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Stable manifolds for non-instantaneous impulsive nonautonomous differential equations

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Abstract. In this paper, we study stable invariant manifolds for a class of nonautonomous non-instantaneous impulsive equations where the homogeneous part has a nonuniform exponential dichotomy. We establish a stable invariant manifold result for sufficiently small perturbations by constructing stable and unstable invariant manifolds and we also show that the stable invariant manifolds are of class C^1 outside the jumping times using the continuous Fiber contraction principle technique.

Keywords: non-instantaneous impulsive equations, nonuniform exponential dichotomy, invariant manifolds.

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1 Introduction

Instantaneous impulsive effects arise naturally in physics, biology and control theory [1–4,14]. Non-instantaneous impulsive differential equations (impulse effects start at an arbitrary point and remain active on a finite time interval) was introduced by Hernández and O'Regan [10] and is an extension of classical instantaneous impulsive differential equations [19,21]; we refer the reader to [9,11,13,15–18,22] and the reference therein for results on qualitative and stability theory.

Invariant manifold theory plays an important role in the theory of dynamical systems. To construct stable and unstable invariant manifolds without assuming the existence of uniform exponential dichotomy for associated linear systems is of interest. As a result it is natural to discuss the notion of nonuniform exponential dichotomy as it seems to be the weakest assumption needed to find weak sufficient conditions to guarantee the existence of stable and unstable invariant manifolds. The concept of invariant manifolds was defined first for nonuniformly hyperbolic trajectories in [12] and in [5] the authors established the existence of stable invariant manifold for nonautonomous differential equations without impulses in

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Banach spaces. The authors in [7] studied the existence of stable invariant manifolds and stable invariant manifolds of C^1 regularity for instantaneous impulsive differential equations. The existence of stable invariant manifold for non-instantaneous impulsive differential equations has not been discussed.

In this paper, we consider the ideas in [5,7] to discuss the existence of stable invariant manifolds for non-instantaneous nonlinear impulsive differential equations, where the linear part has a nonuniform exponential dichotomy. Recently [20] the authors studied Lyapunov regularity, the relation between the Lyapunov characteristic exponent and stability, and nonuniform exponential behavior for the following non-instantaneous linear impulsive differential equations:

$$\begin{cases} y'(t) = A(t)y(t),\ t \in (s_i, t_{i+1}], & i = 0, 1, 2, \dots, \\ y(t_i^+) = B_i(t_i^+)y(t_i^-), & i = 1, 2, \dots, \\ y(t) = B_i(t)y(t_i^-), & t \in (t_i, s_i],\ i = 1, 2, \dots, \\ y(s_i^+) = y(s_i^-), & i = 1, 2, \dots, \end{cases} \tag{1.1}$$

in $\mathbb{R}^n$, where we consider $n \times n$ matrices $A(t)$ and $B_i(t)$ varying continuously for $t \geq 0$ and $i \in \mathbb{N}$ and impulsive point t_i and junction point s_i satisfying the relation $s_{i-1} < t_i < s_i$, $i \in \mathbb{N}$. The symbols $y(\varrho_i^+)$ and $y(\varrho_i^-)$ represent the right and left limits of $y(t)$ at $t = \varrho_i$, respectively and set $y(\varrho_i^-) = y(\varrho_i)$.

In this paper we study the following perturbed equations:

$$\begin{cases} y'(t) = A(t)y(t) + f(t, y(t)),\ t \in (s_i, t_{i+1}], & i = 0, 1, 2, \dots, \\ y(t_i^+) = B_i(t_i^+)y(t_i^-) + g_i(t_i^+, y(t_i^-)), & i = 1, 2, \dots, \\ y(t) = B_i(t)y(t_i^-) + g_i(t, y(t_i^-)),\ t \in (t_i, s_i], & i = 1, 2, \dots, \\ y(s_i^+) = y(s_i^-), & i = 1, 2, \dots, \end{cases} \tag{1.2}$$

where $f : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $g_i : \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy $f(t, 0) = 0$ and $g_i(t, 0) = 0$ for each $t \geq 0, i \in \mathbb{N}$. We assume f is piecewise continuous in t with at most discontinuities of the first kind at t_i and g_i is of class C^1.

We show that for a small deviation from the classical notion of uniform exponential dichotomy for (1.1) and for any sufficiently small perturbation term f and non-instantaneous impulsive conditions g_i, there exists a stable invariant manifold for the perturbed equation (1.2). It was emphasized in [5] that this smallness is a rather common phenomenon at least from the point of view of ergodic theory (almost all linear variational equations obtained from a measure-preserving flow admit a nonuniform exponential dichotomy with arbitrarily small nonuniformity).

The notion of nonuniform hyperbolicity plays an important role in the construction of stable and unstable invariant manifolds and we establish a stable invariant manifold result for sufficiently small perturbations by constructing stable and unstable invariant manifolds and we also show that the stable invariant manifolds are of class C^1 outside the jumping times using the continuous Fiber contraction principle technique.

The rest of the paper is organized as follows. In Section 2, we recall the notion of nonuniform exponential dichotomy and use Example 2.2 to present nonuniform exponential dichotomies for non-instantaneous impulsive differential equations. In Section 3, we establish the existence of stable manifolds under sufficiently small perturbations of a nonuniform exponential dichotomy. Existence of stable manifolds are formulated and proved. In the final section, we establish a C^1 regularity result, Theorem 4.7, for stable manifolds by assuming that (1.1) admits a nonuniform exponential dichotomy.

