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Continuity of solutions to the G-Laplace equation involving measures

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Abstract. We establish local continuity of solutions to the G-Laplace equation involving measures, i.e.,

$$-\mathrm{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu,$$

where μ is a nonnegative Radon measure satisfying $\mu(B_r(x_0)) \leq Cr^m$ for any ball $B_r(x_0) \subset\subset \Omega$ with $r \leq 1$ and $m > n-1-\delta \geq 0$. The function g is supposed to be nonnegative and C^1-continuous on $[0,+\infty)$, satisfying $g(0) = 0$ and

$$\delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \forall t > 0$$

with positive constants δ and g_0, which generalizes the structural conditions of Ladyzhenskaya–Ural'tseva for an elliptic operator.

Keywords: G-Laplace, Radon measure, Orlicz space, Hölder continuity.

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1 Introduction

Let Ω be an open bounded domain of $\mathbb{R}^n (n \geq 2)$, and μ a nonnegative Radon measure in Ω with $\mu(B_r(x_0)) \leq Cr^m$ for some constant $C > 0$ whenever $B_r(x_0) \subset\subset \Omega$. We consider the equation

$$-\Delta_G u = -\mathrm{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu \quad \text{in } \mathcal{D}'(\Omega), \tag{1.1}$$

where g is a nonnegative C^1-function on $[0,+\infty)$, satisfying $g(0) = 0$ and the following structural condition

$$0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \quad \forall\ t > 0,\ \delta, g_0 \text{ are positive constants.} \tag{1.2}$$

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The structural condition of g was introduced by Tolksdorf in 1983 [14], which is a natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations (see [10]). The conditions of g imply that the operator Δ_G includes not only the p-Laplace operator Δ_p where $g(t) = t^{p-1}$ and $\delta = g_0 = p - 1$, but also the case of a variable exponent $p = p(t) > 0$:

$$-\Delta_G u = -\text{div}\,(|\nabla u|^{p(|\nabla u|)-2}\nabla u),$$

corresponding to set $g(t) = t^{p(t)-1}$, for which (1.2) holds if $\delta \leq t(\ln t)p'(t) + p(t) - 1 \leq g_0$ for all $t > 0$. Another typical example of g is $g(t) = t^p \log(at + b)$ with $p, a, b > 0$ where in this case $\delta = p$ and $g_0 = p + 1$. More examples can be found in [2,3,6,17] etc.

Let $G(t) = \int_0^t g(s)\text{d}s$. Under assumption (1.2), G is an increasing, C^2-continuous and convex function, which is an N-function satisfying Δ_2-condition (see [1]). Thus our class of operators will be considered in the setting of Orlicz spaces. We recall the definitions of Orlicz and Orlicz–Sobolev spaces together with their respective norms (see [1])

$$\begin{aligned}
L^G(\Omega) &= \{u \in L^1(\Omega);\ \int_\Omega G(|u(x)|)\text{d}x < +\infty\},\\
\|u\|_{L^G(\Omega)} &= \inf\left\{k > 0;\ \int_\Omega G\left(\frac{|u(x)|}{k}\right)\text{d}x \leq 1\right\},\\
W^{1,G}(\Omega) &= \{u \in L^G(\Omega); |\nabla u| \in L^G(\Omega)\},\\
\|u\|_{W^{1,G}(\Omega)} &= \|u\|_{L^G(\Omega)} + \|\nabla u\|_{L^G(\Omega)}.
\end{aligned}$$

Under the assumption (1.2), $W^{1,G}(\Omega)$ is a reflexive and separable Banach space (see [1]).

We shall call a solution of (1.1) any function $u \in W^{1,G}_{\text{loc}}(\Omega)$ that satisfies

$$\int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla\varphi\text{d}x = \int_\Omega \varphi\text{d}\mu \quad \forall\varphi \in \mathcal{D}(\Omega).$$

If $\mu \equiv 0$ in a domain $D \subset \Omega$, we say that u is G-harmonic in D.

