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On singular p-Laplacian boundary value problems involving integral boundary conditions

Dang Dinh Hai[✉] and **Xiao Wang**

Department of Mathematics and Statistics, Mississippi State University
Mississippi State, MS 39762, USA

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Abstract. We prove the existence of positive solutions for the p-Laplacian equations

$$-(\phi(u'))' = \lambda f(t,u), \qquad t \in (0,1)$$

with integral boundary conditions. Here λ is a positive parameter, $\phi(s) = |s|^{p-2}s$, $p > 1$, $f : (0,1) \times (0,\infty) \to \mathbb{R}$ is p-superlinear or p-sublinear at ∞ in the second variable and is allowed be singular at $t = 0, 1$ and $u = 0$.

Keywords: p-Laplacian, integral boundary conditions, positive solutions.

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1 Introduction

Consider the one-dimensional p-Laplacian equation

$$-(\phi(u'))' = \lambda f(t,u), \qquad t \in (0,1) \tag{1.1}$$

with boundary conditions

$$au(0) - bu'(0) = \int_0^1 g(t)u(t)dt, \qquad u'(1) = 0, \tag{1.2}$$

or

$$au(0) - bu'(0) = \int_0^1 g(t)u(t)dt, \qquad u(1) = 0, \tag{1.3}$$

where $\phi(s) = |s|^{p-2}s, p > 1, a > 0, b \geq 0, g : (0,1) \to [0,\infty), f : (0,1) \times (0,\infty) \to \mathbb{R}$, and λ is a positive parameter.

Equation (1.1) arises in some physical models such as non-Newtonian fluids, chemical reactions, and population biology, see e.g. [2,3,7,8]. The integral boundary conditions occur in thermal conduction, semiconductor and hydrodynamic problems [5,6,11].

[✉] Corresponding author. Email: dang@math.msstate.edu

In [16], Zhang and Feng studied the existence of positive solutions to (1.1)–(1.2)) with $b > 0$ for a certain range for λ when $f(t,u) = \omega(t)F(t,u)$ is nonnegative and nonsingular in u under a variety of assumptions involving $\lim_{u\to 0^+} \frac{F(t,u)}{u^{p-1}}$ and/or $\lim_{u\to\infty} \frac{F(t,u)}{u^{p-1}}$. In particular, they showed in [16, Theorem 1.3] that (1.1)–(1.2) has a positive solution u_λ for all $\lambda > 0$ if either $\lim_{u\to 0^+} \frac{F(t,u)}{u^{p-1}} = 0$ and $\lim_{u\to\infty} \frac{F(t,u)}{u^{p-1}} = \infty$ (p-superlinear) uniformly for $t \in [0,1]$, or $\lim_{u\to 0^+} \frac{F(t,u)}{u^{p-1}} = \infty$ and $\lim_{u\to\infty} \frac{F(t,u)}{u^{p-1}} = 0$ (p-sublinear) uniformly for $t \in [0,1]$. In addition, $\lim_{\lambda\to 0} \|u_\lambda\|_\infty = \infty$ in the former case and $\lim_{\lambda\to 0} \|u_\lambda\|_\infty = 0$ in the latter case. The approach used in [16] was fixed point theory in a cone, and the nonnegativity assumption of f there was essential to ensure that the equivalent fixed point mapping maps the cone of nonnegative continuous functions into itself. In this note, we shall establish the existence of positive solutions to (1.1) with boundary condition (1.2) or (1.3) when $f(t,u)$ can be $\pm\infty$ at $u = 0$ and is either p-superlinear or p-sublinear at ∞, which has not been considered in the literature to the best of our knowledge.

Our results when applied to the model equation

$$-(\phi(u'))' = \frac{\lambda}{t^\alpha}\left(\frac{c}{u^\delta} + u^\rho\right), \qquad t \in (0,1), \tag{1.4}$$

where $c \in \mathbb{R}$, $\alpha, \delta \in [0,1), \rho > 0$, give the existence of a large positive solution to (1.4) with boundary conditions (1.2) or (1.3) for λ small when $\rho > p-1$, or for λ large when $\rho < p-1$. We refer to [4,9,10,12–15] for results related to (1.1) with integral boundary conditions.

Define $q(t) = t$ if (1.2) holds and $q(t) = \min(t, 1-t)$ if (1.3) holds. We shall make the following assumptions:

(A1) $g : (0,1) \to [0,\infty)$ is integrable and $\int_0^1 g(t)dt < a$.

(A2) $f : (0,1) \times (0,\infty) \to \mathbb{R}$ is a Carathéodory function, that is $f(.,z)$ is measurable for $z > 0$ and $f(t,\cdot)$ is continuous for a.e. $t \in (0,1)$.

(A3) There exists a constant $\delta \in [0,1)$ such that for each $k > 0$, there exists a function $\gamma_k \colon (0,1) \to [0,\infty)$ with $\gamma_k/q^\delta \in L^1(0,1)$ such that

$$|f(t,z)| \le \gamma_k(t)z^{-\delta}$$

for a.e. $t \in (0,1)$ and $z \in (0,k]$.

(A4) There exist $\gamma \in L^1(0,1)$ with $\gamma > 0$ a.e. on $(0,1)$ and $\nu \in \{0,\infty\}$ such that

$$\lim_{z\to\infty} \frac{f(t,z)}{\gamma(t)z^{p-1}} = \nu,$$

uniformly for a.e. $t \in (0,1)$.

