ON STABILITY PROPERTIES OF SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS*

L. Hatvani**
Bolyai Institute, Szeged University
Aradi vértanúk tere 1, H-6720 Szeged, Hungary
e-mail: hatvani@math.u-szeged.hu

Dedicated to Árpád Elbert on his 60th birthday

Consider the linear nonautonomous second order differential equation

$$x'' + a(t)x = 0, \tag{L}$$

where $a : \mathbb{R}_+ \to \mathbb{R}_+$ is a positive nondecreasing function. This equation describes the motion of a material point of unit mass under the action of restoring force with changing elasticity coefficient. It is known [11] that every solution x of (L) is oscillatory, the maxima of $|x|$ do not increase and the maxima of $|x'|$ do not decrease as t goes to infinity. The following question arises: Can maxima of $|x|$ tend to zero as t goes to infinity, provided that the elasticity coefficient a tends to infinity?

This is a classical problem of the stability theory of nonautonomous differential systems [11, Ch. XIV]. In 1934, H. Milloux [23] proved that there always exists a solution $x_0 \not\equiv 0$ of (L) with the property

$$\lim_{t\to\infty} x_0(t) = 0, \tag{1}$$

provided that a is continuously differentiable. The celebrated Armellini-Tonelli-Sansone theorem says that if function a "grows regularly" to infinity, then all solutions of (L) have property (1). The "regular growth" means that the function does not increase intermittently; i.e., the increase of the function cannot be located to a sequence of intervals whose density on $\mathbb{R}_+$ is small in some sense. The theorem has been sharpened and extended to various types of equations [4–9, 12–13, 15, 20, 22, 24].

Let us consider the simplest case of "irregular" growth when a is a step function:

$$x'' + a_k x = 0 \quad (t_k \le t < t_{k+1}), \qquad k = 0, 1, 2, \dots, \tag{2}$$

---

* This paper is in final form and no version of it will be submitted for publication elsewhere.

** The author was supported by the Ministry of Education (FKFP 0420/1999) and Hungarian National Foundation for Scientific Research (OTKA T/029188).

where

$$t_0 = 0, \quad \lim_{k\to\infty} t_k = \infty,$$

$$0 < a_0 < a_1 < \ldots < a_k < a_{k+1} < \ldots, \quad \lim_{k\to\infty} a_k = \infty.$$

By a *solution* of (2) we mean a function continuously differentiable on $\mathbb{R}_+$, twice continuously differentiable on the set $I := \cup_{k=0}^{\infty}(t_k, t_{k+1})$, and satisfying equation (2) on set I. Extending Milloux's theorem to equation (2), we proved

THEOREM A [14]. *There exists at least one solution $x_0 \not\equiv 0$ of equation (2) satisfying (1).*

Now let us deal with the problem of finding sufficient conditions on sequence $\{t_k, a_k\}_{k=0}^{\infty}$ guaranteeing that *all* solutions of (2) satisfy (1). In fact, equation (2) is integrable: one can integrate the equation on every interval $[t_k, t_{k+1})$, then one "glues" the pieces together properly to get a solution on $\mathbb{R}_+$. Using this method, Á. Elbert proved the following

THEOREM B [9]. *If*

$$\sum_{k=1}^{\infty} \min\left\{1 - \frac{\sqrt{a_k}}{\sqrt{a_{k+1}}};\ 1 - \frac{\sqrt{a_{k+1}}}{\sqrt{a_{k+2}}}\right\} \sin^2\left(\sqrt{a_{k+1}}(t_{k+1} - t_k)\right) = \infty,$$

then all solutions of equation (2) satisfy (1).

This is a very nice theoretical result. However, it is not easy to check the condition in practice. Roughly speaking, the condition says that the time difference $t_{k+1} - t_k$ between two consecutive jumps should not be near to $m\pi/\sqrt{a_{k+1}}$, $m \in \mathbb{N}$. One can think that this condition "typically" is satisfied. To make this claim more precise, we made the following conjecture [14]: If $\{a_k\}_{k=1}^{\infty}$ is given and $\{t_k\}$ is chosen "at random", then it is almost sure that all solutions of (1) satisfy (2). Recently it has turned out that the conjecture is true for an important class of random sequence $\{t_k\}$:

THEOREM C [15]. *Suppose $\{a_k\}_{k=0}^{\infty}$ is given, and $\{t_k\}_{k=0}^{\infty}$ is random such that for every $k = 0, 1, 2, \ldots$ the difference $t_{k+1} - t_k$ is a random variable uniformly distributed on interval $[0, 1]$.*

Then all solutions of (2) satisfy (1) with probability 1.

