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Maximal L_p-regularity for a second-order differential equation with unbounded intermediate coefficient

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Abstract. We consider the following equation

$$-y'' + r(x)y' + q(x)y = f(x),$$

where the intermediate coefficient r is not controlled by q and it is can be strong oscillate. We give the conditions of well-posedness in $L_p(-\infty, +\infty)$ of this equation. For the solution y, we obtained the following maximal regularity estimate:

$$\|y''\|_p + \|ry'\|_p + \|qy\|_p \le C\|f\|_p,$$

where $\|\cdot\|_p$ is the norm of $L_p(-\infty, +\infty)$.

Keywords: differential equation, unbounded coefficients, well-posedness, maximal regularity, separability.

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1 Introduction and main theorem

Let $C_0^{(2)}(R)$ be the set of all twice continuously differentiable functions with compact support. We study the following differential equation:

$$L_0 y = -y'' + r(x)y' + q(x)y = f(x), \tag{1.1}$$

where $x \in R = (-\infty, +\infty)$ and $f \in L_p(R)$, $1 < p < +\infty$. We assume that r, q are, respectively, continuously differentiable and continuous functions. We denote by L the closure in $L_p(R)$ of the differential operator L_0 defined on the set $C_0^{(2)}(R)$. We call that $y \in L_p(R)$ is a solution of the equation (1.1), if $y \in D(L)$ and $Ly = f$.

Everywhere, in this paper, by $C, C_-, C_+, C_j, \tilde{C}_j$ $(j = 0, 1, 2, \dots)$ etc., we will denote the positive constants, which, generally speaking, are different in the different places.

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The purpose of this work is to find some conditions for the coefficients r and q such that for any $f \in L_p(R)$ there exists a unique solution y of the equation (1.1) and the following estimate holds:

$$\| y'' \|_p + \| ry' \|_p + \| qy \|_p \leq C \| Ly \|_p , \tag{1.2}$$

where $\| \cdot \|_p$ is the norm in $L_p(R)$.

As in [4] and [2], if the estimate (1.2) holds, then we call that the solution y of the equation (1.1) is maximally L_p-regular, and call (1.2) is an maximal L_p-regularity estimate. If (1.2) holds, then the operator L is said to be separable in $L_p(R)$ (see [7]).

The maximal regularity is an important tool in the theory of linear and nonlinear differential equations. For example, from the estimate (1.2) we obtain the following:

a) under mild assumptions on r and q, we obtain the optimal smoothness of a solution and some information about the behavior of y and y' at infinity;

b) we give the domain of the operator L, so that we can use the embedding theory of the weighted function spaces for study of spectrum of the operator L and the approximate characteristics of a solution y of the equation (1.1) (see [19, 20]);

c) we reduce the study of the singular nonlinear second order differential equations via a fixed point argument to the linear equation (1.1) (see [2, 13, 20]).

Moreover, the maximal L_p-regularity estimate (1.2) and the closed smoothness properties of L are useful for the study of the following evolutionary problem:

$$u_t = Lu + F(x, t), \qquad u(0, x) = \phi(x)$$

(see [4, 16, 18] and the references therein).

The equation (1.1) and its multidimensional generalization

$$lu = -\Delta u + \sum_{j=1}^{N} r_j(x) u_{x_j} + q(x)u = F(x) \qquad (x \in R^N), \tag{1.3}$$

with unbounded coefficients have used in stochastic analysis, biology and financial mathematics (see [5, 9, 11]). For this reason, interest in these equations has considerably grown in recent years. A number of researches of (1.3) were devoted to the case that the coefficients r_j ($j = \overline{1, N}$) are controlled by q (see [3, 6, 17, 24]). Without the dominating potential q, the case that r_j grow at most as $|x| \ln(1 + |x|)$ were considered in [10, 14, 15, 23].

In the present work, we study the equation (1.1) in assumption that the coefficient r can quickly grow and fluctuate, and it does not depend on q. We find conditions, which provides the correct solvability of (1.1) and the fulfillment of the maximal L_p-regularity estimate (1.2). In [20–22] the equation (1.1) was investigated in the case that r is a weakly oscillating function.

Let $0 \leq \varepsilon < 1$, $1 < p < \infty$, and $p' = p/(p-1)$. For continuous functions g and $h \neq 0$, we denote

$$\alpha_{g,h,\varepsilon}(t) = \|g\|_{L_p(0,t)} \|1/h\|_{L_{p'}((1-\varepsilon)t, +\infty)} \qquad (t > 0)$$

and

$$\beta_{g,h,\varepsilon}(\tau) = \|g\|_{L_p(\tau,0)} \|1/h\|_{L_{p'}(-\infty, (1+\varepsilon)\tau)} \qquad (\tau < 0) .$$

Let

$$\gamma_{g,h,\varepsilon} = \max \left(\sup_{t>0} \alpha_{g,h,\varepsilon}(t) , \sup_{\tau<0} \beta_{g,h,\varepsilon}(\tau) \right) .$$

If $v(x)$ is a continuous function, we define

$$v^*(x) = \inf_{d>0} \left\{ d^{-1} : d^{-p+1} \geq \int_{\Delta_d(x)} |v(t)|^p \, dt \right\}, \qquad x \in R,$$

where $\Delta_d(x) = (x-d,\, x+d)$ (see [19]). The main result of this paper is the following.