2 Preliminary

Set $\mathbb{R}_0^+ = [0, +\infty)$ and $PC(\mathbb{R}_0^+, \mathbb{R}^n) := \{x : \mathbb{R}_0^+ \to \mathbb{R}^n : x \in C((t_i, t_{i+1}], \mathbb{R}^n), i = 0, 1, 2, \cdots$ and there exist $x(t_i^-)$ and $x(t_i^+)$ with $x(t_i^-) = x(t_i)\}$ with the norm $\|x\|_{PC} := \sup_{t\in R^+} \|x(t)\|$, and $C(\mathbb{R}_0^+, \mathbb{R}^n)$ denotes the Banach space of vector-valued continuous functions from $\mathbb{R}_0^+ \to \mathbb{R}^n$ endowed with the norm $\|x\|_{C(\mathbb{R}_0^+)} = \sup_{t\in\mathbb{R}_0^+} \|x(t)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$.

We assume that

$$0 = s_0 = t_0 < t_1 < s_1 < \cdots < t_i < s_i < \cdots,$$

with $\lim_{i\to\infty} t_i = \infty$, $\lim_{i\to\infty} s_i = \infty$, and

$$\rho := \limsup_{t>s>0} \frac{r(t,s)}{t-s} < \infty, \tag{2.1}$$

where $r(t,s)$ denotes the number of impulsive points which belong to (s,t).

In [20], the authors introduced a bounded linear operator $W(\cdot,\cdot)$ and any nontrivial solution of (1.1) can be formulated by $y(t) = W(t,s)y(s)$ for every $t, s \in \mathbb{R}_0^+$. In addition, the fact that any nontrivial solution of (1.1) has a finite Lyapunov exponent provided (2.1) holds was obtained. Note $W(t,s)W(s,\tau) = W(t,\tau)$ and $W(t,t) = \mathrm{Id}$ for every $t \geq s \geq \tau \geq 0$, where Id denotes the identity operator.

Definition 2.1. (see [7]) We say that (1.1) admits a nonuniform exponential dichotomy if there exist projections $P(t)$ for every $t \geq 0$ satisfying

$$W(t,s)P(s) = P(t)W(t,s), \qquad t \geq s,$$

and there exist some constants $D, a, b, \varepsilon > 0$ such that

$$\|W(t,s)P(s)\| \leq De^{-a(t-s)+\varepsilon s}, \qquad t \geq s, \tag{2.2}$$

and

$$\|W(t,s)Q(s)\| \leq De^{-b(s-t)+\varepsilon s}, \qquad s \geq t, \tag{2.3}$$

where $Q(t) = \mathrm{Id} - P(t)$ is the complementary projection of $P(t)$.

Let $E(t) = P(t)(\mathbb{R}^n)$ and $F(t) = Q(t)(\mathbb{R}^n)$ be the stable and unstable subspaces for each $t \geq 0$ respectively.

Now, we consider the following examples (in the particular case $P(t) = \mathrm{Id}$) of nonuniform exponential dichotomies for non-instantaneous impulsive differential equations.

Example 2.2. Let $\mu, \nu, b > 0$. We consider non-instantaneous impulsive differential equations

$$\begin{cases} y'(t) = (-\mu - \nu\cos(t))y(t),\ t \in (s_i, t_{i+1}], & i = 0, 1, 2, \ldots, \\ y(t_i^+) = (b+1)e^{-\mu t_i} y(t_i^-), & i = 1, 2, \ldots, \\ y(t) = (b+1)e^{-\mu t} y(t_i^-), & t \in (t_i, s_i],\ i = 1, 2, \ldots, \\ y(s_i^+) = y(s_i^-), & i = 1, 2, \ldots, \\ y(s) = y_s, & t_0 < s < t_1, \end{cases} \tag{2.4}$$

with

$$\mu > \nu + \rho \ln(b+1). \tag{2.5}$$

For $s_i < t \le t_{i+1}$, the solutions are given by $y(t) = W(t,s)y_s$, where

$$W(t,s) = (b+1)^{r(t,s)} e^{-\mu \sum_{i=1}^{r(t,s)} t_i} e^{-\mu(t-s)+\nu \sum_{i=1}^{r(t,s)} (\sin s_i - \sin t_i) + \nu(\sin s - \sin t)}.$$

From (2.1) and (2.5), there exists constant $D > 0$ such that

$$\begin{aligned} W(t,s) &= (b+1)^{r(t,s)} e^{-\mu \sum_{i=1}^{r(t,s)} t_i} e^{-\mu(t-s)+\nu \sum_{i=1}^{r(t,s)} (\sin s_i - \sin t_i) + \nu(\sin s - \sin t)} \\ &\le D(b+1)^{r(t,s)} e^{-\mu \sum_{i=1}^{r(t,s)} t_i} e^{-\mu(t-s)+\nu(t-s)+2\nu s} \\ &\le D e^{(-\mu+\nu+\rho\ln(b+1))(t-s)+2\nu s}. \end{aligned} \tag{2.6}$$

For $t_{i+1} < t \le s_{i+1}$, the solutions are given by $y(t) = (b+1)e^{-\mu t} y(t_{i+1}^-) = W(t,s)y_s$, where

$$W(t,s) = (b+1)^{r(t,s)} e^{-\mu \sum_{i=1}^{r(t,s)} t_i} e^{-\mu(t-s)+\nu \sum_{i=1}^{r(t_i,s)} (\sin s_i - \sin t_i) + \nu(\sin s - \sin t_{r(t,s)})}.$$