Consider now the so-called half-linear second order differential equation

$$\text{(HL)} \qquad x''|x'|^{n-1} + a(t)|x|^{n-1}x = 0, \qquad (0 < n \in \mathbb{R}),$$

introduced by I. Bihari [3]. He called this equation half-linear because its solution set is homogeneous, but it is not additive. Many papers have been devoted to the study of asymptotic properties of the solutions of (HL) (see [1–2, 4, 10, 16–19, 21] and the references therein). I. Bihari extended the Armellini-Tonelli-Sansone theorem proving

THEOREM D [4]. *If coefficient a in (HL) is continuously differentiable and it grows to infinity "regularly" as $t \to \infty$, then all solutions of (HL) have property (1).*

The problem of extending Milloux's theorem to (HL) is more difficult since all the tools of the proof were connected with the linearity of (L). Very recently, F.V. Atkinson and Á. Elbert proved

THEOREM E [1–2]. *If function a is continuously differentiable, monotonous, and $\lim_{t\to\infty} a(t) = \infty$, then there exists at least one solution $x_0 \not\equiv 0$ of (HL) satisfying (1).*

Now we show, that Milloux's theorem has an extension to equation (HL) also in the case if a is a step function. Consider the equation

$$x''|x'|^{n-1} + a_k|x|^{n-1}x = 0, \quad (t_k \le t < t_{k+1}), \qquad k = 0, 1, 2 \ldots, \tag{3}$$

where

$$t_0 = 0, \quad \lim_{k\to\infty} t_k = \infty,$$

$$0 < a_0 < a_1 < \ldots < a_k < a_{k+1} < \ldots, \quad \lim_{k\to\infty} a_k = \infty.$$

A solution of (3) is defined in the same way as it was done for equation (2).

THEOREM 1. *There exists a solution $x_0 \not\equiv 0$ of equation (3) satisfying (1), i.e., $\lim_{t\to\infty} x_0(t) = 0$.*

Proof. At first we make the changing a_k disappear from the equation. To this end we rescale the time: $t = t(\tau)$, where τ denotes the "new time" such that $\tau_0 := t_0 = 0,\ t(\tau_k) = t_k$, and

$$t(\tau) = t_k + \alpha_k(\tau - \tau_k) \qquad (\tau_k \le \tau < \tau_{k+1}),$$

$$\alpha_k := a_k^{-\frac{1}{n+1}} \quad (k = 0, 1, 2, \ldots).$$

If we use the notation $y(\tau) := x(t(\tau))$, then

$$x'(t(\tau)) = \frac{1}{\alpha_k} y'(\tau), \quad x''(t(\tau)) = \frac{1}{\alpha_k^2} y''(\tau),$$

and equation (3) with the new variables τ, y has the form

$$y''(\tau)|y'(\tau)|^{n-1} + |y(\tau)|^{n-1} y(\tau) = 0, \quad (\tau_k \le \tau < \tau_{k+1}), \qquad k = 0, 1, 2, \ldots.$$

Any solution x of equation (3) has to be differentiable on $\mathbb{R}_+$; therefore, for every $k \in \mathbb{N},\ x'(t_{k+1} - 0) = x'(t_{k+1} + 0)$, which means, that

$$y'(\tau_{k+1}) = y'(\tau_{k+1} + 0) = \frac{\alpha_{k+1}}{\alpha_k} y'(\tau_{k+1} - 0), \quad k = 1, 2, \ldots.$$

We obtained, that equation (3) is equivalent to the following differential equation with impulses:

$$\begin{cases} y''|y'|^{n-1} + |y|^{n-1}y = 0, \quad \tau \ne \tau_k, \\ y'(\tau_{k+1}) = \left(\dfrac{a_k}{a_{k+1}}\right)^{\frac{1}{n+1}} y'(\tau_{k+1} - 0), \quad k = 0, 1, 2, \ldots. \end{cases} \tag{4}$$