By a solution of (1.1) with boundary condition (1.2) (resp. (1.3)), we mean a function $u \in C^1[0,1]$ with $\phi(u')$ absolutely continuous on $[0,1]$, and satisfying (1.2) (resp. (1.3)). Our main result is the following.

Theorem 1.1. *Let (A1)–(A3) hold.*

(i) If (A4) holds with $\nu = \infty$ then there exists a constant $\lambda_0 > 0$ such that for $\lambda < \lambda_0$, (1.1) with boundary condition (1.2)) or (1.3) has a positive solution u_λ with $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$.

(ii) If (A4) holds with $\nu = 0$ and $\lim_{z\to\infty} f(t,z) = \infty$ uniformly for a.e. $t \in (0,1)$ then there exists a constant $\tilde\lambda_0 > 0$ such that for $\lambda > \tilde\lambda_0$, (1.1) with boundary condition (1.2) or (1.3) has a positive solution u_λ with $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$.

2 Preliminary result

We shall denote the norm in $L^p(0,1)$ by $\|\cdot\|_p$.

Lemma 2.1. *Let $h \in L^1(0,1)$ and let (A1) hold. Then the equation*

$$-(\phi(u'))' = h \quad \text{on } (0,1) \tag{2.1}$$

with boundary condition (1.2) or (1.3) has a unique solution $u \equiv Th \in C^1[0,1]$. Furthermore,

$$\|u\|_\infty \le M\phi^{-1}(\|h\|_1), \tag{2.2}$$

where $M = \max\left(\frac{a+b}{a-\|g\|_1}, 2^{\frac{1}{p-1}}\right)$, and the map $T : L^1(0,1) \to C[0,1]$ is completely continuous.

Proof. Suppose (1.2) holds. By integrating (2.1) and using (1.2), it follows that (2.1) with boundary condition (1.2) has a unique solution u, given by

$$u(t) = C + \int_0^t \phi^{-1}\left(\int_s^1 h\right) ds,$$

where

$$C = \frac{b\phi^{-1}\left(\int_0^1 h\right) + \int_0^1 g(t)\left(\int_0^t \phi^{-1}\left(\int_s^1 h\right) ds\right) dt}{a - \int_0^1 g}. \tag{2.3}$$

Since $|C| \le \frac{(b+\|g\|_1)\phi^{-1}(\|h\|_1)}{a-\|g\|_1}$, it follows that

$$\|u\|_\infty \le \frac{a+b}{a-\|g\|_1}\phi^{-1}(\|h\|_1),$$

i.e. (2.2) holds. Since $|u|_{C^1} = \|u\|_\infty + \|u'\|_\infty \le (M+1)\phi^{-1}(\|h\|_1)$, it follows that T maps bounded sets in $L^1(0,1)$ into bounded sets in $C^1[0,1]$ and hence relatively compact subsets in $C[0,1]$. To show continuity of T, let $(h_n) \subset L^1(0,1)$ and $h \in L^1(0,1)$ be such that $h_n \to h$ in $L^1(0,1)$. Let $u_n = Th_n$ and $u = Th$. Then

$$u_n(t) = C_n + \int_0^t \phi^{-1}\left(\int_s^1 h_n\right) ds,$$

where C_n is given by (2.3) with h replaced by h_n. It is easy to see that $C_n \to C$ and hence $u_n \to u$ in $C[0,1]$.

Suppose next that (1.3) holds. By integrating (2.1) and using (1.3), it follows that (2.1) with boundary condition (1.3) has a unique solution u, given by

$$u(t) = \int_t^1 \phi^{-1}\left(-C + \int_0^s h\right) ds,$$

where $C = \phi(u'(0)) \in \mathbb{R}$ is the unique solution of $H(\xi) = 0$, where

$$H(\xi) = \left(a - \int_0^1 g\right)\int_0^1 \phi^{-1}\left(-\xi + \int_0^s h\right) ds - b\phi^{-1}(\xi) + \int_0^1 \left(\int_t^1 g\right)\phi^{-1}\left(-\xi + \int_0^t h\right) dt.$$

Note that the existence and uniqueness of C follows from the fact that H is continuous, decreasing in ξ with $\lim_{\xi\to\infty} H(\xi) = -\infty$ and $\lim_{\xi\to-\infty} H(\xi) = \infty$. Since $H(C) < 0$ if $C > \|h\|_1$ and $H(C) > 0$ if $C < -\|h\|_1$, it follows that $|C| \le \|h\|_1$. Hence

$$\|u\|_\infty \le \phi^{-1}(2\|h\|_1).$$

i.e. (2.2) holds. Since $|u|_{C^1} = \|u\|_\infty + \|u'\|_\infty \leq 2^{\frac{p}{p-1}}\phi^{-1}(\|h\|_1)$, it follows that T maps bounded sets in $L^1(0,1)$ into bounded sets in $C^1[0,1]$. To show continuity of T, let $(h_n) \subset L^1(0,1)$ and $h \in L^1(0,1)$ be such that $h_n \to h$ in $L^1(0,1)$. Let $u_n = Th_n$ and $u = Th$. Then

$$u_n(t) = \int_t^1 \phi^{-1}\left(-C_n + \int_0^s h_n\right) ds$$

for $t \in [0,1]$, where $C_n \in \mathbb{R}$ is the unique solution of $H_n(\xi) = 0$, where

$$H_n(\xi) = \left(a - \int_0^1 g\right)\int_0^1 \phi^{-1}\left(-\xi + \int_0^s h_n\right) ds - b\phi^{-1}(\xi) + \int_0^1 \left(\int_t^1 g\right)\phi^{-1}\left(-\xi + \int_0^t h_n\right) dt.$$

