemma 3.8. *The functional $d : X \to \mathbb{R}_+$ defined in (D) is Lipschitz.*

Proof. The lemma follows from the fact that μ, ν, γ and g are Lipschitz, and from Lemma 3.6. □

Observe that this delay functional d satisfies the hypotheses (Hd): $d(\varphi)$ is bounded between $\gamma_0 a > 0$ and $b \leq 1$ for all φ, and depends only on the restriction of φ to $[-1, -\alpha]$.

Let us now consider whether (DTI) holds for solutions of Equation (1.1) with segments in X, when the delay functional d is as defined in (D). Suppose that $x : [-1, \infty) \to \mathbb{R}$ is a function with $x_t \in X$ for all $t \geq 0$. We now show that, under appropriate assumptions on the parameters a, b, γ_0, c_1 and c_2,

$$t + \epsilon - d(x_{t+\epsilon}) > t - d(x_t)$$

for all $t \geq 0$ and $\epsilon > 0$. Since f and d satisfy the conditions given in Proposition 2.1, it follows that (DTI) holds for solutions of Equation (1.1) with segments in X.

Lemma 3.9. *Suppose that $x : [-1, \infty) \to \mathbb{R}$ satisfies $x_t \in X$ for all $t \geq 0$. For any $t \geq 0$ and $\epsilon > 0$,*

$$|\nu(x_{t+\epsilon}) - \nu(x_t)| \leq \epsilon.$$

(Thus, while the functional ν does not satisfy $\ell(\nu) \leq 1$, the function

$$s \mapsto \nu(x_s)$$

does have Lipschitz constant no greater than 1.)

Proof. The function $s \mapsto \nu(x_s)$ is continuous; thus the set

$$S = \{s \in [t, t+\epsilon] \ : \ \nu(x_s) = 0\ \}$$

is closed.

Case 1: if S is empty then, by our observation in Remark 3.5, $\mu(x_s)$ is equal to some constant ξ for $s \in [t, t+\epsilon]$. Thus, for s in this range, $\nu(x_s)$ can be written

$$\nu(x_s) = \min\Big\{\xi - |x(s-1)|, \xi - |x(s-\alpha)|\Big\},$$

and in this case the map $s \mapsto \nu(x_s)$ is clearly Lipschitz on $[t, t+\epsilon]$ with Lipschitz constant no greater than 1 (recall part 1 of Lemma 3.6).

Case 2: if S is nonempty, write

$$t + \epsilon_1 = \sup\{s \in [t, t+\epsilon] \ : \ s \in S\}.$$

$\nu(x_{t+\epsilon_1}) = 0$ since S is closed. If $t + \epsilon_1 = t + \epsilon$, then obviously $\nu(x_{t+\epsilon_1}) = \nu(x_{t+\epsilon})$; if $t + \epsilon_1 < t + \epsilon$, then $\nu(x_s)$ is nonzero for $s \in (t+\epsilon_1, t+\epsilon)$ and an argument like that in Case 1 shows that the map $s \mapsto \nu(x_s)$ has Lipschitz constant no more than 1 on $[t+\epsilon_1, t+\epsilon]$. Whether $t + \epsilon_1 = t + \epsilon$ or not, then, we have

$$|\nu(x_{t+\epsilon}) - \nu(x_{t+\epsilon_1})| \leq \epsilon - \epsilon_1.$$

But since $\nu(x_{t+\epsilon_1}) = 0$ and ν is nonnegative, this last estimate tells us that

$$\nu(x_{t+\epsilon}) \in [0, \epsilon - \epsilon_1].$$

Now, if $\nu(x_t) = 0$, we're done. Otherwise, set

$$t + \epsilon_0 = \inf\{s \in [t, t+\epsilon] \ : \ s \in S\}.$$

Then since $s \mapsto \nu(x_s)$ is nonzero on $(t, t+\epsilon_0)$, another argument like that in Case 1 shows that $s \mapsto \nu(x_s)$ has Lipschitz constant no more than 1 on $[t, t+\epsilon_0]$ and so (since $\nu(x_{t+\epsilon_0}) = 0$) that $\nu(x_t) \in [0, \epsilon_0]$. Thus (recalling that $\epsilon_0 \leq \epsilon_1$) we obtain