We now introduce regularities of related elliptic equations involving measures. In 1994, Kilpeläinen considered the situation of the p-Laplacian and proved that if μ satisfies $\mu(B_r) \leq Cr^{n-p+\alpha(p-1)}$ for some positive constants C and $\alpha \in (0, 1]$, then any solution to the p-Laplace equation

$$-\Delta_p u = -\text{div}\,(|\nabla u|^{p-2}\nabla u) = \mu \tag{1.3}$$

is $C^{0,\beta}_{\text{loc}}$-continuous for each $\beta \in (0, \alpha)$ (see [7]). This result was improved by Kilpeläinen and Zhong in 2002, showing that every solution of (1.3) is in fact Hölder continuous with the same exponent α as the one in the assumption $\mu(B_r) \leq Cr^{n-p+\alpha(p-1)}$ (see [8]). In 2010, the p-Laplace problem (1.3) was extended by Lyaghfouri to the case with variable exponents, i.e., considering

$$-\text{div}\,(|\nabla u|^{p(x)-2}\nabla u) = \mu. \tag{1.4}$$

Under certain assumptions on the function $p(x)$ and the assumption $\mu(B_r) \leq Cr^{n-p(x)+\alpha(p(x)-1)}$ for some positive constants C and $\alpha \in (0, 1]$, the author proved that any bounded solution of (1.4) is $C^{0,\alpha}_{\text{loc}}$-continuous with the same exponent α (see [11]).

When focusing on the problem governed by G-Laplacian, if $\mu(B_r(x_0)) \leq Cr^m$ with $m \in [n - 1, n)$, Challal and Lyaghfouri proved that any solution of (1.1) is $C^{0,\alpha}_{\text{loc}}$-continuous with

$\alpha = \frac{m-n+1+\delta}{1+g_0}$ (see [3]). Particularly, if $m = n-1$, any bounded solution is $C^{0,\alpha}_{\text{loc}}$-continuous with any $\alpha \in \big(0, \frac{\delta}{g_0}\big)$ (see Theorem 3.3 in [3]). In 2011, these regularities were improved by Challal and Lyaghfouri in [5], showing that any local bounded solution of (1.1) is $C^{0,\alpha}_{\text{loc}}$-continuous with any $\alpha \in \big(0, \frac{m-n+1+\delta}{g_0}\big)$ provided $m > n-1-\delta$. Note that under the assumption of non-decreasing monotonicity on $\frac{g(t)}{t}$, Zheng, Feng and Zhang obtained local $C^{1,\alpha}$-continuity of solutions for $m > n$ and local Hölder continuity with a small exponent for some $m < n$ in 2015 (see [15]).

In this paper, we continue the work of Challal, Lyaghfouri and Zheng et al. by improving the regularity of solutions of the equation (1.1). Particularly, we prove the $C^{0,\alpha}_{\text{loc}}$-continuity of solutions with any $\alpha \in (0,1)$ if $m = n-1$. More precisely, for any $m > n-1-\delta$ and without any monotonicity assumption on $\frac{g(t)}{t}$, we have the following results.

Theorem 1.1. *Assume that μ satisfies (1.1) with $m > n-1-\delta \geq 0$. For any local bounded solution $u \in W^{1,G}_{\text{loc}}(\Omega)$ of (1.1), we have the following regularities:*

(i) If $m > n$, then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ with any $\alpha \in \big(0, \min\{\frac{\sigma}{1+g_0}, \frac{m-n}{2(1+g_0)}\}\big)$, where σ is the same as in Lemma 2.5.

(ii) If $m \in [n-1, n)$, then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ with any $\alpha \in (0,1)$.

(ii) If $n-1-\delta < m < n-1$, then $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ with any $\alpha \in (0, \frac{m-n+1+\delta}{\delta})$.

Remark 1.2. In [7], the author proved for the p-Laplacian problem that $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ with any $\alpha \in (0,1)$ provided $m = n-1$. In this paper we not only improve the results of [3,5] and [15], but also extend the problem in [7] to general equations governed by a large class of degenerate and singular elliptic operators.

Throughout this paper, without special states, by B_R and B_r we denote the balls contained in Ω with the same center. Moreover, $B_r \subset\subset B_R \subset\subset \Omega$ and $\|u\|_{L^\infty(B_R)} \leq M$ for some constant $M > 0$. $(u)_r = \frac{1}{|B_r|}\int_{B_r} u dx$ be the average value of u on the ball B_r.

2 Preliminary

In this section, we state some auxiliary results which will be used throughout this paper. We begin with some properties of the function G.