Suppose $C_n > C + \||h_n - h\|_1$. Then

$$-C_n + \int_0^s h_n < -C + \int_0^s h$$

for $s \in [0,1]$, which together with (A1) and the fact that ϕ^{-1} is increasing imply $H_n(C_n) < H(C)$. On the other hand, if $C_n < C - \|h_n - h\|_1$ then

$$-C_n + \int_0^s h_n > -C + \int_0^s h$$

for $s \in [0,1]$, which implies $H_n(C_n) > H(C)$. Hence we reach a contradiction in either case. Consequently,

$$|C_n - C| \leq \|h_n - h\|_1,$$

which implies $C_n \to C$ as $n \to \infty$. Using the formulas for u_n and u, it is easily seen that (u_n) converges to u in $C[0,1]$, which completes the proof. □

Next, we establish a comparison principle.

Lemma 2.2. *Let $0 \leq r_0 < r_1 \leq 1$ and let $h_1, h_2 \in L^1(r_0, r_1)$ be such that $h_1 \geq h_2$ on (r_0, r_1). Let $u, v \in C^1[r_0, r_1]$ satisfy*

$$\begin{cases} -(\phi(u'))' = h_1, \quad -(\phi(v'))' = h_2 \quad \text{on } (r_0, r_1), \\ au(r_0) - bu'(r_0) - \int_{r_0}^{r_1} g(t)u(t)dt \geq av(r_0) - bv'(r_0) - \int_{r_0}^{r_1} g(t)v(t)dt, \\ u'(r_1) \geq v'(r_1) \quad \text{or} \quad u(r_1) \geq v(r_1). \end{cases}$$

Then $u \geq v$ on $[r_0, r_1]$.

Proof. Suppose on the contrary that there exists $r^* \in (r_0, r_1)$ such that $u(r^*) < v(r^*)$. Let $(\alpha, \beta) \subset (r_0, r_1)$ be the largest open interval containing r^* such that $u < v$ on (α, β). Then $u(\alpha) \leq v(\alpha)$ and $u(\beta) \leq v(\beta)$. Multiplying the equation

$$-(\phi(u') - \phi(v'))' = h_1 - h_2 \geq 0 \quad \text{on } (r_0, r_1) \tag{2.4}$$

by $u - v$ and integrating on (α, β), we obtain

$$C_\alpha - C_\beta + \int_\alpha^\beta (\phi(u') - \phi(v'))(u' - v') = \int_\alpha^\beta (h_1 - h_2)(u - v) \leq 0, \tag{2.5}$$

where $C_\kappa = (\phi(u'(\kappa)) - \phi(v'(\kappa))(u(\kappa) - v(\kappa)), \kappa \in \{\alpha, \beta\}$. We claim that $C_\alpha = 0$ and $C_\beta \leq 0$.

To show $C_\alpha = 0$, we verify that $u(\alpha) = v(\alpha)$. If $\alpha > r_0$ then clearly $u(\alpha) = v(\alpha)$. Suppose $\alpha = r_0$ and $u(r_0) < v(r_0)$. We show that this will lead to a contradiction.

Case 1: $u'(r_1) \geq v'(r_1)$.

Then, since $\phi(u') - \phi(v')$ is nonincreasing, it follows that $u' \geq v'$ on $[r_0, r_1]$. Hence $\min_{[r_0,r_1]}(u - v) = (u - v)(r_0)$, which together with the boundary inequality at r_0 and (A1) imply

$$0 \leq b(u' - v')(r_0) \leq a(u - v)(r_0) - \int_{r_0}^{r_1} g(u - v)dt \leq \left(a - \int_{r_0}^{r_1} g\right)(u - v)(r_0) < 0, \tag{2.6}$$

a contradiction.

Case 2. $u(r_1) \geq v(r_1)$.

Then $u(\beta) = v(\beta)$ and hence $u'(\beta) \geq v'(\beta)$, which implies $u' \geq v'$ on $[r_0, \beta]$. In particular, $\min_{[r_0,\beta]}(u - v) = (u - v)(r_0)$. If $\beta = r_1$ then we reach a contradiction as in case 1. Suppose $\beta < r_1$. We shall verify that $u \geq v$ on $[\beta, r_1]$. If not, then there exists an interval $(\alpha_0, \beta_0) \subset (\beta, r_1)$ such that $u < v$ on (α_0, β_0) and $(u - v)(\alpha_0) = (u - v)(\beta_0) = 0$.