$$|\nu(x_{t+\epsilon}) - \nu(x_t))| \leq \epsilon_0 + \epsilon - \epsilon_1 \leq \epsilon,$$

as desired. □

With x, t and ϵ as in the statement of Lemma 3.9 just above, let us now consider the quantity $d(x_{t+\epsilon}) - d(x_t)$. Since $s \mapsto \nu(x_s)$ has a Lipschitz constant no more than one on $[t, t+\epsilon]$, we can find a finite set of points

$$t = s_0 < s_1 < \cdots < s_m = t + \epsilon$$

such that, given $j \in \{1, \ldots, m\}$, $\nu(x_s)$ is either in the interval $[0, c_1]$ for all $s \in [s_{j-1}, s_j]$ or in the interval $[c_1/2, \infty)$ for all $s \in [s_{j-1}, s_j]$. In the former case, $\gamma(\nu(x_s))$ is equal to $\gamma(0) = \gamma_0$ on the subinterval $s \in [s_{j-1}, s_j]$ and so the Lipschitz constant of $s \mapsto d(x_s)$ on this subinterval is no more than

$$\gamma_0 \times \ell(g) \times \ell(\mu) \times \ell(s \mapsto x_s),$$

which by the definition of X and Lemma 3.4 is no more than

$$\gamma_0 \ell(g).$$

In the latter case, $\mu(x_s)$ is constant on the subinterval $s \in [s_{j-1}, s_j]$ and so the Lipschitz constant of $s \mapsto d(x_s)$ on the subinterval is no more than $\ell(\gamma) \times \ell(s \mapsto \nu(x_s)) \times b = \ell(\gamma) \times b$.

These estimates establish the following proposition.

Proposition 3.10. *Assume that d is such that*

$$\ell(\gamma) b = \frac{1 - \gamma_0}{c_2 - c_1} b < 1 \quad \textit{and} \quad \gamma_0 \ell(g) < 1.$$

Then, if $x : [-1, \infty) \to \mathbb{R}$ is any function with $x_t \in X$ for all $t \geq 0$, we have

$$t + \epsilon - d(x_{t+\epsilon}) > t - d(x_t)$$

for any $t \geq 0$ and any $\epsilon > 0$. In particular, since d and f satisfy (Hd) and (Hf), respectively, (DTI) holds along all solutions of Equation (1.1) with segments in X.

We shall henceforth consider Equation (1.1) with d and f as defined in this section.

4 The set M and the map R

The construction that follows is motivated by the following observation. The constant-delay equation

$$y'(t) = -\text{sign}(y(t - 1/3))$$

has a periodic solution w whose zeros are separated by $2/3$ and that satisfies $w(0) = 0$ and $w'(0) = 1$. This solution is pictured in Figure 4.1.

The solutions of Equation (1.1) that we will construct are "close" to the solution w.

We define the functional $\lambda : X \to \mathbb{R}$ as follows:

$$\lambda(\varphi) = \max\{|\varphi(s)| \ : s \in [-2/3, 0]\}.$$

The following Lemma is proven very much like Lemma 3.4; accordingly, we omit the proof.

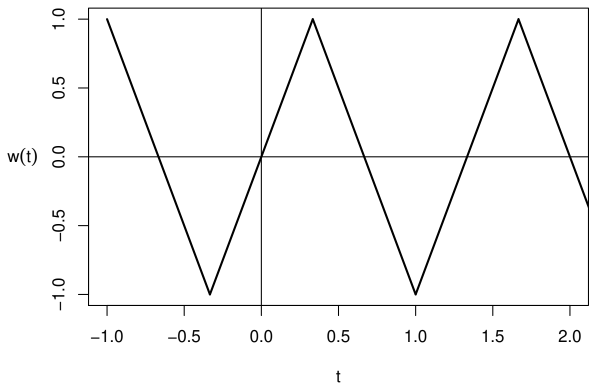

Figure 4.1: The solution w.

Lemma 4.1. $\ell(\lambda) \leq 1$.

For the reader's convenience we recall the following notation and parameters.