Theorem 1.1. *Assume that $1 < p < \infty$. Let r be a continuously differentiable function, q be a continuous function and the following conditions hold:*

a) $r \geq 1$ and $\gamma_{1,\sqrt[p]{r},0} < \infty$;

b) If $x, \eta \in R$ satisfy $|x-\eta| \leq \frac{k(\eta)}{r(\eta)}$, then

$$C^{-1} \leq \frac{r(x)}{r(\eta)} \leq C,$$

where $k(\eta)$ is a continuous function satisfies $k(\eta) \geq 4$ and $\lim_{|\eta|\to+\infty} k(\eta) = +\infty$;

c) $\gamma_{q,r^*,0} < \infty$.

Then for any $f \in L_p(R)$ there exists a unique solution y of the equation (1.1). Moreover, for y the following estimate holds:

$$\|y''\|_p + \|ry'\|_p + \|qy\|_p \leq C \|f\|_p . \tag{1.4}$$

Remark 1.2. We will prove Theorem 1.1 in the assumption $r(x) \geq 1$. The case $r(x) \leq -1$ can easily be reduced to the case $r(x) \geq 1$ by replacing of the variable x.

Remark 1.3. Conditions of Theorem 1.1 are close to the necessary.

i) If $\gamma_{1,\sqrt[p]{|r|},0} = \infty$ in the condition a) and $q = 0$, then the equation (1.1) has not a solution from $L_p(R)$. Using the well-known weighted Hardy inequality (see Theorem 5 in Chapter 3 of [19]) one easily prove it;

ii) If performed a) and b), as well as the estimate (1.4), then for a wide class of coefficients q and r (for example, they may be power functions) holds the condition c). This fact follows from Theorem 6.3 in [1] (in the case $n = 2$ and $k = 1$**).**

Example 1.4. The following equation:

$$-y'' - \left(15 + 9x^2 + e^{\sqrt{1+x^2}} \cos^2 x^{11}\right) y' + x^7 y = f(x), \qquad f \in L_p(R), \tag{1.5}$$

satisfies the conditions of Theorem 1.1, hence, the equation (1.5) is uniquely solvable, and for the solution y of (1.5), the following maximal regularity estimate holds:

$$\|y''\|_p + \left\| \left(15 + 9x^2 + e^{\sqrt{1+x^2}} \cos^2 x^{11}\right) y' \right\|_p + \|x^7 y\|_p \leq C \|f\|_p .$$

2 Weighted integral inequalities

We denote by $C_0^{(2)}[0,+\infty)$ (resp. $C_0^{(2)}(-\infty,0]$) the set of all twice continuously differentiable in $[0,+\infty)$ (resp. $(-\infty,0]$) functions with compact support. The following Lemma 2.1 and Lemma 2.2 are special cases of Theorem 6.1 and Theorem 6.3 in [1], respectively.

Lemma 2.1. *Let*

$$\sup_{t>0}\ \alpha_{g,h^*,\varepsilon}(t)<\infty \tag{2.1}$$

for some $\varepsilon\in(0,1)$. *Then for any* $y\in C_0^{(2)}[0,+\infty)$,

$$\left(\int_0^{+\infty}|g(t)\,y(t)|^p\,dt\right)^{\frac{1}{p}}\le C_+\left(\int_0^{+\infty}\left[|y''(t)|^p+|h(t)\,y'(t)|^p\right]dt\right)^{\frac{1}{p}} \tag{2.2}$$

and $C_+\le C_1\ \sup_{t>0}\ \alpha_{g,h^*,\varepsilon}(t)$. *Conversely, if* (2.2) *holds with some* C_+, *then* $\sup_{t>0}\ \alpha_{g,h^*,0}(t)<\infty$ *and*

$$C_+\ge C_0\sup_{t>0}\ \alpha_{g,h^*,0}(t). \tag{2.3}$$

Lemma 2.2. *Let for some* $\varepsilon\in(0,1)$ *the condition* (2.1) *and at least one of the following relationships* (2.4) *and* (2.5):