Multiplying (2.4) by $u - v$ and integrating on (α_0, β_0) gives

$$\int_{\alpha_0}^{\beta_0} (\phi(u') - \phi(v'))(u' - v') = 0,$$

which implies $u' = v'$ on $[\alpha_0, \beta_0]$. Hence $u = v$ on $[\alpha_0, \beta_0]$, a contradiction. Thus $u - v \geq 0$ on $[\beta, r_1]$, which, together with $\min_{[r_0,\beta]}(u - v) = (u - v)(r_0)$ and $(u - v)(r_0) < 0$, gives $\min_{[r_0,r_1]}(u - v) = (u - v)(r_0)$ and again (2.6) holds, a contradiction.

Thus $u(\alpha) = v(\alpha)$ in both cases. Next, we claim that $C_\beta \leq 0$. If $u(r_1) \geq v(r_1)$ then $u(\beta) = v(\beta)$ while if $u'(r_1) \geq v'(r_1)$ then $u(\beta) = v(\beta)$ if $\beta < r_1$ and $C_\beta = (\phi(u'(\beta)) - \phi(v'(\beta))(u(\beta) - v(\beta)) \leq 0$ if $\beta = r_1$. This proves the claim. Hence (2.5) gives

$$\int_{\alpha}^{\beta} (\phi(u') - \phi(v'))(u' - v') = 0,$$

which implies $u' = v'$ on $[\alpha, \beta]$ and so $u = v + c$ on $[\alpha, \beta]$, where c is a negative constant. Hence $\alpha = r_0$ and $\beta = r_1$. Using the assumption on the boundary at r_0, we obtain $\left(a - \int_{r_0}^{r_1} g\right)c \geq 0$. Thus $c \geq 0$, a contradiction. Hence $u \geq v$ on $[r_0, r_1]$, which completes the proof. □

Lemma 2.3. *Let $h \in L^1(0, 1)$ with $h \geq 0$ and let $u \in C^1[0, 1]$ with $\phi(u')$ absolutely continuous on $[0, 1]$ satisfying*

$$(\phi(u'))' \leq h \quad \text{on } (0, 1), \tag{2.7}$$

with either

$$au(0) - bu'(0) \geq \int_0^1 g(t)u(t)dt, \qquad u'(1) \geq 0, \tag{2.8}$$

or

$$au(0) - bu'(0) \geq \int_0^1 g(t)u(t)dt, \qquad u(1) \geq 0. \tag{2.9}$$

Suppose $\|u\|_\infty > L\phi^{-1}(\|h\|_1)$, where $L = \frac{2ma+b}{a-\|g\|_1}$ and $m = 2^{\left(\frac{2-p}{p-1}\right)+}$. Then

$$u(t) \geq c\|u\|_\infty q(t) \tag{2.10}$$

for $t \in [0, 1]$, where $c = 1/L$.

Proof. By Lemma 2.2, $u \geq v$ on $[0,1]$, where v satisfies

$$(\phi(v'))' = h \quad \text{on } (0,1)$$

with

$$av(0) - bv'(0) = \int_0^1 g(t)v(t)dt, \qquad v'(1) = 0$$

if (2.8) holds, and

$$av(0) - bv'(0) = \int_0^1 g(t)v(t)dt, \qquad v(1) = 0$$

if (2.9) holds. Suppose $\|u\|_\infty = |u(\tau)|$ for some $\tau \in [0,1]$. Then $u(\tau) > 0$. Indeed, if $u(\tau) \leq 0$ then in view of (2.2), we get

$$\|u\|_\infty = -u(\tau) \leq -v(\tau) \leq M\phi^{-1}(\|h\|_1) \leq L\phi^{-1}(\|h\|_1),$$

where M is defined in Lemma 2.1. This contradicts the assumption on $\|u\|_\infty$.

Suppose $\tau \in (0,1)$. Let $w \in C^1[0,\tau]$ be the solution of

$$\begin{cases} (\phi(w'))' = h \quad \text{on } (0,\tau), \\ aw(0) - bw'(0) = \int_0^1 g(t)w(t)dt, \quad w(\tau) = \|u\|_\infty. \end{cases}$$

A calculation shows that

$$w(t) = K_C + \int_0^t \phi^{-1}\left(C + \int_0^s h\right) ds \tag{2.11}$$

for $t \in [0,1]$, where

$$K_C = \frac{b\phi^{-1}(C) + \int_0^1 g(t)\left(\int_0^t \phi^{-1}\left(C + \int_0^s h\right) ds\right) dt}{a - \|g\|_1}, \tag{2.12}$$

and $C = \phi(w'(0))$ is the unique solution of $H_\tau(\rho) = \|u\|_\infty$, where

$$H_\tau(\rho) = K_\rho + \int_0^\tau \phi^{-1}\left(\rho + \int_0^s h\right) ds.$$

Note that the existence and uniqueness of C follows from the fact that H_τ is increasing in ρ and $\lim_{\rho\to\infty} H_\tau(\rho) = \infty$, $\lim_{\rho\to-\infty} H_\tau(\rho) = -\infty$.

Using Lemma 2.2 with $r_0 = 0$ and $r_1 = \tau$, we deduce that $u \geq w$ on $[0,\tau]$. If $w'(0) \leq 0$ then $C \leq 0$ and hence

$$\begin{aligned}\|u\|_\infty = H_\tau(C) &\leq \frac{\int_0^1 g(t)\left(\int_0^t \phi^{-1}\left(\int_0^s h\right) ds\right) dt}{a - \|g\|_1} + \int_0^\tau \phi^{-1}\left(\int_0^s h\right) ds \\ &\leq \frac{a}{a - \|g\|_1}\phi^{-1}(\|h\|_1) < L\phi^{-1}(\|h\|_1),\end{aligned}$$

a contradiction. Hence $w'(0) > 0$ i.e. $C > 0$.