- $\eta > 0$ – the feedback function f is linear on $[-\eta, \eta]$, and $f(u) = -\text{sign}(u)$ for $u \notin [-\eta, \eta]$.
- $\ell(g)$ – the Lipschitz constant of g.
- $0 < a < b \leq 1$ – the function g maps $\mathbb{R}$ to $[a, b]$.
- $\alpha \in (0, 1)$ – $\mu(\varphi)$ and $\nu(\varphi)$ (and hence $d(\varphi)$) are determined by the restriction of φ to $[-1, -\alpha]$.
- $0 < c_1 < c_2$, and $\gamma_0 \in (0, 1)$ – the function γ is equal to γ_0 on $[0, c_1]$, is linear on $[c_1, c_2]$, and is equal to 1 on $[c_2, \infty)$.

Recall also our formula for d:

$$d(\varphi) = \gamma(\nu(\varphi))g(\mu(\varphi) + \eta/2).$$

We shall henceforth set and fix $\alpha = 1/3$. We impose the following additional requirements on our parameters:

(i) $\gamma_0 \ell(g) < 1$.

(ii) $b(1 - \gamma_0)/(c_2 - c_1) < 1$.

(iii) $\eta < \gamma_0 a$.

(iv) $6\eta < 1/3$.

(v) $1/3 - \eta \leq a < b \leq 1/3 + \eta$.

(vi) $1/3 - 6\eta > c_2$.

Conditions (i) and (ii) are just the conditions given in Proposition 3.10 for (DTI) to hold. We shall need the other conditions below. The following particular choice of values shows that conditions (i)–(vi) can be satisfied.

$$\begin{cases} \eta = 1/100; \\ \ell(g) = 2; \\ a = 1/3 - \eta \text{ and } b = 1/3 + \eta; \\ \gamma_0 = 9/20; \\ c_2 = 1/4; \\ c_1 = 1/100. \end{cases} \tag{C}$$

We now define the following subset $M \subseteq X$.

$$M = \left\{ \varphi \in X \ : \begin{cases} (I)\ \varphi(0) = 0; \\ (II)\ \varphi'(s) = 1 \text{ for all } s \in (-\eta, 0); \\ (III)\ |\varphi(-2/3)| \leq 2\eta; \\ (IV)\ \varphi(s) \leq -\eta \text{ for all } s \in [-2/3 + 3\eta, -\eta]; \\ (V)\ \lambda(\varphi) \geq 1/3 - 2\eta \end{cases} \right\}.$$

A typical element φ of M is pictured in Figure 4.2. The size of η is exaggerated in the figure.

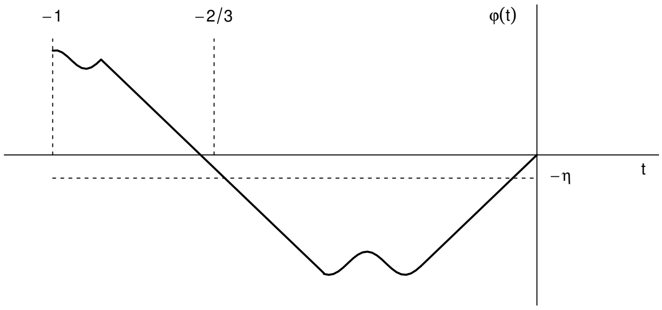

Figure 4.2: An element φ of M.

There are two piecewise linear elements ψ^1 and ψ^2 of M with the following properties. ψ^1 has

- slope 1 on $[-1/3 + 2\eta, 0]$;
- slope 0 on $[-1/3 - 2\eta, -1/3 + 2\eta]$;
- slope -1 on $[-1, -1/3 - 2\eta]$.

ψ^2 has

- slope 1 on $[-1/3 - \eta, 0]$;
- slope -1 on $[-1, -1/3 - \eta]$.

These two points of M make it easy to see the following lemma, whose proof we omit.

Lemma 4.2. $[1/3 - 2\eta, 1/3 + \eta] \subseteq \lambda(M)$.

Assume that $\varphi = x_0 \in M$ has continuation $x : [-1, \infty) \to \mathbb{R}$ as a solution of Equation (1.1). We now study x.

Proposition 4.3. *Assume that points (i)–(vi) above hold, and that $\varphi = x_0 \in M$ has continuation x as a solution of Equation (1.1). Then x has a first positive zero z, and the following hold:*

(a) $z = 2g(\lambda(\varphi) + \eta/2)$;

(b) $-x_z \in M$, *with* $\lambda(-x_z) = g(\lambda(\varphi) + \eta/2) - \eta/2$;

(c) if $y_0 \in M$ has continuation y as a solution of Equation (1.1) and $\lambda(y_0) = \lambda(x_0)$, then $y|_{[0,z]} = x|_{[0,z]}$.