From (2.1) and (2.5), there exists constant $D > 0$ such that

$$\begin{aligned} W(t,s) &= (b+1)^{r(t,s)} e^{-\mu \sum_{i=1}^{r(t,s)} t_i} e^{-\mu(t-s)+\nu \sum_{i=1}^{r(t_i,s)} (\sin s_i - \sin t_i) + \nu(\sin s - \sin t_{r(t,s)})} \\ &\le D e^{(-\mu+\nu+\rho\ln(b+1))(t-s)+2\nu s}. \end{aligned} \tag{2.7}$$

Throughout the paper, we will always denote the norm $\|(x,y)\| = \|x\| + \|y\|$ for $(x,y) \in \mathbb{R}^n$. We assume that there exists sufficiently small $\delta > 0$ such that for each $t \ge 0, i \in \mathbb{N}$, we have

$$\begin{cases} \|f(t,x) - f(t,y)\| \le \delta e^{-2\varepsilon t} \|x - y\|, \\ \|g_i(t,x) - g_i(t,y)\| \le \delta e^{-(a+2\varepsilon)t} \|x - y\|. \end{cases} \tag{2.8}$$

Note in (2.8) the constant $\delta > 0$ is sufficiently small so that some constants in the following Lemmas can be appropriately chosen.

Now we assume that (1.1) admits a nonuniform exponential dichotomy and the unique solution $(P(t)y(t), Q(t)y(t)) = (u(t), v(t)) \in E(t) \times F(t)$ of (1.2) with initial condition $(\xi, \eta) \in E(s) \times F(s)$ and fixed point s with $s_j < s < t_{j+1} < \infty, j = 0, 1, 2, \ldots$ satisfies the following conditions:

Let $s_{j+r(t,s)} < t \le t_{j+r(t,s)+1}$ and $r(t,s) \ge 1$, and we have

$$\begin{aligned} u(t) &= W(t,s)\xi + \int_s^{t_{j+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),v(\tau))d\tau \\ &+ \int_{s_{j+r(t,s)}}^t W(t,\tau)P(\tau)f(\tau,u(\tau),v(\tau))d\tau \\ &+ \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),v(\tau))d\tau \\ &+ \sum_{k=1}^{r(t,s)} W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k}, u(t_{j+k}), v(t_{j+k})), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} v(t) &= W(t,s)\eta + \int_s^{t_{j+1}} W(t,\tau)Q(\tau)f(\tau,u(\tau),v(\tau))d\tau \\ &+ \int_{s_{j+r(t,s)}}^{t} W(t,\tau)Q(\tau)f(\tau,u(\tau),v(\tau))d\tau \\ &+ \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q(\tau)f(\tau,u(\tau),v(\tau))d\tau \\ &+ \sum_{k=1}^{r(t,s)} W(t,s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),v(t_{j+k})). \end{aligned} \tag{2.10}$$

Let $t_{j+r(t,s)} < t \leq s_{j+r(t,s)}$ and $r(t,s) \geq 1$, and we have

$$\begin{aligned} u(t) &= W(t,s)\xi + \int_s^{t_{j+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),v(\tau))d\tau \\ &+ P(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),v(t_{j+r(t,s)})) \\ &+ \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),v(\tau))d\tau \\ &+ \sum_{k=1}^{r(t,s)-1} W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),v(t_{j+k})), \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} v(t) &= W(t,s)\eta + \int_s^{t_{j+1}} W(t,\tau)Q(\tau)f(\tau,u(\tau),v(\tau))d\tau \\ &+ Q(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),v(t_{j+r(t,s)})) \\ &+ \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q(\tau)f(\tau,u(\tau),v(\tau))d\tau \\ &+ \sum_{k=1}^{r(t,s)-1} W(t,s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),v(t_{j+k})). \end{aligned} \tag{2.12}$$

For each $(s,\xi,\eta) \in \mathbb{R}_0^+ \times E(s) \times F(s)$ we consider the semiflow

$$\Psi_t(s,\xi,\eta) = (s+t, u(s+t), v(s+t)).$$

3 Stable manifold results

In this section, using ideas from [7], we consider the existence of stable manifolds under sufficiently small perturbations of a nonuniform exponential dichotomy. We first describe a certain class of functions (in fact each stable manifold is a graph of one of these functions (see [5])).

Let $\mathcal{Z}$ be the space of functions $\psi : \mathbb{R}_0^+ \times E(\cdot) \to F(\cdot)$ having at most discontinuities of the first kind in the first variable such that for each $s \geq 0$, and $x, y \in E(s)$ we have:

1. $\psi(s,0) = 0$ and $\psi(s,E(s)) \subset F(s)$;
2. There exists a constant $L > 0$ such that

$$\|\psi(s,x) - \psi(s,y)\| \leq L\|x-y\|. \tag{3.1}$$

We equip the space $\mathcal{Z}$ with the distance

$$d(\psi,\varphi)=\sup\{\|\psi(s,x)-\varphi(s,x)\|/\|x\|: s\in\mathbb{R}_0^+ \text{ and } x\in E(s)\backslash\{0\}\},$$

and note $\mathcal{Z}$ is a complete metric space. Given a $\psi\in\mathcal{Z}$ we consider the set

$$\mathcal{W}_\psi^s=\{(s,\xi,\psi(s,\xi)):(s,\xi)\in\mathbb{R}_0^+\times E(s)\}. \tag{3.2}$$

Definition 3.1. $\mathcal{W}_\psi^s$ is called the stable manifold of (1.2) if the semiflow

$$\Psi_t(s,\xi,\psi(s,\xi))\in\mathcal{W}_\psi^s, \text{ for every } t\geq 0,$$

where $\psi\in\mathcal{Z}$ and $\xi\in E(s)$.