$$\sup_{x>0}\int_{(1-\varepsilon)x}^{x}|h^*(t)|^{-p'}\,dt\left(\int_x^{+\infty}|h^*(\eta)|^{-p'}\,d\eta\right)^{-1}<\infty, \tag{2.4}$$

$$\sup_{x>0}\left(\int_0^{x}|g(\eta)|^p d\eta\right)^{-1}\int_x^{(1+\varepsilon)x}|g(t)|^p dt<\infty,\quad g(t)\ne 0\quad (t\in[0,+\infty)) \tag{2.5}$$

be fulfilled. Then the inequality (2.2) *holds for any* $y\in C_0^{(2)}[0,+\infty)$ *if and only if*

$$\sup_{t>0}\ \alpha_{g,h^*,0}(t)<\infty,$$

and for a constant C_+ *in* (2.2) *the following estimates hold:*

$$C_2\sup_{t>0}\ \alpha_{g,h^*,0}(t)\le C_+\le C_3\sup_{t>0}\ \alpha_{g,h^*,0}(t).$$

Using Lemma 2.1 and Lemma 2.2, we prove the following Lemma 2.3 and Lemma 2.4, respectively.

Lemma 2.3. *Assume that for some* $\varepsilon\in(0,1)$

$$\sup_{\tau<0}\ \beta_{g,h^*,\varepsilon}(\tau)<\infty. \tag{2.6}$$

Then for any $y\in C_0^{(2)}(-\infty,0]$ *the following inequality holds:*

$$\left(\int_{-\infty}^{0}|g(t)\,y(t)|^p\,dt\right)^{\frac{1}{p}}\le C_-\left(\int_{-\infty}^{0}\left[|y''(t)|^p+|h(t)\,y'(t)|^p\right]dt\right)^{\frac{1}{p}}, \tag{2.7}$$

where $C_-\le\tilde C_1\ \sup_{\tau<0}\ \beta_{g,h^*,\varepsilon}(\tau)$. *Conversely, if* (2.7) *holds for some* C_-, *then* $\sup_{\tau<0}\ \beta_{g,h^*,0}(\tau)<\infty$ *and*

$$C_-\ge\tilde C_0\sup_{\tau<0}\ \beta_{g,h^*,0}(\tau). \tag{2.8}$$

Lemma 2.4. *Let for some $\varepsilon \in (0,1)$ the condition (2.6) be fulfilled and at least one of the following relationships (2.9) and (2.10) holds:*

$$\sup_{x<0} \int_{(1+\varepsilon)x}^{x} |h^*(t)|^{-p'} dt \left(\int_{-\infty}^{x} |h^*(\eta)|^{-p'} d\eta \right)^{-1} < \infty, \tag{2.9}$$

$$\sup_{x<0} \int_{(1+\varepsilon)x}^{x} |g(t)|^{p} dt \left(\int_{x}^{0} |g(\eta)|^{p} d\eta \right)^{-1} < \infty, \tag{2.10}$$

where $g(\eta) \neq 0$ for each $\eta \in (-\infty, 0]$. Then the inequality (2.7) holds for any $y \in C_0^{(2)}(-\infty, 0]$ if and only if

$$\sup_{\tau<0} \beta_{g,h^*,0}(\tau) < \infty,$$

and for a constant C_- in (2.7) the following estimates hold:

$$\tilde{C}_2 \sup_{\tau<0} \beta_{g,h^*,0}(\tau) \le C_- \le \tilde{C}_3 \sup_{\tau<0} \beta_{g,h^*,0}(\tau).$$

Lemma 2.5. *Assume that for some $\varepsilon \in (0,1)$,*

$$\gamma_{g,h^*,\varepsilon} < \infty.$$

Then for any $y \in C_0^{(2)}(R)$, the following inequality holds:

$$\left(\int_{-\infty}^{+\infty} |g(t)\, y(t)|^p dt \right)^{1/p} \le C \left(\int_{-\infty}^{+\infty} \left[|y''(t)|^p + |h(t)\, y'(t)|^p \right] dt \right)^{1/p},$$

where

$$C_4 \min \left[\alpha_{g,h^*,0}, \beta_{g,h^*,0} \right] \le C \le C_5 \gamma_{g,h^*,\varepsilon}. \tag{2.11}$$