Observe that point (b) says that $\lambda(\varphi)$ (and the fact that $\varphi \in M$) determines $\lambda(-x_z)$. Moreover, point (c) says that $\lambda(\varphi)$ (and the fact that $\varphi \in M$) completely determines the restriction of x to $[0, z]$. Proceeding inductively, we see that $\lambda(\varphi)$ determines all of $x|_{[0,\infty)}$. Moreover, we shall see that the sequence of (absolute values) of local extrema of x is an orbit of a discrete dynamical system determined by a function h as in Lemma 3.2. The properties of solutions of Equation (1.1) described in Theorem 1.1 follow from corresponding properties of h^2; the rest of the paper amounts to a detailed explanation of this idea.

We now prove Proposition 4.3.

Proof. Observe first that, by points (iii), (iv), and (v),

$$d(x_0) \in [\gamma_0 a, b] \subseteq (\eta, 1/3 + \eta] \subseteq (\eta, 2/3 - 3\eta),$$

and so in particular $-d(x_0) \in (-2/3 + 3\eta, -\eta)$ and (by (IV)) $x(-d(x_0)) \leq -\eta$.

Now let us consider the interval $[1/3 - 2\eta, 1/3 + 2\eta]$. As t runs over this interval, $t - 1$ runs over the interval $[-2/3 - 2\eta, -2/3 + 2\eta]$. Since $|x(-2/3)| \leq 2\eta$ (by (III)) and x has Lipschitz constant 1, we see that $|x(t-1)| \leq 4\eta$ as t runs over $[1/3 - 2\eta, 1/3 + 2\eta]$. Similarly, $|x(t - 1/3)| \leq 2\eta$ as t runs over $[1/3 - 2\eta, 1/3 + 2\eta]$. We can draw three conclusions. First, since $\lambda(\varphi) \geq 1/3 - 2\eta > 4\eta$ (by (iv) and (V)), $\max\{|\varphi(s)| : s \in [-2/3, 0]\}$ is actually attained at some $s \in [-2/3 + 2\eta, -2\eta]$. Second and similarly, $\mu(x_t)$ is constant (and equal to $\lambda(\varphi)$) across all $t \in [1/3 - 2\eta, 1/3 + 2\eta]$. Finally, as t runs across $[1/3 - 2\eta, 1/3 + 2\eta]$, since

$$\mu(x_t) - |x(t-1)| \geq 1/3 - 2\eta - 4\eta = 1/3 - 6\eta > c_2$$

and

$$\mu(x_t) - |x(t - 1/3)| \geq 1/3 - 2\eta - 2\eta = 1/3 - 4\eta > c_2$$

(by (vi)) we have that $\gamma(\nu(x_t)) = 1$ for all such t. We conclude that *as t runs over the interval* $[1/3 - 2\eta, 1/3 + 2\eta]$, *$d(x_t)$ is constant* and is equal to

$$g(\lambda(\varphi) + \eta/2) =: d_*.$$

See Figure 4.3. The solution x is in black, and the value of $d(x_t)$ is in blue.

Now, since $[a, b] \subseteq [1/3 - \eta, 1/3 + \eta]$, $d_* \in [1/3 - \eta, 1/3 + \eta]$. As t traverses the interval $[1/3 - 2\eta, 1/3 + 2\eta]$, $t - d_*$ traverses an interval that contains the interval $[-\eta, \eta]$; more particularly, $t - d_*$ will traverse the interval $[-\eta, \eta]$ precisely as t traverses the interval $[d_* - \eta, d_* + \eta] \subseteq [1/3 - 2\eta, 1/3 + 2\eta]$. Since $-d(x_0) \geq -2/3 + 3\eta$, $d_* - \eta - d(x_{d_* - \eta}) = -\eta$, and (DTI) holds, we actually can now conclude that $t - d(x_t) \in [-2/3 + 3\eta, -\eta]$ for all $t \in [0, d_* - \eta]$, and hence (by (IV)) that $x'(t) = 1$ for all $t \in [0, d_* - \eta]$. Thus, in particular, $x(d_* - \eta) = d_* - \eta \geq 1/3 - 2\eta$ and $x(-\eta + s) = -\eta + s$ for all $s \in [0, 2\eta]$.