Using the inequality $(x+y)^r \leq 2^{(r-1)^+}(x^r + y^r)$ for $x, y \geq 0$ with $r = (p-1)^{-1}$, we obtain

$$\phi^{-1}\left(\phi(w'(0)) + \int_0^s h\right) \leq m\left(w'(0) + \phi^{-1}(\|h\|_1)\right),$$

where $m = 2^{\left(\frac{2-p}{p-1}\right)+}$. Hence it follows from (2.11)–(2.12) that

$$\begin{aligned}\|u\|_\infty &\le \frac{bw'(0) + m\|g\|_1\left(w'(0) + \phi^{-1}(\|h\|_1)\right)}{a - \|g\|_1} + m\left(w'(0) + \phi^{-1}(\|h\|_1)\right)\\ &= m_1 w'(0) + M_1\phi^{-1}(\|h\|_1),\end{aligned}$$

where $m_1 = \frac{b+ma}{a-\|g\|_1}$ and $M_1 = \frac{ma}{a-\|g\|_1}$. Consequently,

$$w'(0) \ge \frac{\|u\|_\infty - M_1\phi^{-1}(\|h\|_1)}{m_1} \ge \frac{\|u\|_\infty}{m_1}\left(1 - \frac{M_1}{L}\right) = \frac{\|u\|_\infty}{L}, \tag{2.13}$$

where we have used the assumption $\|u\|_\infty > L\phi^{-1}(\|h\|_1)$.

Since $K_C, h \ge 0$, it follows from (2.11) and (2.13), that

$$u(t) \ge w(t) \ge \phi^{-1}(C)t = w'(0)t \ge \frac{\|u\|_\infty t}{L} \tag{2.14}$$

for $t \in [0, \tau]$.

Next, we establish a lower bound estimate for $u(t)$ in terms of $\|u\|_\infty$ on $[\tau, 1]$. By Lemma 2.2, $u \ge z$ on $[\tau, 1]$, where $z \in C^1[\tau, 1]$ satisfies

$$\begin{cases}(\phi(z'))' = h & \text{on } (\tau, 1),\\ z(\tau) = \|u\|_\infty, & z'(1) = 0\end{cases}$$

if (2.8) holds, and

$$\begin{cases}(\phi(z'))' = h & \text{on } (\tau, 1),\\ z(\tau) = \|u\|_\infty, & z(1) = 0\end{cases}$$

if (2.9) holds.

Suppose first that (2.8) holds. Then

$$z(t) = D + \int_t^1 \phi^{-1}\left(\int_s^1 h\right) ds,$$

where $D = \|u\|_\infty - \int_\tau^1 \phi^{-1}(\int_s^1 h)ds$. Since $L \ge 2m \ge 2$, it follows from Lemma 2.2 with $r_0 = \tau, r_1 = 1$, $b = 0, g \equiv 0$ that

$$u(t) \ge z(t) \ge \|u\|_\infty - \phi^{-1}(\|h\|_1) \ge \|u\|_\infty/2 \tag{2.15}$$

for $t \in [\tau, 1]$. Next, suppose (2.9) holds. Then

$$z(t) = \int_t^1 \phi^{-1}\left(-D - \int_0^s h\right) ds, \tag{2.16}$$

where $D = \phi(z'(0))$ is the unique solution of

$$\int_\tau^1 \phi^{-1}\left(D + \int_0^s h\right) ds = -\|u\|_\infty. \tag{2.17}$$

Since $h \ge 0$, it follows from (2.17) that

$$(1 - \tau)\phi^{-1}(D) \le -\|u\|_\infty,$$

which implies $D \leq -\phi(\|u\|_\infty)$. Hence, since $\|u\|_\infty \geq 2m\phi^{-1}(\|h\|_1)$, it follows that

$$-D - \int_0^s h \geq \phi(\|u\|_\infty) - \|h\|_1 \geq \left(1 - \frac{1}{\phi(2m)}\right)\phi(\|u\|_\infty) \geq \frac{\phi(\|u\|_\infty)}{2}. \tag{2.18}$$

Using (2.16)–(2.18), we obtain

$$u(t) \geq z(t) \geq (1/2)^{\frac{1}{p-1}} \|u\|_\infty (1-t) \tag{2.19}$$

for $t \in [\tau, 1]$. Since

$$L \geq 2m = \begin{cases} 2 & \text{if } p \geq 2, \\ 2^{\frac{1}{p-1}} & \text{if } 1 < p < 2 \end{cases} = \max\left(2^{\frac{1}{p-1}}, 2\right),$$

it follows that $\min\left(L^{-1}, 2^{-1}, 2^{\frac{1}{1-p}}\right) = L^{-1}$, it follows from (2.14), (2.15), and (2.19) that (2.10) holds for the case $\tau \in (0,1)$.