Lemma 2.1 ([13, Lemma 2.1, Remark 2.1])**.** *The function G has the following properties:*

(G_1) *G is convex and C^2-continuous.*

(G_2) $\frac{tg(t)}{1+g_0} \leq G(t) \leq tg(t), \ \forall t \geq 0.$

(G_3) $\min\{s^{\delta+1}, s^{g_0+1}\}\frac{G(t)}{1+g_0} \leq G(st) \leq (1+g_0)\max\{s^{\delta+1}, s^{g_0+1}\}G(t), \ \forall s,t \geq 0.$

(G_4) $G(a+b) \leq 2^{g_0}(1+g_0)(G(a)+G(b)), \ \forall a,b \geq 0.$

For much more properties of G and problems governed by the operator Δ_G, please see [2–6, 13, 15, 16, 18, 19] etc.

Lemma 2.2 ([9, Lemma 2.7]). *Let $\phi(s)$ be a non-negative and non-decreasing function. Suppose that*

$$\phi(r) \le C_1 \left(\frac{r}{R}\right)^{\alpha} \phi(R) + C_1 R^{\beta},$$

for all $r \le R \le R_0$, with positive constants α, β and C_1. Then, for any $\tau < \min\{\alpha, \beta\}$, there exists a constant $C_2 = C_2(C_1, \alpha, \beta, \tau)$ such that for any $r \le R \le R_0$ we have

$$\phi(r) \le C_2 r^{\tau}.$$

The following lemmas are some properties of G-harmonic functions.

Lemma 2.3 ([13, Theorem 2.3]). *Assume $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$. Let h be a weak solution of*

$$\Delta_G h = 0 \quad \text{in } B_R, \qquad h - u \in W_0^{1,G}(B_R),$$

then

$$\int_{B_R} (G(|\nabla u|) - G(|\nabla h|))dx \ge C\left(\int_{A_2} G(|\nabla u - \nabla h|)dx + \int_{A_1} \frac{g(|\nabla u|)}{|\nabla u|} |\nabla u - \nabla h|^2 dx \right),$$

where $A_1 = \{x \in B_R; |\nabla u - \nabla h| \le 2|\nabla u|\}$, $A_2 = \{x \in B_R; |\nabla u - \nabla h| > 2|\nabla u|\}$, and $C = C(\delta, g_0) > 0$.

Lemma 2.4 ([13, Lemma 2.7]). *Let $h \in W^{1,G}(B_R)$ be a weak solution of $\Delta_G h = 0$ in B_R. Then $h \in C^{1,\alpha}_{\mathrm{loc}}(B_R)$. Moreover, for every $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta, g_0) > 0$ such that*

$$\int_{B_r} G(|\nabla h|)dx \le C r^{\lambda}, \qquad \forall r \in (0, R].$$

Proof. Indeed, we have (see [10, p. 345])

$$\begin{aligned}\int_{B_r} G(|\nabla h|)\mathrm{d}x &\le C\left(\frac{r}{R}\right)^n \int_{B_R} G(|\nabla h|)\mathrm{d}x \\ &\le C\left(\frac{r}{R}\right)^n \int_{B_R} G(|\nabla h|)\mathrm{d}x + C R^n, \qquad \forall r \in (0, R].\end{aligned}$$

Then for any $\lambda \in (0, n)$, we obtain by Lemma 2.3

$$\int_{B_r} G(|\nabla h|)\mathrm{d}x \le C r^{\lambda}, \qquad \forall r \in (0, R],$$

which completes the proof. □

Lemma 2.5 (Comparison with G-harmonic functions [15, Lemma 3.1]). *Assume $u \in W^{1,G}(B_R)$. Let $h \in W^{1,G}(B_R)$ be a weak solution of $\Delta_G h = 0$ in B_R. Then there exist $\sigma \in (0, 1)$ and $C = C(n, \delta, g_0) > 0$ such that*

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)dx \le C\left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|)dx + C\int_{B_R} G(|\nabla u - \nabla h|)dx, \quad \forall r \in (0, R].$$

Lemma 2.6. *Assume $u \in W^{1,G}_{\mathrm{loc}}(\Omega)$. Let $B_R \subset\subset \Omega$ and $h \in W^{1,G}(B_R)$ be a weak solution of*

$$\Delta_G h = 0 \quad \text{in } B_R, \quad h - u \in W_0^{1,G}(B_R).$$

Then for any $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta, g_0, \|u\|_{L^\infty(B_R)}) > 0$ such that

$$\int_{B_R} G(|\nabla u - \nabla h|)dx \le C R^m + C R^{\frac{m+\lambda}{2}}.$$

Proof. Firstly, convexity of G gives

$$\begin{aligned}\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x &\leq \int_{B_R}\frac{g(|\nabla u|)}{|\nabla u|}\nabla u(\nabla u-\nabla h)\mathrm{d}x \\ &= \int_{B_R}(u-h)\mathrm{d}\mu \qquad (2.1)\\ &\leq C\mu(B_R) \\ &\leq CR^m, \qquad (2.2)\end{aligned}$$

where we used the boundedness of u which forces h to be bounded too.