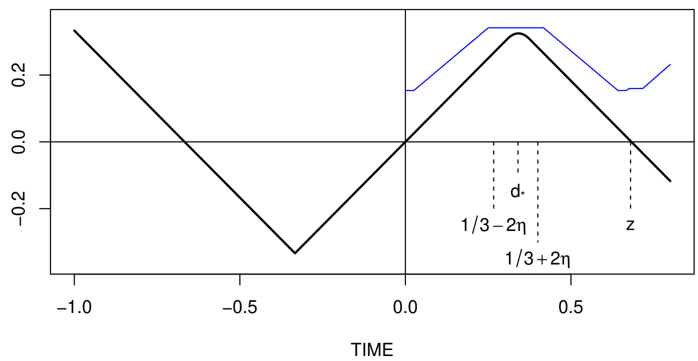

Figure 4.3: The solution x and the function $t \mapsto d(x_t)$.

On $[d_* - \eta, d_* + \eta]$, then, x solves the ODE

$$x(d_* - \eta) = d_* - \eta, \quad x'(d_* - \eta + s) = f(-\eta + s) = 1 - s/\eta \text{ for all } s \in [0, 2\eta].$$

Direct computation shows that, for $s \in [0, 2\eta]$, we have

$$x(d_* - \eta + s) = d_* - \eta + s - \frac{s^2}{2\eta}$$

and so in particular

$$x(d_*) = d_* - \eta/2$$

– this is the maximum value of x on $[d_* - \eta, d_* + \eta]$. x will traverse a symmetric arc as t runs over $[d_* - \eta, d_* + \eta]$; we have $x(d_* + \eta) = d_* - \eta$ and $x'(d_* + \eta) = -1$.

Since (DTI) holds, we will have $x'(t) = -1$ at least from time $t = d_* + \eta$ until such time as $t - d(x_t) > \eta$ and $x(t - d(x_t)) = \eta$. Before this can occur, since η is less than the smallest possible delay (by (iii)), x will attain a first zero $z = 2d_*$. This completes the proof of part (a) of the proposition. Part (c) is also clear: $\lambda(\varphi)$ (and the fact that $\varphi \in M$) completely determines the restriction of x to $[0, z]$.

It remains to check part b). We certainly have $x_z(0) = 0$, and, since $z = 2d_* \geq 2/3 - 2\eta > 1/3 + 3\eta \geq d_* + 2\eta$, we certainly have that $x_z'(s) = -1$ for all $s \in [-\eta, 0]$. $z - 2/3$ is in the interval $[-2\eta, 2\eta]$. Since $x(0) = 0$ and x has Lipschitz constant 1, we certainly have that $|x_z(-2/3)| \leq 2\eta$, as desired. Similarly, $z - 2/3 + 3\eta \geq \eta$, so $x_z(s) \geq \eta$ for all $s \in [-2/3 + 3\eta, -\eta]$. Finally, since $x(d_*) = d_* - \eta/2 > 2\eta$, the maximum of $|x(t)|$ as t runs over $[z - 2/3, z]$ is clearly attained at d_*, and $x(d_*) \geq 1/3 - \eta - \eta/2 \geq 1/3 - 2\eta$. Thus $-x_z \in M$, as desired; and it is clear that $\lambda(-x_z) = d_* - \eta/2 = g(\lambda(\varphi) + \eta/2) - \eta/2$. □

Since d is even and f is odd, similar considerations let us see that x will have an infinite sequence of simple positive zeros $z = z_1 < z_2 < z_3 < \cdots$ with

$$(-1)^n x_{z_n} \in M \quad \text{for all } n \in \mathbb{N}.$$

We have the following formulas:

$$\begin{aligned}
\lambda(-x_{z_1}) &= g(\lambda(x_0)+\eta/2)-\eta/2;\\
\lambda(x_{z_2}) &= g(\lambda(-x_{z_1})+\eta/2)-\eta/2 = g^2(\lambda(x_0)+\eta/2)-\eta/2;\\
\lambda(-x_{z_3}) &= g(\lambda(x_{z_2})+\eta/2)-\eta/2 = g^3(\lambda(x_0)+\eta/2)-\eta/2;\\
&\vdots\\
\lambda((-1)^n x_{z_n}) &= g^n(\lambda(x_0)+\eta/2)-\eta/2;\\
&\vdots
\end{aligned}$$

Let us define the map $R : M \to M$ by the formula $R(x_0) = x_{z_2}$. Since $z_2 > 4/3 - 4\eta > 1$ for any $x_0 \in M$, we observe that

$(**)$ $\lambda(x_0)$ completely determines $R(x_0)$.