Proof. Let $y \in C_0^{(2)}(R)$. By Lemmas 2.1 and 2.3 and estimates (2.2) and (2.7), we have

$$\begin{aligned}
\|g(t)\, y(t)\|_p = {} & \|g(t)\, y(t)\|_{L_p(-\infty,0)} + \|g(t)\, y(t)\|_{L_p(0,+\infty)} \\
& \le C_- \left(\int_{-\infty}^{0} \left[|y''(t)|^p + |h(t)\, y'(t)|^p \right] dt \right)^{1/p} \\
& \quad + C_+ \left(\int_{0}^{+\infty} \left[|y''(t)|^p + |h(t)\, y'(t)|^p \right] dt \right)^{1/p} \\
& \le \tilde{C}_1(\varepsilon) \sup_{\tau<0} \beta_{g,h^*,\varepsilon}(\tau) \left(\|y''\|_{L_p(-\infty,0)} + \|hy'\|_{L_p(-\infty,0)} \right) \\
& \quad + C_1(\varepsilon) \sup_{t>0} \alpha_{g,h^*,\varepsilon}(t) \left(\|y''\|_{L_p(0,+\infty)} + \|hy'\|_{L_p(0,+\infty)} \right) \\
& \le C \left(\|y''\|_p + \|hy'\|_p \right),
\end{aligned}$$

where $C = \max\{\tilde{C}_1(\varepsilon) \sup_{\tau<0} \beta_{g,h^*,\varepsilon}(\tau), C_1(\varepsilon) \sup_{t>0} \alpha_{g,h^*,\varepsilon}(t)\}$. This implies the right-hand side of (2.11). Left-hand side of these inequalities follows from (2.3) and (2.8). □

Lemma 2.6. *Assume that for some $\varepsilon \in (0,1)$ either relations (2.4) and (2.9), or (2.5) and (2.10) are fulfilled. Then the inequality*

$$\left(\int_{-\infty}^{+\infty} |g(t)\, y(t)|^p dt \right)^{\frac{1}{p}} \le C \left(\int_{-\infty}^{+\infty} \left[|y''(t)|^p + |h(t)\, y'(t)|^p \right] dt \right)^{\frac{1}{p}} \tag{2.12}$$

holds for any $y \in C_0^{(2)}(R)$ *if and only if* $\gamma_{g,h^*,0} < \infty$. *Furthermore, for a constant* C *in* (2.12) *the following estimates hold:*

$$C_6\gamma_{g,h^*,0} \le C \le C_7\gamma_{g,h^*,0}. \tag{2.13}$$

Similarly to Lemma 2.5, using Lemma 2.2, Lemma 2.4 and the fact that the quantities $\gamma_{g,h^*,\varepsilon}$ and $\gamma_{g,h^*,0}$ are equivalent to each other under the conditions of this lemma, we can prove this lemma.

3 Auxiliary estimates for two-term differential operator

In this section, we will study the following two-term equation

$$l_0 y = -y'' + r(x)\, y' = F(x), \tag{3.1}$$

where $F \in L_p(R)$ $(1 < p < +\infty)$. We denote by l the closure in $L_p(R)$ of the differential operator l_0 defined on the set $C_0^{(2)}(R)$. If $y \in D(l)$ and $ly = F$, then we call that y is a solution of the equation (3.1).

Lemma 3.1. *Let* r *be continuously differentiable and*

$$r(x) \ge 1, \qquad \gamma_{1,\sqrt[p]{r},0} < \infty.$$

Then for any $F \in L_p(R)$ $(1 < p < +\infty)$ *there exists a unique solution* y *of the equation* (3.1) *and for* y *the following estimate holds:*

$$\|\sqrt[p]{r}y'\|_p^p + \|y\|_p^p \le \left(1 + C^p\gamma^p_{1,\sqrt[p]{r},0}\right)\|F\|_p^p. \tag{3.2}$$

Proof. Let $\beta > -1$, and $y \in C_0^{(2)}(R)$. Integrating by parts, we have

$$\left(l_0 y, y'\left[(y')^2\right]^{\beta/2}\right) = \int_R r\left[(y')^2\right]^{\beta/2+1} dx.$$

We take a number $\alpha > 0$, then

$$\int_R r\left[(y')^2\right]^{\beta/2+1} dx \le \left(\int_R r^{-\alpha p}\,|l_0 y|^p\,dx\right)^{1/p}\left(\int_R r^{\alpha p'}\,|y'|^{(\beta+1)p'}\,dx\right)^{1/p'}. \tag{3.3}$$

We choose α and β such that $(\beta+1)\,p' = \beta + 2$ and $\alpha p' = 1$, where $p' = \frac{p}{p-1}$. Then $-\alpha p = -\frac{p}{p'}$ and (3.3) implies that

$$\|\sqrt[p]{r}y'\|_p^p \le \left\|\frac{1}{\sqrt[p']{r}}l_0 y\right\|_p^p. \tag{3.4}$$

It is well known (see Theorem 5 in Chapter 3 of [19]) that for any $y \in C_0^{(2)}[0,\infty)$ the following inequality holds:

$$\|y\|^p_{L_p(0,\infty)} \le C_0^p\alpha^p_{1,\sqrt[p]{r},0}\,\|\sqrt[p]{r}y'\|^p_{L_p(0,\infty)},$$

moreover $1 \le C_0 \le p^{1/p}\,(p')^{1/p'}$. From this, as in [21], we obtain for any $y \in C_0^{(2)}(R)$

$$\|y\|_p^p \le C^p\gamma^p_{1,\sqrt[p]{r},0}\,\|\sqrt[p]{r}y'\|_p^p.$$

This inequality and (3.4) imply (3.2).