The expression

$$|y'|^{n+1} + |y|^{n+1}$$

is a first integral of (4) on every interval $[\tau_k, \tau_{k+1}),\ k = 0, 1, 2, \ldots$. So the dynamics of (4) is the following. Any point (y_0, y_0') starts moving at $t_0 = \tau_0 = 0$ along the Minkowski circle

$$|y'|^{n+1} + |y|^{n+1} = |y_0'|^{n+1} + |y_0|^{n+1}.$$

At $\tau = \tau_1$ it jumps from the point $(y(\tau_1 - 0),\ y'(\tau_1 - 0))$ of this circle to the point

$$(y(\tau_1), y'(\tau_1)) := \left(y(\tau_1 - 0), \left(\frac{a_0}{a_1}\right)^{\frac{1}{n+1}} y'(\tau_1 - 0) \right),$$

and starts turning again around the origin along another Minkowski circle on (τ_1, τ_2) and this is repeated as $k = 0, 1, 2, \ldots$. To prove our theorem it is enough to find a point (y_{0*}, y_{0*}') on the Minkowski unit circle

$$C_0 : |y'|^{n+1} + |y|^{n+1} = 1 \tag{5}$$

from which there starts a trajectory $\tau \mapsto (y(\tau; y_{0*}, y_{0*}'),\ y'(\tau; y_{0*}, y_{0*}'))$ such that

$$\lim_{\tau\to\infty} y(\tau; y_{0*}, y_{0*}') = 0. \tag{6}$$

Now we construct such a point (y_{0*}, y_{0*}').

Let us start a trajectory from every point of unit circle C_0. The points of this trajectories at τ_k form a Minkowski ellipse C_k around the origin. The area A_k of the interior of C_k tends to 0 as $k \to \infty$, since

$$A_k = \left(\frac{a_0}{a_1}\frac{a_1}{a_2}\cdots\frac{a_k}{a_{k+1}}\right)^{\frac{1}{n+1}} A_0 = \left(\frac{a_0}{a_{k+1}}\right)^{\frac{1}{n+1}} A_0 \to 0, \quad \text{as } k \to \infty.$$

Consequently, there is a sequence $\{P_k\}_{k=1}^{\infty}$ such that

$$P_k \in C_k, \qquad \lim_{k\to\infty} \operatorname{dist}(P_k, O) = 0,$$

where $O = (0, 0)$ denotes the origin. Let $Q_k \in C_0$ denote the pre-image of P_k. Since C_0 is compact, we can assume that $\{Q_k\}_{k=1}^{\infty}$ converges: $\lim_{k\to\infty} Q_k =: Q_* = (y_{0*}, y_{0*}') \in C_0$. We prove that (6) is satisfied for this point Q_*.

In fact, if we define

$$\rho\left[(y, y'); O\right] = \left(|y'|^{n+1} + |y|^{n+1}\right)^{\frac{1}{n+1}}$$

(Minkowski norm), then

$$\begin{aligned}
&\rho\left[(y(\tau_k; Q_*), y'(\tau_k; Q_*))\,; O\right] \\
&\qquad \le \rho[P_k; O] + \big\{\rho\left[(y(\tau_k; Q_*), y'(\tau_k; Q_*))\,; O\right] \\
&\qquad - \rho\left[(y(\tau_k; Q_k), y'(\tau_k; Q_k))\,; O\right]\big\} \to 0, \text{ as } k \to \infty,
\end{aligned}$$

since $\rho[P_k;O] \to 0$ by the definition of P_k, and $\{\dots\}$ also tends to zero because the phase plane (y,y') contracts with respect to ρ by the dynamics of (4). But the function $t \mapsto \rho\left[(y(\tau;Q_*),y'(\tau;Q_*))\,;O\right]$ is nonincreasing, so we have

$$|y(\tau;Q_*)| \le \rho\left[(y(\tau;Q_*),y'(\tau;Q_*))\,;O\right] \to 0 \text{ as } \tau \to \infty,$$

which was to be proved.

Finally we mention, that the probabilistic approach in Theorem C can be extended also for the half-linear equation (3).

REFERENCES

[1] F.V. Atkinson and Á. Elbert, Extension of Prodi-Trevisan theorem to a half-linear differential equation, 1997 (preprint).

[2] F.V. Atkinson and Á. Elbert, An extension of Milloux's theorem to half-linear differential equation, *Proceedings of the Sixth Colloquium on the Qualitative Theory of Differential Equations*, August 10-14, 1999, Szeged, Hungary.