We also have the following lemma.

Lemma 4.4. *$R : M \to M$ is Lipschitz continuous.*

Proof. Our work so far shows that the map $Z : x_0 \mapsto z_2$ is bounded by $4/3 + 4\eta$, and is Lipschitz because λ and g are Lipschitz. Thus the kind of estimate in the proof of Proposition 2.1 shows that, for some $\tilde{m} > 0$,

$$\begin{aligned}
\|R(x_0) - R(y_0)\| &\le \|x_{Z(x_0)} - y_{Z(y_0)}\| \le \|x_{Z(x_0)} - y_{Z(x_0)}\| + \|y_{Z(x_0)} - y_{Z(y_0)}\|\\
&\le \tilde{m}\|x_0 - y_0\| + \ell(Z)\|x_0 - y_0\| =: B\|x_0 - y_0\|.
\end{aligned}$$ □

Recalling Lemma 3.2, let us define the function

$$h : \mathbb{R} \to [a-\eta/2, b-\eta/2] \;\text{ by }\; h(u) = g(u+\eta/2) - \eta/2.$$

We know that, given any $k \in \mathbb{N}$, h^2 has a periodic point q^k in $[a-\eta/2, b-\eta/2]$ with minimal period k. We also know that there is a point $q \in [a-\eta/2, b-\eta/2]$ with the feature that, for any k, there is a sequence $n_j \to \infty$ with $h^{2n_j}(q) \to q^k$ as $j \to \infty$.

Note that, by Lemma 4.2, the entire interval $[a-\eta/2, b-\eta/2]$ is contained in the image of λ.

The work we have done so far shows that we have the following semiconjugacy:

$$\lambda \circ R = h^2 \circ \lambda;$$

similarly,

$$\lambda \circ R^k = h^{2k} \circ \lambda \quad \text{for all } k \in \mathbb{N}.$$

Our approach to proving Theorem 1.1 now follows standard lines, and hinges on the following observation. If x_0 is a periodic point of R with minimal period m, then x is a nontrivial periodic solution of Equation (1.1) with minimal period z_{2m}, and $\lambda(x_0)$ is a periodic point of h^2 with minimal period m (minimal because $\lambda(y_0)$ determines the continuation of y_0 as a solution of Equation (1.1) for all $y_0 \in M$). Similarly, if x_0 and p_0 are points of M with $R^{n_j}(x_0) \to p_0$ for some sequence $n_j \to \infty$, then there is a sequence $t_j \to \infty$ such that $x_{t_j} \to p_0$. To finish the proof of Theorem 1.1, then, we need to show that R has periodic points of all minimal periods, as well as an orbit that comes arbitrarily close to all these periodic points.

We require one more lemma.

Lemma 4.5. *For any $x_0, y_0 \in M$, $\|R(x_0) - R(y_0)\| \leq \ell(R)|\lambda(x_0) - \lambda(y_0)|$.*

Proof. Suppose that $\lambda(y_0) \leq \lambda(x_0)$. Then consider the following point w_0 of M:

$$w_0(s) = \begin{cases} x_0(s), \ s \in [-1, -2/3]; \\ \max\{x_0(s), -\lambda(y_0)\}, \ s \in [-2/3, 0]. \end{cases}$$

Observe that

- w_0 does in fact lie in M; and
- $\lambda(w_0) = \lambda(y_0)$; and
- $\|x_0 - w_0\| \leq |\lambda(x_0) - \lambda(y_0)|$.

Since R is Lipschitz and by $(**)$, then, we have

$$\|R(x_0) - R(y_0)\| = \|R(x_0) - R(w_0)\| \leq \ell(R)\|x_0 - w_0\| \leq \ell(R)|\lambda(x_0) - \lambda(y_0)|. \qquad \square$$

We are now ready to prove our main theorem (Theorem 1.1).

Proof. Points (i)–(iii) have already been proven. We now prove (iv) and (v).