If $\tau = 1$ then $w \in C^1[0,1]$ and (2.14) holds for $t \in [0,1]$, which implies (2.10) since $t \geq q(t)$ for $t \in [0,1]$. Finally, if $\tau = 0$ then $z \in C^1[0,1]$ and (2.15), (2.19) hold for $t \in [0,1]$, which implies (2.10). This completes the proof of Lemma 2.3. □

3 Proof of the main result

Proof. Let $E = C[0,1]$ be equipped with $\|\cdot\|_\infty$. For the rest of the proof, we set $\tilde{\gamma}_k = \gamma_k / q^\delta$, where γ_k is defined in (A3), and recall that $\tilde{\gamma}_k \in L^1(0,1)$. For $v \in C[0,1]$ with $\|v\|_\infty \leq k$ for some $k \geq 1$, it follows from (A3) that there exists $\gamma_k \in L^1(0,1)$ with $\gamma_k \geq 0$ such that

$$|f(t, \tilde{v})| \leq \gamma_k(t) \tilde{v}^{-\delta} \leq \tilde{\gamma}_k(t) \tag{3.1}$$

for a.e. $t \in (0,1)$, where $\tilde{v} = \max(v, q)$. Let $\lambda > 0$. Then, by Lemma 2.1, the equation

$$-(\phi(u'))' = \lambda f(t, \tilde{v}), \qquad t \in (0,1)$$

with boundary condition (1.2) or (1.3) has a unique solution $u \equiv A_\lambda v \in C^1[0,1]$. Let $S_\lambda : E \to L^1(0,1)$ be defined by $S_\lambda v = \lambda f(t, \tilde{v})$. Then S_λ is continuous by the Lebesgue dominated convergence Theorem. By (3.1), S_λ maps bounded sets in E into bounded sets in $L^1(0,1)$. Since $A_\lambda = T \circ S_\lambda$, where T is defined in Lemma 2.1, it follows that $A_\lambda : E \to E$ is completely continuous.

(i) Suppose (A4) holds with $\nu = \infty$.

Let $\lambda \in (0,1)$ satisfy $M\left(\lambda \|\tilde{\gamma}_L\|_1\right)^{\frac{1}{p-1}} < L$, where M and L are defined in Lemma 2.1 and Lemma 2.3 respectively. We claim that

(a) *If $u \in E$ satisfies $u = \theta A_\lambda u$ for some $\theta \in (0,1]$ then $\|u\|_\infty \neq L$.*

Indeed, let $u \in E$ satisfy $u = \theta A_\lambda u$ for some $\theta \in (0,1]$, and suppose $\|u\|_\infty = L$. Then $u/\theta = T(S_\lambda u)$ and (2.2) gives

$$\|u\|_\infty \leq M\theta\phi^{-1}(\|S_\lambda u\|_1) \leq M\left(\lambda \|\tilde{\gamma}_L\|_1\right)^{\frac{1}{p-1}} < L,$$

a contradiction, which proves the claim.

Next, we show that

(b) *There exists a constant $R_\lambda > L$ such that if $u \in E$ satisfies $u = A_\lambda u + \xi$ for some $\xi \geq 0$ then $\|u\|_\infty \neq R_\lambda$.*

Let $u \in E$ satisfy $u = A_\lambda u + \xi$ for some $\xi \geq 0$. Then $u - \xi = A_\lambda u$ and therefore u satisfies

$$-(\phi(u'))' = \lambda f(t, \tilde{u}) \quad \text{on } (0,1)$$

with

$$au(0) - bu'(0) - \int_0^1 g(t)u(t)dt = \left(a - \int_0^1 g\right)\xi \geq 0, \qquad u'(1) = 0 \tag{3.2}$$

if (1.2) holds, and

$$au(0) - bu'(0) - \int_0^1 g(t)u(t)dt = \left(a - \int_0^1 g\right)\xi \geq 0, \qquad u(1) = \xi \geq 0 \tag{3.3}$$

if (1.3) holds.

Let $K > 0$ be such that $f(t,z) > 0$ for a.e. $t \in (0,1)$ and all $z \geq K$. For $z \in (0,K)$, $|f(t,z)| \leq \gamma_K(t)z^{-\delta}$ for a.e. $t \in (0,1)$ in view of (A3). Consequently,

$$f(t,\tilde{u}) \geq -\gamma_K(t)\tilde{u}^{-\delta} \geq -\tilde{\gamma}_K(t) \tag{3.4}$$

for a.e. $t \in (0,1)$. Hence Lemma 2.3 holds with $h = \lambda\tilde{\gamma}_K$ if $\|u\|_\infty > L\phi^{-1}(\lambda\|\tilde{\gamma}_K\|_1)$.

Let $\omega \in C^1\left[\frac{1}{4}, \frac{1}{2}\right]$ satisfy

$$\begin{cases} -(\phi(\omega'))' = \gamma(t) \quad \text{on } (1/4, 1/2), \\ \omega(1/4) = \omega(1/2) = 0, \end{cases}$$

and let $R_0 > 0$ be such that $(\lambda R_0)^{\frac{1}{p-1}} L^{-1}\|\omega\|_\infty > 4$.

Since $\lim_{z\to\infty} \frac{f(t,z)}{\gamma(t)z^{p-1}} = \infty$ uniformly for $t \in (0,1)$, there exists $k_0 > 1$ such that

$$f(t,z) \geq R_0\gamma(t)z^{p-1}$$

for a.e. $t \in (0,1)$ and all $z \geq k_0$. Suppose $|u\|_\infty = R_\lambda > L\max(\phi^{-1}(\lambda\|\tilde{\gamma}_K\|_1), 4k_0)$. Then (2.10) holds.