Given a constant $c>0$ we define

$$R_j^c=\sup_{t>s}\sum_{k=1}^{r(t,s)}e^{-c(t_{j+k}-s)}<\infty,\qquad j\in\mathbb{N}. \tag{3.3}$$

Using Definition 3.1, each solution in $\mathcal{W}_\psi^s$ must be of the form $(t,u(t),\psi(t,u(t)))$ for $t\geq s$. In particular, the equations in (2.9); (2.10) for $s_{j+r(t,s)}<t\leq t_{j+r(t,s)+1}$ and (2.11); (2.12) for $t_{j+r(t,s)}<t\leq s_{j+r(t,s)}$ can be replaced by

$$\begin{aligned}u(t)=&W(t,s)\xi+\int_s^{t_{j+1}}W(t,\tau)P(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau\\&+\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}W(t,\tau)P(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau\\&+\int_{s_{j+r(t,s)}}^{t}W(t,\tau)P(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau\\&+\sum_{k=1}^{r(t,s)}W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\psi(t_{j+k},u(t_{j+k}))),\end{aligned} \tag{3.4}$$

$$\begin{aligned}\psi(t,u(t))=&W(t,s)\psi(s,u(s))+\int_s^{t_{j+1}}W(t,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau\\&+\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}W(t,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau\\&+\int_{s_{j+r(t,s)}}^{t}W(t,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau\\&+\sum_{k=1}^{r(t,s)}W(t,s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\psi(t_{j+k},u(t_{j+k}))).\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}u(t)=&W(t,s)\xi+\int_s^{t_{j+1}}W(t,\tau)P(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau\\&+\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}W(t,\tau)P(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau\\&+\sum_{k=1}^{r(t,s)-1}W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\psi(t_{j+k},u(t_{j+k})))\\&+P(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),\psi(t_{j+r(t,s)},u(t_{j+r(t,s)}))),\end{aligned} \tag{3.6}$$

$$\begin{aligned}\psi(t,u(t)) &= W(t,s)\psi(s,u(s)) + \int_s^{t_{j+1}} W(t,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ &+ \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ &+ \sum_{k=1}^{r(t,s)-1} W(t,s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\psi(t_{j+k},u(t_{j+k}))) \\ &+ Q(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),\psi(t_{j+r(t,s)},u(t_{j+r(t,s)}))).\end{aligned} \tag{3.7}$$

Define the function $u = u_\psi$ for $\psi \in \mathcal{Z}$. We need the following impulsive Gronwall's inequality results.

Lemma 3.2. *Let $x : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a piecewise continuous function at most with discontinuities of the first kind at the points t_i. If*

$$x(t) \le \alpha + \int_s^t w(\tau)x(\tau)d\tau + \sum_{s\le t_i<t} \gamma_i x(t_i), \qquad t \ge s$$

for some constants $\alpha, \gamma_i \ge 0$, and some function $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, then the following estimate holds

$$x(t) \le \alpha \prod_{s\le t_i<t} (1+\gamma_i) \exp\left(\int_s^t w(\tau)d\tau\right).$$

Lemma 3.3. *Assume that (1.1) admits a nonuniform exponential dichotomy. Given $\delta > 0$ sufficiently small and $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$, for each $\psi \in \mathcal{Z}$ there exists a unique function $u_\psi : [s,+\infty) \to \mathbb{R}^n$ with $u_\psi(s) = \xi$ and $u_\psi(t) \in E(t)$ satisfying (3.4) and (3.6) with $t \ge s$. Moreover,*

$$\|u(t)\| \le 2De^{-a(t-s)+\varepsilon s}\|\xi\| \quad \text{for } t \ge s. \tag{3.8}$$

Proof. Given $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$ with $\xi \ne 0$, and $\psi \in \mathcal{Z}$, we consider the space $\Omega := \{u(\cdot) : [s,+\infty) \to \mathbb{R}^n\}$ such that $u(s) = \xi$ and $u(t) \in E(t)$ for each $t > s$ and u is piecewise continuous with at most discontinuities of the first kind at t_i with $\|u\|' = \sup_{t\ge s}\left\{\frac{\|u(t)\|}{\|\xi\|}e^{a(t-s)-\varepsilon s}\right\} \le 2D$. One can easily verify that Ω is a Banach space with the norm $\|\cdot\|'$. For arbitrary $t \ge s$, we consider the operator Λ (see below) defined in the two intervals $(s_{j+r(t,s)}, t_{j+r(t,s)+1}]$ and $(t_{j+r(t,s)}, s_{j+r(t,s)}]$.

Case 1. For $s_{j+r(t,s)} < t \le t_{j+r(t,s)+1}$, we consider

$$\begin{aligned}(\Lambda u)(t) &= W(t,s)\xi + \int_s^{t_{j+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ &+ \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)P(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ &+ \int_{s_{j+r(t,s)}}^{t} W(t,\tau)P(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ &+ \sum_{k=1}^{r(t,s)} W(t,s_{j+k})P(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\psi(t_{j+k},u(t_{j+k}))).\end{aligned}$$