Let A_1 and A_2 be defined as in Lemma 2.3. By Lemma 2.3, there exists a constant $C = C(\delta, g_0) > 0$ such that

$$\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x \geq C\int_{A_2}G(|\nabla u-\nabla h|)\mathrm{d}x \qquad (2.3)$$

and

$$\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x \geq C\int_{A_1}\frac{g(|\nabla u|)}{|\nabla u|}|\nabla u-\nabla h|^2\mathrm{d}x. \qquad (2.4)$$

By (G_2), $\frac{G(t)}{t}$ is increasing in $t>0$. It follows from (G_2), (G_3), (2.2), (2.3), (2.4), Lemma 2.3 and 2.4 that

$$\begin{aligned}\int_{A_1}G(|\nabla u-\nabla h|)\mathrm{d}x &= \int_{A_1}\frac{G(|\nabla u-\nabla h|)}{|\nabla u-\nabla h|}(|\nabla u-\nabla h|)\mathrm{d}x \\ &\leq \int_{A_1}\frac{G(2|\nabla u|)}{2|\nabla u|}|\nabla u-\nabla h|\mathrm{d}x \\ &\leq C\int_{A_1}\frac{G(|\nabla u|)}{|\nabla u|}|\nabla u-\nabla h|\mathrm{d}x \\ &\leq C\left(\int_{A_1}\frac{G(|\nabla u|)}{|\nabla u|^2}|\nabla u-\nabla h|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{A_1}G(|\nabla u|)\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{A_1}\frac{g(|\nabla u|)|\nabla u|}{|\nabla u|^2}|\nabla u-\nabla h|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{A_1}G(|\nabla u|)\mathrm{d}x\right)^{\frac{1}{2}} \\ &= C\left(\int_{A_1}\frac{g(|\nabla u|)}{|\nabla u|}|\nabla u-\nabla h|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{B_R}G(|\nabla u|)\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{B_R}G(|\nabla u|)\mathrm{d}x\right)^{\frac{1}{2}} \\ &= C\left(\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\right)^{\frac{1}{2}} \\ &\quad\cdot\left(\int_{B_R}(G(|\nabla u|)-G(|\nabla h|)+G(|\nabla h|))\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq C\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x \\ &\quad + C\left(\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{B_R}G(|\nabla h|)\mathrm{d}x\right)^{\frac{1}{2}}, \\ &\leq CR^m + CR^{\frac{m+\lambda}{2}}, \qquad (2.5)\end{aligned}$$

where in the last inequality but one we used $(a+b)^\gamma \le a^\gamma + b^\gamma$ for any $a \ge 0, b \ge 0$ and $\gamma \in (0,1)$. By (2.2), (2.3) and (2.5), we have

$$\begin{aligned}\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x &= \int_{A_2} G(|\nabla u - \nabla h|)\mathrm{d}x + \int_{A_1} G(|\nabla u - \nabla h|)\mathrm{d}x \\ &\le C\int_{B_R} (G(|\nabla u|) - G(|\nabla h|))\mathrm{d}x + CR^m + CR^{\frac{m+\lambda}{2}} \\ &\le CR^m + CR^{\frac{m+\lambda}{2}}. \qquad \square\end{aligned}$$

3 Proof of Theorem 1.1

Proof of Theorem 1.1. Let h be a G-harmonic function in B_R that agrees with u on the boundary, i.e.,

$$\operatorname{div} \frac{g(|\nabla h|)}{|\nabla h|}\nabla h = 0 \text{ in } B_R \quad \text{and} \quad h - u \in W_0^{1,G}(B_R).$$

By Lemma 2.5 and Lemma 2.6, for any $r \le R$ there holds

$$\begin{aligned}\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x &\le C\left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + C\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x \\ &\le C\left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + CR^m + CR^{\frac{m+\lambda}{2}},\end{aligned}$$

where λ is an arbitrary constant in $(0,n)$.