Now, if $y \in D(l)$, then there exists the sequence $\{y_n\}_{n=1}^{\infty} \subset C_0^{(2)}(R)$ such that $\|y_n - y\|_p \to 0$, $\|l_0 y_n - ly\|_p \to 0$ as $n \to \infty$. For y_n $(n \in N)$ the inequality (3.2) holds, so the sequence $\left\{\sqrt[p]{r}\,(y_n)'\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $L_p(R)$. By virtue of completeness of $L_p(R)$ and closedness of the differentiation operation, it converges to $\sqrt[p]{r}y' \in L_p(R)$. So, (3.2) holds for any solution of (3.1).

(3.2) implies the uniqueness of solution of the equation (3.1). Let us prove the existence of solution. By inequality (3.2), the range $R\,(l)$ of l is closed. Therefore, it is enough to prove that $R(l) = L_p(R)$. Indeed, let $R(l) \neq L_p(R)$. Then there exists the non-zero element $z \in L_{p'}\,(R)$ such that $(ly, z) = 0$ for any $y \in C_0^{(2)}\,(R)$ (see [25]). Taking into account the equality

$$(ly,\, z) = \int_R y \left(-\left[\bar{z}\right]'' - \left[r\,(x)\,\bar{z}\right]' \right) dx,$$

we obtain

$$-\,z'' - rz = C_1. \tag{3.5}$$

It is clear that z is a twice differentiable function. Let $C_1 \neq 0$. By properties of $L_p\,(R)$-norm, without loss of generality we can assume that $C_1 = 1$. Hence,

$$z' + r(x)z = -1, \qquad x \in R.$$

Then

$$\left[z\,(x) \exp \int_{x_0}^{x} r\,(t)\,dt\right]' = -\exp \int_{x_0}^{x} r\,(t)\,dt,$$

where $x_0 \in R$. Consequently, $z(x) \exp \int_{x_0}^{x} r(t)dt$ on (x_0, ∞) is monotonously decreases function and

$$z\,(x-k) > \exp k \cdot z\,(x) \qquad (x \in (x_0, +\infty))$$

for each $k > 0$. Therefore there exists $x_1 \in R$ such, that $z\,(x) \leq \theta < 0$ for any $x \in (x_1, +\infty)$. So $z \notin L_{p'}(R)$.

If $C_1 = 0$, then by (3.5),

$$z\,(x) = \exp\left[-\int_a^x r\,(t)\,dt\right],$$

therefore $z \notin L_{p'}(R)$. This is a contradiction. □

Remark 3.2. Lemma 3.1 remains valid, if $|r(x)| \geq \delta > 0$.

Remark 3.3. Lemma 3.1 remains valid, if (3.1) is replaced by

$$l_{0,\lambda}y = -y'' + (1+\lambda)r\,(x)\,y' = F,$$

where $\lambda \geq 0$. In this case, instead of (3.2) we have the estimate

$$\left\|(1+\lambda)ry'\right\|_p^p + \|y\|_p^p \leq c_0 \left\|l_{0,\lambda}y\right\|_p^p, \tag{3.6}$$

where c_0 depends on λ.

Lemma 3.4. *Assume that $\lambda \geq 0$ and r satisfies the conditions of Lemma 3.1. Let $k(\eta)$ be a continuous function such that $k(\eta) \geq 4$ and $\lim_{|\eta| \to +\infty} k(\eta) = +\infty$. If for any $(x, \eta) \in \{(x, \eta) : x, \eta \in R,$ $|x - \eta| \leq \frac{k(\eta)}{r(\eta)}\}$, we have that*

$$c^{-1} \leq \frac{r\,(x)}{r\,(\eta)} \leq c, \tag{3.7}$$

then for the solution y of the equation (3.1)*, the following estimate*

$$\left\|y''\right\|_p^p + \left\|ry'\right\|_p^p + \|y\|_p^p \le C\,\|ly\|_p^p \tag{3.8}$$

holds.