Given $u_1, u_2 \in \Omega$ and $\tau \ge s$. Note that (2.8) and (3.1), we obtain

$$\begin{aligned}\varpi(\tau) &= \|f(\tau,u_1(\tau),\psi(\tau,u_1(\tau))) - f(\tau,u_2(\tau),\psi(\tau,u_2(\tau)))\| \\ &\le \delta(1+L)\|\xi\|e^{-a(\tau-s)+\varepsilon s}e^{-2\varepsilon\tau}\|u_1-u_2\|',\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\omega_i &= \|g_i(s_i,u_1(t_i),\psi(t_i,u_1(t_i)))-g_i(s_i,u_2(t_i),\psi(t_i,u_2(t_i)))\|\\ &\le \delta(1+L)\|\xi\|e^{-a(t_i-s)+\varepsilon s}e^{-(a+2\varepsilon)s_i}\|u_1-u_2\|',\ i=j+k,k=1,\cdots,r(t,s),\\ \omega_i(t) &= \|g_i(t,u_1(t_i),\psi(t_i,u_1(t_i)))-g_i(t,u_2(t_i),\psi(t_i,u_2(t_i)))\|\\ &\le \delta(1+L)\|\xi\|e^{-a(t_i-s)+\varepsilon s}e^{-(a+2\varepsilon)t}\|u_1-u_2\|',\ i=j+r(t,s).\end{aligned} \tag{3.10}$$

Therefore, we obtain

$$\begin{aligned}&\|(\Lambda u_1)(t)-(\Lambda u_2)(t)\|\\ &\le \int_s^{t_{j+1}}\|W(t,\tau)P(\tau)\|\varpi(\tau)d\tau+\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}\|W(t,\tau)P(\tau)\|\varpi(\tau)d\tau\\ &\quad+\int_{s_{j+r(t,s)}}^{t}\|W(t,\tau)P(\tau)\|\varpi(\tau)d\tau+\sum_{k=1}^{r(t,s)}\|W(t,s_{j+k})P(s_{j+k})\|\omega_{j+k}\\ &\le \int_s^t\|W(t,\tau)P(\tau)\|\varpi(\tau)d\tau+\sum_{k=1}^{r(t,s)}\|W(t,s_{j+k})P(s_{j+k})\|\omega_{j+k}\\ &\le \frac{D\delta(1+L)}{\varepsilon}\|\xi\|\|u_1-u_2\|'e^{-a(t-s)}+D\delta(1+L)\|\xi\|\|u_1-u_2\|'e^{-a(t-s)+\varepsilon s}\sum_{k=1}^{r(t,s)}e^{-a(t_{j+k}-s)}\\ &\le \frac{D\delta(1+L)}{\varepsilon}\|\xi\|\|u_1-u_2\|'e^{-a(t-s)}+D\delta(1+L)\|\xi\|\|u_1-u_2\|'e^{-a(t-s)+\varepsilon s}R_j^a,\end{aligned}$$

which implies that

$$\|\Lambda u_1-\Lambda u_2\|'\le\theta\|u_1-u_2\|',$$

where $\theta=D\delta(1+L)(\frac{1}{\varepsilon}+R_j^a)$. Take δ sufficiently small so that $\theta<\frac{1}{2}$. Therefore, the operator Λ becomes a contraction mapping. Moreover

$$\|\Lambda u\|'\le\|W(\cdot,s)\xi\|'+\theta\|u\|'\le D+\theta\|u\|'\le 2D,$$

and hence, $\Lambda(\Omega)\subset\Omega$. Therefore, Λ has a unique fixed point $u\in\Omega$ such that $u=\Lambda u$. Moreover, for $t\ge s$ we have

$$\|u(t)\|\le 2De^{-a(t-s)+\varepsilon s}\|\xi\|.$$

Case 2. For $t_{j+r(t,s)}<t\le s_{j+r(t,s)}$, we have

$$\begin{aligned}(\Lambda u)(t)&=W(t,s)\xi+\int_s^{t_{j+1}}W(t,\tau)P(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau\\ &\quad+\sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}}W(t,\tau)P(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau\\ &\quad+\sum_{k=1}^{r(t,s)-1}W(t,s_{j+k})P(s_{j+k})g_{j+k}(u(t_{j+k}),\psi(t_{j+k},u(t_{j+k})))\\ &\quad+P(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),\psi(t_{j+r(t,s)},u(t_{j+r(t,s)}))).\end{aligned}$$

From (2.2) and (2.3) with $t=s\ge t_0$, we obtain

$$\|P(t)\|\le De^{\varepsilon t}\quad\text{and}\quad\|Q(t)\|\le De^{\varepsilon t}.$$

Using (3.9) and (3.10), we have

$$
\begin{aligned}
&\|(\Lambda u_1)(t)-(\Lambda u_2)(t)\| \\
&\le \int_s^{t_{j+1}} \|W(t,\tau)P(\tau)\|\varpi(\tau)d\tau + \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} \|W(t,\tau)P(\tau)\|\varpi(\tau)d\tau \\
&\quad + \sum_{k=1}^{r(t,s)-1} \|W(t,s_{j+k})P(s_{j+k})\|\omega_{j+k} + \|P(t)\omega_{j+r(t,s)}(t)\| \\
&\le \int_s^{t} \|W(t,\tau)P(\tau)\|\varpi(\tau)d\tau + \sum_{k=1}^{r(t,s)-1} \|W(t,s_{j+k})P(s_{j+k})\|\omega_{j+k} + \|P(t)\|\omega_{j+r(t,s)}(t) \\
&\le \frac{D\delta(1+L)}{\varepsilon}\|\xi\|\|u_1-u_2\|'e^{-a(t-s)} + D\delta(1+L)\|\xi\|\|u_1-u_2\|'e^{-a(t-s)+\varepsilon s}\sum_{k=1}^{r(t,s)} e^{-a(t_{j+k}-s)} \\
&\le \frac{D\delta(1+L)}{\varepsilon}\|\xi\|\|u_1-u_2\|'e^{-a(t-s)} + D\delta(1+L)\|\xi\|\|u_1-u_2\|'e^{-a(t-s)+\varepsilon s}R_j^a,
\end{aligned}
$$