(i) If $m > n$, then we have

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + CR^{\frac{m+\lambda}{2}}.$$

Since $m > n$ and λ is an arbitrary constant in $(0,n)$, one may choose λ satisfying $\frac{m+\lambda}{2} > n$. In view of Lemma 2.2, we conclude that for any $\tau < \min\{\sigma, \frac{m+\lambda}{2} - n\}$ there holds

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le Cr^{n+\tau}, \qquad \forall\, r \le R. \tag{3.1}$$

Now we claim that

$$\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x \le Cr^{n+\frac{\tau}{1+g_0}}, \qquad \forall\, r \le R. \tag{3.2}$$

Indeed, for r satisfying $r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x \le r^{\frac{\tau}{1+g_0}}$, (3.2) holds with $C = 1$. Now for r satisfying $r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x > r^{\frac{\tau}{1+g_0}}$, we infer from the increasing monotonicity of $\frac{G(t)}{t}$ in $t > 0$,

$$\frac{G\left(r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x\right)}{r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x} \ge \frac{G\left(r^{\frac{\tau}{1+g_0}}\right)}{r^{\frac{\tau}{1+g_0}}}.$$

It follows from (G_2) and (G_3)

$$\begin{aligned}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x&\le\frac{r^{n+\frac{\tau}{1+g_0}}}{G\big(r^{\frac{\tau}{1+g_0}}\big)}G\Big(r^{-n}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\Big)\\&\le\frac{Cr^{n+\frac{\tau}{1+g_0}}}{r^{\tau}G(1)}G\Big(r^{-n}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\Big)\\&\le\frac{Cr^{n+\frac{\tau}{1+g_0}}}{r^{\tau}g(1)}G\Big(r^{-n}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\Big).\end{aligned}\tag{3.3}$$

Note that convexity of G and (3.1) imply that

$$G\Big(\frac{1}{|B_r|}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\Big)\le\frac{1}{|B_r|}\int_{B_r}G(|\nabla u-(\nabla u)_r|)\mathrm{d}x\le Cr^{\tau}.\tag{3.4}$$

By (G_3), (3.3) and (3.4), one may get

$$\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\le Cr^{n+\frac{\tau}{1+g_0}},$$

where C depends only on $g(1), g_0$ and the volume of the unit ball. Now we have proven that (3.2) holds for any $r\le R$. Thus $u\in C^{1,\frac{\tau}{1+g_0}}_{\mathrm{loc}}(\Omega)$ by Campanato's embedding theorem. Due to the arbitrariness of $\lambda\in(0,n)$, $\tau>0$ can be arbitrary with $\tau<\min\{\sigma,\frac{m-n}{2}\}$, which guarantees that Theorem 1.1 (i) holds true.

(ii) If $m\in[n-1,n]$, we only prove for $m=n-1$ due to the fact that $\mu(B_r)\le Cr^m\le Cr^{n-1}$ with small r. By (G_4), Lemma 2.4 and Lemma 2.6, we get

$$\begin{aligned}\int_{B_r}G(|\nabla u|)\mathrm{d}x&\le C\int_{B_r}G(|\nabla u-\nabla h|)\mathrm{d}x+C\int_{B_r}G(|\nabla h|)\mathrm{d}x\\&\le Cr^m+Cr^{\frac{m+\lambda}{2}}+Cr^{\lambda}\\&\le Cr^m,\end{aligned}$$

where in the last inequality we let $n>\lambda>n-1=m$.

We claim that for any $r\le R<1$ with $B_R\subset\subset\Omega$ and some positive constant C independent of r, there holds

$$\int_{B_r}|\nabla u|\mathrm{d}x\le Cr^{n-1+\alpha_0},\tag{3.5}$$

with some $\alpha_0\in(0,1)$.

Indeed, for $r\le R$ satisfying

$$r^{-n+1-\alpha_0}\int_{B_r}|\nabla u|\mathrm{d}x\le 1,\tag{3.6}$$

(3.5) holds with $C=1$. For $r\le R$ satisfying

$$r^{-n+1-\alpha_0}\int_{B_r}|\nabla u|\mathrm{d}x\ge 1,$$

due to the increasing monotonicity of $F(t)=G(t)-G(1)t$ in $t\ge 1$, it follows

$$G\Big(r^{-n+1-\alpha_0}\int_{B_r}|\nabla u|\mathrm{d}x\Big)\ge G(1)\cdot r^{-n+1-\alpha_0}\int_{B_r}|\nabla u|\mathrm{d}x.$$