Recall that, given any $k \in \mathbb{N}$, h^2 has a periodic point q^k in $[a - \eta/2, b - \eta/2]$ with minimal period k; and that there is a point $q \in [a - \eta/2, b - \eta/2]$ with the feature that, for any k, there is a sequence $n_j \to \infty$ with $h^{2n_j}(q) \to q^k$ as $j \to \infty$.

Given $k \in \mathbb{N}$, there is a point $x_0 \in M$ with $\lambda(x_0) = q^k$ by Lemma 4.2. We have

$$\lambda(R^k(x_0)) = h^{2k}(q^k) = q^k.$$

By $(**)$, it follows that $R^k(x_0) = p_0^k$ is periodic point of R of minimal period k. By the discussion right before the statement of Lemma 4.5, this establishes point (iv) of Theorem 1.1.

Let $q \in [a - \eta/2, b - \eta/2]$ be as above, and pick $v_0 \in M$ with $\lambda(v_0) = q$. Choose $k \in \mathbb{N}$. Let p_0^k and q^k be as above; recall that $q^k = \lambda(p_0^k)$. Given any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|h^{2N}(q) - q^k| < \epsilon/\ell(R)^k$.

By Lemma 4.5,

$$\begin{aligned} \|R^{N+k}(v_0) - p_0^k\| &= \|R^k(R^N(v_0)) - R^k(p_0^k)\| \\ &\leq \ell(R)^k|\lambda(R^N(v_0)) - \lambda(p_0^k)| = \ell(R)^k|h^{2N}(\lambda(v_0)) - \lambda(p_0^k)| \\ &= \ell(R)^k|h^{2N}(q) - q^k| < \ell(R)^k \frac{\epsilon}{\ell(R)^k} = \epsilon. \end{aligned}$$

Again by the discussion right before the statement of Lemma 4.5, this establishes point (v) of Theorem 1.1. $\square$

We close with a numerical example. Figure 4.4 shows an approximate solution of Equation (1.1) with an initial condition in M, where the parameters are as in (C) above. The delay functional is derived from the function $G_{1/2}$ introduced in Remark 3.3. The solution x is the black line; the blue line shows the value of the delay $d(x_t)$.

The "irregular" behavior of the solution is not really visible at this scale. Figure 4.5 shows the (numerically approximated) absolute values of x at the first several positive critical points of x, where x is the solution graphed in Figure 4.4.

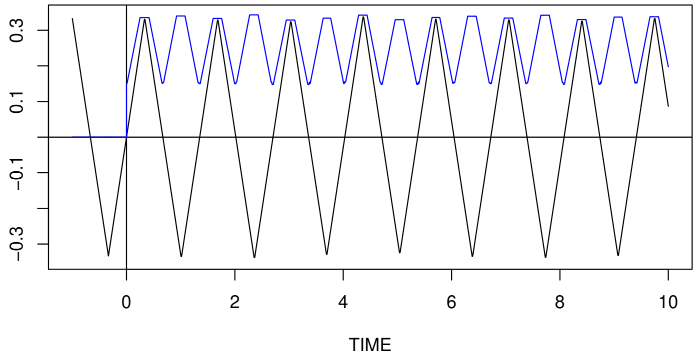

Figure 4.4: Numerical approximation of a solution x and the function $t \mapsto d(x_t)$.

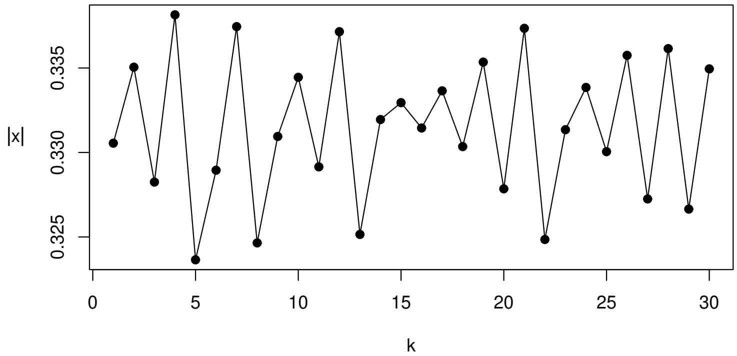

Figure 4.5: Numerically approximated value of $|x|$ at the kth positive critical point of x, $1 \leq k \leq 30$.

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