In particular, $u \geq c\|u\|_\infty q \geq q$ on $[0,1]$ and $u \geq (c/4)\|u\|_\infty \geq k_0$ on $[1/4,1/2]$. Hence $\tilde{u} \equiv u$ and

$$-(\phi(u'))' = \lambda f(t,u) \geq \lambda R_0\gamma(t)u^{p-1} \geq \lambda R_0\left(\frac{c\|u\|_\infty}{4}\right)^{p-1}\gamma(t)$$

for a.e. $t \in (1/4,1/2)$. By Lemma 2.2 with $r_0 = 1/4, r_1 = 1/2, b = 0, g \equiv 0$, we obtain

$$u \geq (\lambda R_0)^{\frac{1}{p-1}}(c/4)\|u\|_\infty\omega$$

on $[1/4,1/2]$, which implies $(\lambda R_0)^{\frac{1}{p-1}}c\|\omega\|_\infty \leq 4$, a contradiction with the choice of R_0 $(c = L^{-1})$. Hence $\|u\|_\infty \neq R_\lambda$ i.e. (b) holds.

Let λ also be small enough so that $\phi^{-1}(\lambda\|\tilde{\gamma}_K\|_1) < 1$. Then it follows from Lemma A in the Appendix that A_λ has a fixed point $u_\lambda \in E$ with $\|u_\lambda\|_\infty > L > L\phi^{-1}(\lambda\|\tilde{\gamma}_K\|_1)$. Hence Lemma 2.3 gives, $u_\lambda \geq c\|u_\lambda\|_\infty q \geq q$ on $[0,1]$ and so $u_\lambda = \tilde{u}_\lambda$ is a positive solution of (1.1) under boundary condition (1.2) or (1.3). We verify next that $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$. Suppose on the contrary that $\|u_\lambda\|_\infty \not\to \infty$ as $\lambda \to 0^+$. Then there exist a constant $C > 0$ and a sequence (λ_n) converging to 0 such that $\|u_{\lambda_n}\|_\infty \leq C$ for all n. By (A3),

$$|f(t,u_{\lambda_n})| \leq \gamma_C(t)u_{\lambda_n}^{-\delta} \leq \tilde{\gamma}_C(t)$$

for a.e. $t \in (0,1)$, from which (2.2) gives

$$L < \|u_{\lambda_n}\|_\infty \leq M\phi^{-1}(\lambda_n \|f(t, u_{\lambda_n})\|_1) \leq M\phi^{-1}(\lambda_n \|\tilde{\gamma}_C\|_1),$$

a contradiction for n large. Thus $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$, which completes the proof of (i).

(ii) Suppose (A4) holds with $\nu = 0$.

Let $L_0 > L\phi^{-1}(\|\tilde{\gamma}_K\|_1)$, where K is defined in (3.4). Let $C_0 > (L_0/\|\bar{\omega}\|_\infty)^{p-1}$, where $\bar{\omega}$ is the solution of

$$\begin{cases} -(\phi(\bar{\omega}))' = 1 & \text{on } (1/4, 1/2), \\ \bar{\omega}(1/4) = \bar{\omega}(1/2) = 0. \end{cases}$$

Since $\lim_{z\to\infty} f(t,z) = \infty$ uniformly for $t \in (0,1)$, there exists a constant $c_0 > 1$ such that

$$f(t,z) \geq C_0 \tag{3.5}$$

for a.e. $t \in (0,1)$ and $z \geq c_0$.

Suppose $\lambda > \tilde{\lambda}_0$, where $\tilde{\lambda}_0 = \left(\frac{4c_0 L}{L_0}\right)^{p-1}$. We claim that

(c) *If $u \in E$ satisfies $u = A_\lambda u + \xi$ for some $\xi \geq 0$ then $\|u\|_\infty \neq \lambda^{\frac{1}{p-1}} L_0$.*

Let $u \in E$ satisfy $u = A_\lambda u + \xi$ for some $\xi \geq 0$. Then (3.4) and either (3.2) or (3.3) hold. Suppose $\|u\|_\infty = \lambda^{\frac{1}{p-1}} L_0$. Then $\|u\|_\infty > L\phi^{-1}(\lambda\|\tilde{\gamma}_K\|_1)$ and therefore Lemma 2.3 with $h = \lambda\tilde{\gamma}_K$ gives

$$u(t) \geq c\|u\|_\infty q(t) = c\lambda^{\frac{1}{p-1}} L_0 q(t) > 4c_0 q(t)$$

for $t \in (0,1)$. In particular, $u \geq c_0$ on $[1/4, 1/2]$, which, together with (3.5), implies

$$-(\phi(u'))' = \lambda f(t, \tilde{u}) = \lambda f(t,u) \geq \lambda C_0 \quad \text{on } (1/4, 1/2).$$

Lemma 2.2 then gives $u \geq (\lambda C_0)^{\frac{1}{p-1}} \bar{\omega}$ on $[1/4, 1/2]$.