Then we have

$$\begin{aligned}\int_{B_r}|\nabla u|\mathrm{d}x &\le Cr^{n-1+\alpha_0}(r^{1-\alpha_0})^{1+\delta}G\left(r^{-n}\int_{B_r}|\nabla u|\mathrm{d}x\right)\\ &\le Cr^{n-1+\alpha_0}\cdot(r^{1-\alpha_0})^{1+\delta}\frac{1}{|B_r|}\int_{B_r}G(|\nabla u|)\mathrm{d}x\\ &\le Cr^{n-1+\alpha_0+(1-\alpha_0)(1+\delta)}\cdot r^{-n}\cdot r^m\\ &= Cr^{n-1+\alpha_0+(1-\alpha_0)(1+\delta)+m-n}.\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we may choose $\alpha_0=\alpha_0+(1-\alpha_0)(1+\delta)+m-n$, i.e., $\alpha_0=1-\frac{n-m}{1+\delta}$ such that (3.5) holds for all $r\le R$.

For $m=n-1$, we conclude that $u\in C^{0,\alpha_0}_{\text{loc}}(\Omega)$ by Morrey Theorem (see page 30, [12]) with $\alpha_0=\frac{\delta}{1+\delta}$.

Note that $\inf_{B_r}u\le\inf_{B_r}h$ (see the proof of Theorem 3.3 in [3]). Then by (2.1) and Lemma 2.4, for λ larger than $m+\alpha_0$, we have

$$\begin{aligned}\int_{B_r}G(|\nabla u|)\mathrm{d}x &\le \int_{B_r}(u-h)\mathrm{d}\mu+\int_{B_r}G(|\nabla h|)\mathrm{d}x\\ &\le (\sup_{B_r}u-\inf_{B_r}h)\mu(B_r)+\int_{B_r}G(|\nabla h|)\mathrm{d}x\\ &\le (\sup_{B_r}u-\inf_{B_r}u)\mu(B_r)+\int_{B_r}G(|\nabla h|)\mathrm{d}x\\ &\le C\operatorname{osc}(u,B_r)r^m+Cr^\lambda\\ &\le Cr^{\alpha_0+m}+Cr^\lambda\\ &\le Cr^{m+\alpha_0},\end{aligned}$$

where $\operatorname{osc}(u,B_r)=\sup_{B_r}u-\inf_{B_r}u$. Arguing as (3.5), we get $u\in C^{0,\alpha_1}_{\text{loc}}(\Omega)$ with

$$\alpha_1=1-\frac{n-(m+\alpha_0)}{1+\delta}=\frac{\delta}{1+\delta}+\frac{\alpha_0}{1+\delta}.$$

Repeating this process, we get $u\in C^{0,\alpha_k}_{\text{loc}}(\Omega)$ with

$$\alpha_k=\frac{\delta}{1+\delta}+\frac{\alpha_{k-1}}{1+\delta}.$$

Finally, we have $\alpha_k=\frac{\alpha_0}{(1+\delta)^k}+\delta\sum_{j=1}^k\frac{1}{(1+\delta)^j}$, which leads to $\lim_{k\to\infty}\alpha_k$=1, and the result follows.

(iii) If $n-1-\delta<m<n-1$, checking the proof and repeating the process as above, we may get $\alpha_0=1-\frac{n-m}{1+\delta}$, $\alpha_1=\frac{1+\delta+m-n}{1+\delta}+\frac{\alpha_0}{1+\delta},\ldots,\alpha_k=\frac{1+\delta+m-n}{1+\delta}+\frac{\alpha_{k-1}}{1+\delta}$. Finally, one has $u\in C^{0,\alpha}_{\text{loc}}(\Omega)$ for any $\alpha\in(0,\frac{1+\delta+m-n}{\delta})$. □

References

[1] R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, Pure and Applied Mathematics (Amsterdam), Vol. 140, Elsevier/Academic Press, Amsterdam, 2003. MR2424078

[2] J. E. M. Braga, D. R. Moreira, Uniform Lipschitz regularity for classes of minimizers in two phase free boundary problems in Orlicz spaces with small density on the negative phase, *Ann. Inst. H. Poincaré, Anal. Non Linéaire.* **31**(2014), No. 4, 823–850. https://doi.org/10.1016/j.anihpc.2013.07.006; MR3249814; Zbl 1301.49097