Proof. We consider the minimal closed operator $l_\lambda y = -y'' + (r+\lambda)\,y'$ $(\lambda \ge 0)$ corresponding to the equation (3.1). By Lemma 3.1 and Remark 3.3 we know that $D(l_\lambda) \subseteq W_p^1(R)$, where $W_p^1(R)$ is the Sobolev space with norm $\|y\|_{W_p^1(R)} = \left(\|y'\|_p^p + \|y\|_p^p\right)^{1/p}$. If we denote $y' = z$, then $l_\lambda y$ become the following form

$$\theta_\lambda z = -z' + [r(x)+\lambda]\,z \qquad (z \in L_p(R)).$$

We choose two systems of concentric intervals $\{\Omega_j\}_{j=-\infty}^{+\infty}$ and $\{\Delta_j\}_{j=-\infty}^{+\infty}$ with centers at the points x_j, and radius of Δ_j does not exceed $\frac{k(x_j)}{10\,r(x_j)}$, as well as the sequence $\{\phi_j(x)\}_{j=-\infty}^{+\infty}$ satisfying the following conditions a) and b):

a) $\Delta_j = (a_j, b_j)$, $a_j < b_j$, $\overline{\Delta_j} \subset \Omega_j \subset \overline{\Delta_{j-1}} \bigcup \overline{\Delta_j} \bigcup \overline{\Delta_{j+1}}$, $|\Omega_j| = 2\,(b_j - a_j)$ $(j \in Z)$, $\lim_{j\to+\infty} a_j = +\infty$, $\lim_{j\to-\infty} b_j = -\infty$, $\Delta_j \bigcap \Delta_k = \emptyset$ $(j \ne k)$, $\bigcup_{j=-\infty}^{+\infty} \overline{\Delta_j} = R$;

b) $\phi_j \in C_0^\infty(\Omega_j)$, $0 \le \phi_j(x) \le 1$, $\phi_j(x) = 1$ $\forall x \in \Delta_j$ $(j \in Z)$, $\sum_{j=-\infty}^{\infty} \phi_j(x) = 1$, $\sup_{j\in Z} \max_{x\in\Delta_j} \left|\phi_j'(x)\right| \le M$.

Sequences $\{\Omega_j\}_{j=-\infty}^{+\infty}$, $\{\Delta_j\}_{j=-\infty}^{+\infty}$ and $\{\phi_j(x)\}_{j=-\infty}^{+\infty}$ with such properties exist by virtue of our assumptions with respect to r and results of [8].

We extend $r(x)$ from Δ_j to all of R so that it extensions $r_j(x)$ are continuously differentiable and satisfy the following inequalities:

$$\frac{1}{2}\inf_{t\in\Omega_j} r(t) \le r_j(x) \le 2\sup_{t\in\Omega_j} r(t)\,. \tag{3.9}$$

By properties of $r(x)$, this extension exists. We denote by $\theta_{j,\lambda}$ $(j \in Z)$ the closure in $L_p(R)$ of the differential expression $\theta_{j,\lambda} z = -z' + \left[r_j(x) + \lambda\right] z$ defined on $C_0^{(2)}(R)$. It is easy to see that $r_j(x) \ge 1/2$ $(j \in Z)$ satisfy the conditions of Lemma 3.1. By Remark 3.2, the operators $\theta_{j,\lambda}$ are boundedly invertible and for any $z \in D\left(\theta_{j,\lambda}\right)$ the following estimate is valid:

$$\left\|\sqrt[p]{r_j+\lambda}\,z\right\|_p^p \le \left\|\frac{1}{\sqrt[p']{r_j+\lambda}}\theta_{j,\lambda} z\right\|_p^p.$$

By (3.9), we obtain

$$\begin{aligned}\left\|\,(r_j+\lambda)z\right\|_p^p &\le 2^p \sup_{j\in Z}\sup_{t\in\Omega_j}\left[r_j(t)+\lambda\right]^{p/p'}\left\|\sqrt[p]{r_j+\lambda}\,z\right\|_p^p \\ &\le 2^p \sup_{j\in Z}\left[\sup_{t\in\Omega_j}\left[r_j(t)+\lambda\right]^{p/p'}\left(\frac{2^p}{\inf\limits_{t\in\Omega_j}\left[r_j(t)+\lambda\right]^{p/p'}}\right)\right]\left\|\theta_{j,\lambda} z\right\|_p^p.\end{aligned}$$

The length of the interval Ω_j does not exceed $\frac{k(x_j)}{2r(x_j)}$, so, by condition (3.7), we have

$$\begin{aligned} \|\,(r_j+\lambda)z\|_p^p &\le 4^p \sup_{j\in Z}\ \sup_{t,\eta\in\Omega_j} \left[\frac{r_j(t)+\lambda}{r_j(\eta)+\lambda}\right]^{p/p'} \|\theta_{j,\lambda}z\|_p^p \\ &\le 4^p(c+1)^{p/p'}\,\|\theta_{j,\lambda}z\|_p^p \qquad \left(z\in D\left(\theta_{j,\lambda}\right),\ \ j\in Z\right). \end{aligned} \tag{3.10}$$