which implies that

$$
\|\Lambda u_1-\Lambda u_2\|' \le \theta\|u_1-u_2\|',
$$

where $\theta = D\delta(1+L)(\frac{1}{\varepsilon}+R_j^a)$. Take δ sufficiently small so that $\theta < \frac{1}{2}$. Therefore, the operator Λ becomes a contraction. Moreover

$$
\|\Lambda u\|' \le \|W(\cdot,s)\xi\|' + \theta\|u\|' \le D + \theta\|u\|' \le 2D.
$$

and hence, $\Lambda(\Omega)\subset\Omega$. Therefore, Λ has a unique fixed point $u\in\Omega$ such that $u=\Lambda u$. Moreover, for $t\ge s$ we have

$$
\|u(t)\| \le 2De^{-a(t-s)+\varepsilon s}\|\xi\|.
$$

The proof is complete. □

Now, we establish some auxiliary results for the function u_ψ. Given $\delta>0$ sufficiently small, $\psi\in\mathcal{Z}, s\ge 0$, and $\xi,\bar{\xi}\in E(s)$, from Lemma 3.3, we consider the unique functions u_ψ and $\bar{u}_\psi$ such that $u_\psi(s)=\xi$ and $\bar{u}_\psi(s)=\bar{\xi}$.

Lemma 3.4. *Assume that* (1.1) *admits a nonuniform exponential dichotomy. Given $\delta>0$ sufficiently small and $\xi,\bar{\xi}\in E(s)$, we have*

$$
\|u_\psi(t)-\bar{u}_\psi(t)\| \le 2De^{(-a+\rho\ln(1+D\delta(1+L)))(t-s)+\varepsilon s}\|\xi-\bar{\xi}\|
$$

for each $\psi\in\mathcal{Z}$ and $t\ge s\ge 0$.

Proof. For each $\tau\ge s$, we have

$$
\|f(\tau,u_\psi(\tau),\psi(\tau,u_\psi(\tau)))-f(\tau,\bar{u}_\psi(\tau),\psi(\tau,\bar{u}_\psi(\tau)))\| \le \delta(1+L)e^{-2\varepsilon\tau}\|u_\psi(\tau)-\bar{u}_\psi(\tau)\|,
$$

and

$$
\begin{aligned}
&\|g_i(s_i,u_\psi(t_i),\psi(t_i,u_\psi(t_i)))-g_i(s_i,\bar{u}_\psi(t_i),\psi(t_i,\bar{u}_\psi(t_i)))\| \\
&\qquad \le \delta(1+L)e^{-(a+2\varepsilon)s_i}\|u_\psi(t_i)-\bar{u}_\psi(t_i)\|, \qquad i=j+k, k=1,2,\dots,r(t,s), \\
&\|g_i(t,u_\psi(t_i),\psi(t_i,u_\psi(t_i)))-g_i(t,\bar{u}_\psi(t_i),\psi(t_i,\bar{u}_\psi(t_i)))\| \\
&\qquad \le \delta(1+L)\|e^{-(a+2\varepsilon)t}\|u_\psi(t_i)-\bar{u}_\psi(t_i)\|, \qquad i=j+r(t,s).
\end{aligned}
$$

Set

$$\phi(t) = \|u_\psi(t) - \bar{u}_\psi(t)\|.$$

Using (3.4) and (3.6), we have two cases to consider:

Case 1. For $s_{j+r(t,s)} < t \leq t_{j+r(t,s)+1}$, we have

$$\begin{aligned}
\phi(t) &\leq \|W(t,s)P(s)\|\|\xi - \bar{\xi}\| + \delta(1+L)\int_s^t \|W(t,\tau)P(\tau)\|e^{-2\varepsilon\tau}\phi(\tau)d\tau \\
&\quad + \delta(1+L)\sum_{k=1}^{r(t,s)} \|W(t,s_{j+k})P(s_{j+k})\|e^{-(a+2\varepsilon)s_{j+k}}\phi(t_{j+k}) \\
&\leq De^{-a(t-s)+\varepsilon s}\|\xi - \bar{\xi}\| + D\delta(1+L)\Big(\int_s^t e^{-a(t-\tau)-\varepsilon\tau}\phi(\tau)d\tau + \sum_{k=1}^{r(t,s)} e^{-at}\phi(t_{j+k})\Big) \\
&\leq De^{-a(t-s)+\varepsilon s}\|\xi - \bar{\xi}\| + D\delta(1+L)\Big(\int_s^t e^{-a(t-\tau)-\varepsilon\tau}\phi(\tau)d\tau + \sum_{k=1}^{r(t,s)} e^{-a(t-t_{j+k})}\phi(t_{j+k})\Big).
\end{aligned}$$