[3] S. Challal, A. Lyaghfouri, Hölder continuity of solutions to the A-Laplace equation involving measures, *Common. Pure Appl. Anal.* **8**(2009), No. 5, 1577–1583. https://doi.org/10.3934/cpaa.2009.8.1577; MR2505287; Zbl 1179.35336

[4] S. Challal, A. Lyaghfouri, Porosity of free boundaries in A-obstacle problems, *Nonlinear Anal.* **70**(2009), No. 7, 2772–2778. https://doi.org/10.1016/j.na.2008.04.002; MR2499745; Zbl 1166.35385

[5] S. Challal, A. Lyaghfouri, Removable sets for A-harmonic functions, *Z. Anal. Anwend.* **30**(2011), No. 4, 421–433. https://doi.org/10.4171/ZAA/1442; MR2853964; Zbl 1260.35046

[6] S. Challal, A. Lyaghfouri, J. F. Rodrigues, On the A-obstacle problem and the Hausdorff measure of its free boundary, *Ann. Mat. Pura Appl. (4)* **191**(2012), No. 1, 113–165. https://doi.org/10.1007/s10231-010-0177-7; MR2886164; Zbl 1235.35285

[7] T. Kilpeläinen, Hölder continuity of solutions to quasilinear elliptic equations involving measures, *Potential Anal.* **3**(1994), No. 3, 265–272. https://doi.org/10.1007/bf01468246; MR1290667; Zbl 0813.35016

[8] T. Kilpeläinen, X. Zhong, Removable set for continuous solutions of quasilinear elliptic equations, *Proc. Amer. Math. Soc.* **130**(2002), No. 6, 1681–1688. https://doi.org/10.2307/2699762; MR1887015

[9] R. Leitão, O. S. de Queiroz, E. V. Teixeira, Regularity for degenerate two-phase free boundary problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32**(2015), No. 4, 741–762. https://doi.org/10.1016/j.anihpc.2014.03.004; MR3390082

[10] G. M. Lieberman, The natural generalization of the natural conditions of Ladyzhensaya and Ural'tseva for elliptic equations, *Comm. Partial Differ. Equ.* **16**(1991), No. 2–3, 311–361. https://doi.org/10.1080/03605309108820761; MR1104103

[11] A. Lyaghfouri, Hölder continuity of $p(x)$-superharmonic functions, *Nonlinear Anal.* **73**(2010), No. 8, 2433–2444. https://doi.org/10.1016/j.na.2010.06.016; MR2674081; Zbl 1194.35482

[12] J. Malý, W. P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs, Vol. 51, Providence (RI): Amer. Math. Soc., 1997. https://doi.org/10.1090/surv/051; MR1461542

[13] S. Martínez, N. Wolanski, A minimum problem with free boundary in Orlicz spaces, *Adv. Math.* **218**(2008), No. 6, 1914–1971. https://doi.org/10.1016/j.aim.2008.03.028; MR2431665; Zbl 1170.35030

[14] P. Tolksdorf, On the Dirichlet problem for quasilinear equations in domains with conical boundary points, *Comm. Partial Differ. Equ.* **8**(1983), No. 7, 773–817. https://doi.org/10.1080/03605308308820285; MR0700735

[15] J. Zheng, B. Feng, Z. Zhang, Regularity of solutions to the G-Laplace equation involving measures, *Z. Anal. Anwend.* **34**(2015), No. 2, 165–174. https://doi.org/10.4171/ZAA/1534; MR3336258; Zbl 1323.35191

[16] J. Zheng, B. Feng, P. Zhao, Regularity of minimizers in the two-phase free boundary problems in Orlicz–Sobolev spaces, *Z. Anal. Anwend.* **36**(2017), No. 1, 37–47. https://doi.org/10.4171/ZAA/1578; MR3638967; Zbl 1359.35066

[17] J. Zheng, X. Guo, Lyapunov-type inequalities for ψ-Laplacian equations, chinaXiv:201805.00171, 2018. https://doi.org/10.12074/201805.00171

[18] J. Zheng, L. S. Tavares, C. O. Alves, A minimum problem with free boundary and subcritical growth in Orlicz spaces, preprint published on arXiv:1809.08518v2, 2018.

[19] J. Zheng, Z. Zhang, P. Zhao, A minimum problem with two-phase free boundary in Orlicz spaces, *Monatsh. Math.* **172**(2013), No. 3–4, 441–475. https://doi.org/10.1007/s00605-013-0557-3; MR3128005; Zbl 1285.35135