Let χ_j be the characteristic function of Δ_j. We introduce the following operators M_λ and B_λ:

$$\begin{aligned} M_\lambda f &= \sum_{j=-\infty}^{+\infty} \phi_j\theta_{j,\,\lambda}^{-1}\left(\chi_j f\right), \\ B_\lambda f &= -\sum_{j=-\infty}^{+\infty} \phi_j'\theta_{j,\,\lambda}^{-1}\left(\chi_j f\right), \qquad f\in C_0^\infty\left(R\right). \end{aligned}$$

Since the support of f is compact, the sums in these expressions contain only finitely many terms. In Δ_j the coefficients of θ_λ and $\theta_{j,\,\lambda}$ coincide. Consequently, by properties of φ_j $(j\in Z)$, we have

$$\begin{aligned} \theta_\lambda\left(M_\lambda f\right) &= \sum_{j=-\infty}^{+\infty}\theta_\lambda\left(\phi_j\theta_{j,\,\lambda}^{-1}\left(\chi_j f\right)\right) = \sum_{j=-\infty}^{+\infty}(-\phi_j)'\theta_{j,\,\lambda}^{-1}\left(\chi_j f\right) + \sum_{j=-\infty}^{+\infty}\phi_j l_\lambda\theta_{j,\,\lambda}^{-1}\left(\chi_j f\right) \\ &= f - \sum_{j=-\infty}^{+\infty}\phi_j'\theta_{j,\,\lambda}^{-1}\left(\chi_j f\right) = \left(E+B_\lambda\right)f, \end{aligned} \tag{3.11}$$

where E is the identity operator. Now, we estimate the norm $\|B_\lambda f\|_p$. Since the interval Ω_j $(j\in Z)$ intersects only with Ω_{j-1} and Ω_{j+1}, we obtain

$$\begin{aligned} \|B_\lambda f\|_p^p &= \sum_{j=-\infty}^{+\infty}\int_{\Delta_j}|B_\lambda f|^p\,dx \le \sum_{j=-\infty}^{+\infty}\int_{\Delta_j}\left[\sum_{k=j-1}^{j+1}\left|\phi_k'\left(x\right)\right|\left|\theta_{k,\,\lambda}^{-1}\left(\chi_k f\right)\right|\right]^p dx \\ &\le 3^p\sum_{j=-\infty}^{+\infty}\int_{\Delta_j}\sum_{k=j-1}^{j+1}\left|\phi_k'\left(x\right)\right|^p\left|\theta_{k,\,\lambda}^{-1}\left(\chi_j f\right)\right|^p dx \\ &\le 9^p M^p\sum_{j=-\infty}^{+\infty}\int_R\left|\theta_{j,\,\lambda}^{-1}\left(\chi_j f\right)\right|^p dx. \end{aligned}$$

By (3.10),

$$\left\|\theta_{k,\,\lambda}^{-1}f\right\|_p \le \frac{4(c+1)^{1/p'}}{\inf\limits_{x\in\Delta_k}\left(r_k\left(x\right)+\lambda\right)}\,\|f\|_p\,,$$

consequently

$$\|B_\lambda f\|_p \le \frac{72M(c+1)^{1/p'}}{1+2\lambda}\,\|f\|_p\,.$$

We choose $\lambda_0 = 72M(c+1)^{1/p'}$. Then for any $\lambda\ge\lambda_0$ there exists the inverse operator $(E+B_\lambda)^{-1}$, and the inequalities $2/3\le\|\,(E+B_\lambda)^{-1}\,\|_{L_p\to L_p}\le 2$ fulfilled. By (3.11),

$$\theta_\lambda^{-1} = M_\lambda\left(E+B_\lambda\right)^{-1}, \qquad \lambda\ge\lambda_0. \tag{3.12}$$

We prove the estimate (3.8). By (3.12), $\left\|(r+\lambda)\theta_\lambda^{-1}\right\|_p \le 2\left\|(r+\lambda)M_\lambda\right\|_p$ $(\lambda \ge \lambda_0)$, and

$$\begin{aligned}\|(r+\lambda)M_\lambda f\|_p^p &= \sum_{k=-\infty}^{+\infty}\int_{\Delta_k}\left|\sum_{j=k-1}^{k+1}(r_j+\lambda)\phi_j\theta_{j,\lambda}^{-1}(\chi_j f)\right|^p dx\\ &\le 3^p\sum_{k=-\infty}^{+\infty}\int_{\Delta_k}\sum_{j=k-1}^{k+1}\left|(r_j+\lambda)\phi_j\theta_{j,\lambda}^{-1}(\chi_j f)\right|^p dx\\ &\le 9^p\sum_{k=-\infty}^{+\infty}\int_R\left|(r_k+\lambda)\phi_k\theta_{k,\lambda}^{-1}(\chi_k f)\right|^p dx.\end{aligned}$$