Case 2. For $t_{j+r(t,s)} < t \leq s_{j+r(t,s)}$, we have

$$\begin{aligned}
\phi(t) &\leq \|W(t,s)P(s)\|\|\xi - \bar{\xi}\| + \delta(1+L)\int_s^{t_{j+r(t,s)}} \|W(t,\tau)P(\tau)\|e^{-2\varepsilon\tau}\phi(\tau)d\tau \\
&\quad + \delta(1+L)\sum_{k=1}^{r(t,s)-1} \|W(t,s_{j+k})P(s_{j+k})\|e^{-(a+2\varepsilon)s_{j+k}}\phi(t_{j+k}) \\
&\quad + D\delta(1+L)e^{\varepsilon t}e^{-(a+2\varepsilon)t}\phi(t_{j+r(t,s)}) \\
&\leq De^{-a(t-s)+\varepsilon s}\|\xi - \bar{\xi}\| \\
&\quad + D\delta(1+L)\Big(\int_s^t e^{-a(t-\tau)-\varepsilon\tau}\phi(\tau)d\tau + \sum_{k=1}^{r(t,s)-1} e^{-at}\phi(t_{j+k}) + e^{-at}\phi(t_{j+r(t,s)})\Big) \\
&\leq De^{-a(t-s)+\varepsilon s}\|\xi - \bar{\xi}\| + D\delta(1+L)\Big(\int_s^t e^{-a(t-\tau)-\varepsilon\tau}\phi(\tau)d\tau + \sum_{k=1}^{r(t,s)} e^{-at}\phi(t_{j+k})\Big) \\
&\leq De^{-a(t-s)+\varepsilon s}\|\xi - \bar{\xi}\| + D\delta(1+L)\Big(\int_s^t e^{-a(t-\tau)-\varepsilon\tau}\phi(\tau)d\tau + \sum_{k=1}^{r(t,s)} e^{-a(t-t_{j+k})}\phi(t_{j+k})\Big),
\end{aligned}$$

where we use $\|P(t)\| \leq De^{\varepsilon t}$.

Setting $\varpi_1(t) = e^{a(t-s)}\phi(t)$, we obtain

$$\begin{aligned}
\varpi_1(t) &\leq De^{\varepsilon s}\|\xi - \bar{\xi}\| + D\delta(1+L)\Big(\int_s^t e^{a(\tau-s)}e^{-\varepsilon\tau}\phi(\tau)d\tau + \sum_{k=1}^{r(t,s)} e^{a(t_{j+k}-s)}\phi(t_{j+k})\Big) \\
&\leq De^{\varepsilon s}\|\xi - \bar{\xi}\| + D\delta(1+L)\Big(\int_s^t e^{-\varepsilon\tau}\varpi_1(\tau)d\tau + \sum_{k=1}^{r(t,s)} \varpi_1(t_{j+k})\Big).
\end{aligned}$$

Therefore, using Lemma 3.2, we have

$$\begin{aligned}
\varpi_1(t) &\leq De^{\varepsilon s}\|\xi - \bar{\xi}\| \prod_{k=1}^{r(t,s)} (1 + D\delta(1+L)) \exp\Big(\int_s^t D\delta(1+L)e^{-\varepsilon\tau}d\tau\Big) \\
&\leq De^{\frac{D\delta(1+L)}{\varepsilon}}e^{\varepsilon s}\|\xi - \bar{\xi}\|(1 + D\delta(1+L))^{r(t,s)}.
\end{aligned}$$

Using (2.1) and taking δ sufficiently small so that $e^{\frac{D\delta(1+L)}{\varepsilon}} \leq 2$, we obtain

$$\varpi_1(t) \leq 2De^{\rho\ln(1+D\delta(1+L))(t-s))+\varepsilon s}\|\xi - \bar{\xi}\|.$$

Therefore, we have

$$\phi(t) = \|u_\psi(t) - \bar{u}_\psi(t)\| \leq 2De^{(-a+\rho\ln(1+D\delta(1+L)))(t-s)+\varepsilon s}\|\xi - \bar{\xi}\|.$$

The proof is complete. □

Lemma 3.5. *Assume that* (1.1) *admits a nonuniform exponential dichotomy. Given* $\delta > 0$ *sufficiently small and* $\psi_1, \psi_2 \in \mathcal{Z}$ *and* $(s, \xi) \in \mathbb{R}_0^+ \times E(s)$*, there exists* $\beta > 0$ *such that*

$$\|u_{\psi_1}(t) - u_{\psi_2}(t)\| \leq 2\beta e^{(-a+\rho\ln(1+D\delta(1+L)))(t-s)}\|\xi\|d(\psi_1, \psi_2),$$

for every $t \geq s$.

Proof. For each $\tau \geq s$, we have

$$\begin{aligned}
&\|f(\tau, u_{\psi_1}(\tau), \psi_1(\tau, u_{\psi_1}(\tau))) - f(\tau, u_{\psi_2}(\tau), \psi_2(\tau, u_{\psi_2}(\tau)))\| \\
&\quad \leq \delta e^{-2\varepsilon\tau}\|(u_{\psi_1}(\tau) - u_{\psi_2}(\tau), \psi_1(\tau, u_{\psi_1}(\tau)) - \psi_2(\tau, u_{\psi_2}(\tau)))\| \\
&\quad \leq \delta e^{-2\varepsilon\tau}(\|u_{\psi_1}(\tau) - u_{\psi_2}(\tau)\| \\
&\qquad\qquad + \|\psi_1(\tau, u_{\psi_1}(\tau)) - \psi_2(\tau, u_{\psi_1}(\tau)) + \psi_2(\tau, u_{\psi_1}(\tau)) - \psi_2(\tau, u_{\psi_2}(\tau))\|) \\
&\quad \leq \delta e^{-2\varepsilon\tau}(\|u_{\psi_1}(\tau)\|d(\psi_1, \psi_2) + (1+L)\|u_{\psi_1}(\tau) - u_{\psi_2}(\tau)\|),
\end{aligned}$$