Consequently,

$$\lambda^{\frac{1}{p-1}} L_0 = \|u\|_\infty \geq (\lambda C_0)^{\frac{1}{p-1}} \|\bar{\omega}\|_\infty$$

i.e. $C_0 \leq (L_0/\|\bar{\omega}\|_\infty)^{p-1}$, a contradiction with the choice of C_0. Hence (c) holds. Next, we verify

(d) *There exists $R_\lambda \gg 1$ such that if $u \in E$ satisfies $u = \theta A_\lambda u$ for some $\theta \in (0,1]$ then $\|u\|_\infty \neq R_\lambda$.*

Let $u \in E$ satisfy $u = \theta A_\lambda u$ for some $\theta \in (0,1]$. Then

$$-(\phi(u'))' = \lambda\theta^{p-1} f(t, \tilde{u}) \quad \text{on } (0,1).$$

Using (3.4), we see that (2.7) holds with $h = \lambda\theta^{p-1}\tilde{\gamma}_K$. Let $\sigma > 1$ be such that $0 < f(t,z) \leq \gamma(t) z^{p-1}$ for a.e. $t \in (0,1)$ and all $z \geq \sigma$.

Let $f_1(z) = \sup_{t\in(0,1)} \frac{f(t,z)}{\gamma(t)}$ for $z \geq \sigma$ and $f_1(z) = f_1(\sigma)$ for $z \in (0,\sigma)$. Then $f_1 > 0$ and therefore (A3) gives

$$f(t,z) \leq \gamma_\sigma(t) z^{-\delta} + \gamma(t) f_1(z) \tag{3.6}$$

for a.e. $t \in (0,1)$ and all $z > 0$.

Suppose $\|u\|_\infty = R_\lambda > \max((\lambda\|\tilde{\gamma}_K\|_1)^{\frac{1}{p-1}}, L, \lambda^{\frac{1}{p-1}} L_0)$. Then $u \geq q$ on $(0,1)$ by Lemma 2.3 and therefore (3.6) gives

$$f(t, \tilde{u}) \leq \tilde{\gamma}_\sigma(t) + \gamma(t)\hat{f}_1(\|u\|_\infty),$$

where $\hat{f}_1(z) = \sup_{0<t\le z} f_1(t)$. This, together with (2.2), implies

$$\|u\|_\infty \le M\phi^{-1}(\lambda\|\tilde{\gamma}_\sigma\|_1 + \lambda\|\gamma\|_1\hat{f}_1(\|u\|_\infty)),$$

i.e.

$$\frac{\|\tilde{\gamma}_\sigma\|_1 + \|\gamma\|_1\hat{f}_1(\|u\|_\infty)}{\|u\|_\infty^{p-1}} \ge \frac{1}{\lambda M^{p-1}}. \tag{3.7}$$

Since $\lim_{z\to\infty} \frac{\hat{f}_1(z)}{z^{p-1}} = 0$, the left side of (3.7) goes to 0 as $\|u\|_\infty \to \infty$, we reach a contradiction if R_λ is large enough. Hence (d) holds.

By Lemma A in the Appendix, A_λ has a fixed point u_λ with $\|u_\lambda\|_\infty \ge \lambda^{\frac{1}{p-1}} L_0$. Using Lemma 2.3, we see that u_λ is a positive solution of (1.1) under boundary condition (1.2) or (1.3). Clearly, $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$, which completes the proof of Theorem 1.1. $\square$

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Appendix

We shall state a version of Krasnoselskii's fixed point theorem in a Banach space. The proof presented here is essentially done in [1, Theorem 12.3], but with no cones involved.

Lemma A. *Let E be a Banach space and $T : E \to E$ be a completely continuous operator. Suppose there exist $h \in E, h \ne 0$ and positive constants r, R with $r \ne R$ such that*

(a) If $y \in E$ satisfies $y = \theta Ty$, $\theta \in (0,1]$ then $\|y\| \ne r$,

(b) If $y \in E$ satisfies $y = Ty + \xi h$, $\xi \ge 0$ then $\|y\| \ne R$.

Then T has a fixed point $y \in E$ with $\min(r,R) < \|y\| < \max(r,R)$.

Proof. Define $H : [0,1] \times E \to E$ by $H(\theta, y) = \theta Ty$. Then H is completely continuous and $H(\theta, y) \ne y$ for $\|y\| = r$ in view of (a).

By the homotopy invariance property,

$$\deg(I - H(1,\cdot), B_r, 0) = \deg(I - H(0,\cdot), B_r, 0) = \deg(I, B_r, 0) = 1,$$

where B_r denotes the open ball centered at 0 with radius r in E. Hence

$$\deg(I - T, B_r, 0) = 1.$$

By (b) and the homotopy invariance property,

$$\deg(I - (T + \xi h), B_R, 0) = C$$

for all $\xi \ge 0$. We claim that $C = 0$. Suppose on the contrary that $C \ne 0$. Let $M = \sup\{\|Ty\| : \|y\| \le R\}$ and choose $\xi = \frac{M+R}{\|h\|}$. Then there exists $y \in B_R$ such that $y = Ty + \xi h$. Hence

$$\|y\| \ge \xi\|h\| - \|Ty\| \ge \xi\|h\| - M = R,$$

a contradiction which proves the claim. In particular, $\deg(I - T, B_R, 0) = 0$, and Lemma A follows from the excision property of degree theory. $\square$

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