Taking into account (3.10), we have

$$\begin{aligned}\left\|(r+\lambda)\theta_\lambda^{-1}f\right\|_p^p &\le 2^p 9^p 4^p (c+1)^{p/p'}\sum_{k=-\infty}^{+\infty}\int_R|\chi_k f|^p dx\\ &= 72^p(c+1)^{p/p'}\|f\|_p^p.\end{aligned}$$

Therefore, for any $z \in D(\theta_\lambda)$

$$\|z'\|_p^p \le \|(r+\lambda)z\|_p^p + \|\theta_\lambda z\|_p^p \le \left[72^p(c+1)^{p/p'+1}\right]\|\theta_\lambda z\|_p^p,$$

that implies

$$\|z'\|_p^p + \|(r+\lambda)z\|_p^p \le \left[2\cdot 72^p(c+1)^{p/p'}+1\right]\|\theta_\lambda z\|_p^p, \qquad z \in D(\theta_\lambda).$$

By (3.6), we obtain the desired estimate (3.8). □

4 Proof of Theorem 1.1

In the equation (1.1) we assume that $x = at$, where $a > 0$. If we introduce the notations

$$\tilde{y}(t) = y(at), \quad \tilde{r}(t) = r(at), \quad \tilde{q}(t) = q(at), \quad \tilde{f}(t) = a^2 f(at) \quad (t \in R),$$

then (1.1) become the following form:

$$\tilde{L}\tilde{y} = -\tilde{y}'' + a\tilde{r}\tilde{y}' + a^2\tilde{q}\tilde{y} = \tilde{f}(t). \tag{4.1}$$

We denote by l_a the closure of $l_{0,a}$ in $L_p(R)$, where $l_{0,a}$ is the differential expression

$$l_{0,a}\tilde{y} = -\tilde{y}'' + a\,\tilde{r}(t)\tilde{y}'$$

defined on the set $C_0^{(2)}(R)$. Note that $a|\tilde{r}(t)| \ge a > 0$. By Lemma 3.1, Lemma 3.4 and Remark 3.2, the operator l_a is continuously invertible, moreover the following estimate holds:

$$\left\|\tilde{y}''\right\|_p + \left\|a\,\tilde{r}\tilde{y}'\right\|_p \le C_{l_a}\|l_a\tilde{y}\|_p, \qquad \forall\tilde{y} \in D(l_a). \tag{4.2}$$

By Theorem 6.3 in [1], taking into account the condition c) of Theorem 1.1, we have

$$\left\|a^2\tilde{q}\tilde{y}\right\|_p \le a^2\gamma_{\tilde{q},\tilde{r}^*,0}C_{l_a}\|l_a\tilde{y}\|_p. \tag{4.3}$$

If we choose

$$a = \left[2\gamma_{\tilde{q},\tilde{r}^*,0}C_{l_a}\right]^{-\frac{1}{2}},$$

then, by (4.3),

$$\left\|a^2\tilde{q}\tilde{y}\right\|_p \le \theta\, \|l_a\tilde{y}\|_p \tag{4.4}$$

holds, where $\theta \in \left(0, \frac{1}{2}\right]$. From this inequality, and the well-known perturbation theorem (for example, see Theorem 1.16 in Chapter 4 of [12]), it follows that there exists the inverse operator $\left(l_a + a^2\tilde{q}E\right)^{-1}$, as well as the equality $R\left(l_a + a^2\tilde{q}E\right) = L_p\left(R\right)$ fulfilled. So, denoting $t = a^{-1}x$, we obtain that for any $f \in L_p(R)$ there exists a solution y of the equation (4.1) and it is unique.

By estimates (4.2) and (4.4),

$$\left\|\tilde{y}''\right\|_p + \left\|a\,\tilde{r}\tilde{y}'\right\|_p + \left\|a^2\tilde{q}\tilde{y}\right\|_p \le \left(\frac{1}{2} + C_{l_a}\right) \|l_a\tilde{y}\|_p . \tag{4.5}$$

Taking into account (4.4), we get

$$\|l_a\tilde{y}\|_p \le \left\|\left(l_a + a^2\tilde{q}E\right)\tilde{y}\right\|_p + \frac{1}{2}\|l_a\tilde{y}\|_p . \tag{4.6}$$

The estimates (4.5) and (4.6) imply

$$\left\|\tilde{y}''\right\|_p + \left\|a\,\tilde{r}\tilde{y}'\right\|_p + \left\|a^2\tilde{q}\tilde{y}\right\|_p \le C\left\|\tilde{f}\right\|_p, \qquad C = 2\left(\frac{1}{2} + C_{l_a}\right).$$

By replacing $t = a^{-1}x$, we get the estimate (1.2). □

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