[3] I. Bihari, Ausdehnung der Sturmschen Oszillations und Vergleichssätze auf die Lösungen gewisser nichtlinearen Differentialgleichungen zweiter Ordnung, *Publ. Math. Inst. Hungar. Acad. Sci.*, **2**(1957), 159–173.

[4] I. Bihari, An asymptotic statement concerning the solutions of the differential equation $x'' + a(t)x = 0$, *Studia Sci. Math. Hungar.*, **20**(1985), 11–13.

[5] T.A. Burton and J.W. Hooker, On solutions of differential equations tending to zero, *J. Reine Angew. Math.*, **267**(1974), 154–165.

[6] Z. Došlá and I. Kiguradze, On vanishing at infinity solutions of second order linear differential equations with advanced arguments, *Funkcial. Ekvac.*, **41**(1998), 1889–2005.

[7] Z. Došlá and I. Kiguradze, On boundedness and stability of solutions of second order linear differential equations with advanced arguments, *Adv. Math. Sci. Appl.*, **9**(1999), 1–24.

[8] Á. Elbert, Stability of some difference equations, In *Advances in Difference Equations* (Proceedings of Second International Conference on Difference Equations, Veszprém, Hungary, August 7-11, 1993), Gordon and Breach Science Publishers, London, 1997, 155–178.

[9] Á. Elbert, On asymptotic stability of some Sturm-Liouville differential equations, *General Seminar in Mathematics*, University of Patras, 1996/97, 22-23, 57–66.

[10] W. Eberhardt and Á. Elbert, Half-linear eigenvalue problems, *Math. Nachr.*, **183**(1997), 55–72.

[11] P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964.

[12] P. Hartman, On a theorem of Milloux, *Amer. J. Math.,* **70**(1948), 395–399.

[13] L. Hatvani, On the Armellini - Tonelli - Sansone theorem, *Mem. Differential Equations Math. Phys.*, **12**(1997), 76–81.

[14] L. Hatvani, On the existence of a small solution of linear second order differential equations with step function coefficients, *Dynam. Contin. Discrete Impuls. Systems,* **4**(1998), 321–330.

[15] L. Hatvani and L. Stachó, On small solutions of second order differential equations with random coefficients, *Arch. Math. (Brno)*, **34**(1998), 119–126.

[16] J. Karsai, On the existence of a solution tending to zero of nonlinear differential equations (a nonlinear extension of a theorem of Milloux, Prodi, Trevisan and Hartman), *Dynam. Systems Appl.*, **6**(1997), 429–440.

[17] J. Karsai and J. Graef, Behavior of solutions of impulsively perturbed non-halflinear oscillator equations, *J. Math. Anal. Appl.*, **244**(2000), 77–96.

[18] J. Karsai J., J. R. Graef and M. Y. Li, A generalization of the Milloux–Hartman theorem for nonlinear systems, *Communications in Difference Equations*, Proceedings of the 4th International Conference on Difference Equations and Applications, Poznan, Poland, August 27–31, 1998, 203–213.

[19] J. Karsai, J. Graef and M.Y. Li, On the phase volume method for nonlinear difference equations, *Internat. J. Differential Equations Appl.*, **1**(2000), 17–35.

[20] I.T. Kiguradze and T.A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Acad. Publ., Dordrecht, 1993.

[21] T. Kusano, J. Jarŏs and N. Yoshida, A Picone identity and Sturmian comparison theorem for a class of half-linear partial differential equations of second order, *Nonlinear Anal.* (to appear).

[22] J. W. Macki, Regular growth and zero tending solutions, In *Ordinary Differential Equations and Operators,* (Dundee, 1982), Lecture Notes in Math. 1032, 358–399.

[23] H. Milloux, Sur l'equation differentielle $x'' + A(t)x = 0$, *Prace Mat.-Fiz.*, **41**(1934), 39–54.

[24] P. Pucci and J. Serrin, Asymptotic stability for ordinary differential systems with time dependent restoring potentials, *Arch. Rational Mech. Anal.,* **132**(1995), 207–232.

[25] J. Terjéki, On the conversion of a theorem of Milloux, Prodi and Trevisan, In *Differential Equations,* Lecture Notes in Pure and Appl. Math., **118**(1982), 661–665.