and

$$\begin{aligned}
&\|g_i(s_i, u_{\psi_1}(t_i), \psi_1(t_i, u_{\psi_1}(t_i))) - g_i(s_i, u_{\psi_2}(t_i), \psi_2(t_i, u_{\psi_2}(t_i)))\| \\
&\quad \leq \delta e^{-(a+2\varepsilon)s_i}(\|u_{\psi_1}(t_i)\|d(\psi_1, \psi_2) + (1+L)\|u_{\psi_1}(t_i) - u_{\psi_2}(t_i)\|), \quad i = j+k, k = 1, \ldots, r(t,s), \\
&\|g_i(t, u_{\psi_1}(t_i), \psi_1(t_i, u_{\psi_1}(t_i))) - g_i(t, u_{\psi_2}(t_i), \psi_2(t_i, u_{\psi_2}(t_i)))\| \\
&\quad \leq \delta e^{-(a+2\varepsilon)t}(\|u_{\psi_1}(t_i)\|d(\psi_1, \psi_2) + (1+L)\|u_{\psi_1}(t_i) - u_{\psi_2}(t_i)\|), \quad i = j + r(t,s).
\end{aligned}$$

Set

$$\bar{\phi}(t) = \|u_{\psi_1}(t) - u_{\psi_2}(t)\|.$$

Using (3.4), (3.6) and Lemma 3.3, we have two cases to consider:

Case 1. For $s_{j+r(t,s)} < t \leq t_{j+r(t,s)+1}$, we have

$$\begin{aligned}
\bar{\phi}(t) &\leq \delta \int_s^t \|W(t,\tau)P(\tau)\|e^{-2\varepsilon\tau}\|u_{\psi_1}(\tau)\|d(\psi_1, \psi_2)d\tau \\
&\quad + \delta(1+L)\int_s^t \|W(t,\tau)P(\tau)\|e^{-2\varepsilon\tau}\bar{\phi}(\tau)d\tau \\
&\quad + \delta\sum_{k=1}^{r(t,s)} \|W(t, s_{j+k})P(s_{j+k})\|e^{-(a+2\varepsilon)s_{j+k}}\|u_{\psi_1}(t_{j+k})\|d(\psi_1, \psi_2) \\
&\quad + \delta(1+L)\sum_{k=1}^{r(t,s)} \|W(t, s_{j+k})P(s_{j+k})\|e^{-(a+2\varepsilon)s_{j+k}}\bar{\phi}(t_{j+k}) \\
&\leq 2D^2\delta\|\xi\|d(\psi_1, \psi_2)e^{-a(t-s)}\int_s^t e^{-\varepsilon(\tau-s)}d\tau + D\delta(1+L)\int_s^t e^{-a(t-\tau)-\varepsilon\tau}\bar{\phi}(\tau)d\tau \\
&\quad + 2D^2\delta\|\xi\|d(\psi_1, \psi_2)e^{-a(t-s)}\sum_{k=1}^{r(t,s)} e^{-\varepsilon s_{j+k}+\varepsilon s-a(t_{j+k}-s)} + D\delta(1+L)\sum_{k=1}^{r(t,s)} e^{-at}\bar{\phi}(t_{j+k})
\end{aligned}$$

$$
\begin{aligned}
&\leq 2D^2\delta\|\xi\|d(\psi_1,\psi_2)e^{-a(t-s)}\int_s^t e^{-\varepsilon(\tau-s)}d\tau + D\delta(1+L)\int_s^t e^{-a(t-\tau)-\varepsilon\tau}\bar{\phi}(\tau)d\tau \\
&\quad + 2D^2\delta\|\xi\|d(\psi_1,\psi_2)e^{-a(t-s)}\sum_{k=1}^{r(t,s)} e^{-(a+\varepsilon)(t_{j+k}-s)} + D\delta(1+L)\sum_{k=1}^{r(t,s)} e^{-at}\bar{\phi}(t_{j+k}) \\
&\leq 2D^2\delta\|\xi\|d(\psi_1,\psi_2)e^{-a(t-s)}\left(\int_s^t e^{-\varepsilon(\tau-s)}d\tau + R_j^{a+\varepsilon}\right) \\
&\quad + D\delta(1+L)\int_s^t e^{-a(t-\tau)-\varepsilon\tau}\bar{\phi}(\tau)d\tau + D\delta(1+L)\sum_{k=1}^{r(t,s)} e^{-a(t-t_{j+k})}\bar{\phi}(t_{j+k}).
\end{aligned}
$$

Case 2. For $t_{j+r(t,s)} < t \leq s_{j+r(t,s)}$, we have

$$
\begin{aligned}
\bar{\phi}(t) \leq &\int_s^{t_{j+r(t,s)}} \|W(t,\tau)P(\tau)\|\delta e^{-2\varepsilon\tau}(\|u_{\psi_1}(\tau)\|d(\psi_1,\psi_2) + (1+L)\bar{\phi}(\tau))d\tau \\
&+ \sum_{k=1}^{r(t,s)-1} \|W(t,s_{j+k})P(s_{j+k})\|\delta e^{-(a+2\varepsilon)s_{j+k}}\left(\|u_{\psi_1}(t_{j+k})\|d(\psi_1,\psi_2) + (1+L)\bar{\phi}(t_{j+k})\right) \\
&+ De^{\varepsilon t}\delta e^{-(a+2\varepsilon)t}(\|u_{\psi_1}(t_{j+r(t,s)})\|d(\psi_1,\psi_2) + (1+L)\bar{\phi}(t_{j+r(t,s)})) \\
\leq &\, 2D^2\delta\|\xi\|d(\psi_1,\psi_2)e^{-a(t-s)}\int_s^t e^{-\varepsilon(\tau-s)}d\tau + D\delta(1+L)\int_s^t e^{-a(t-\tau)-\varepsilon\tau}\bar{\phi}(\tau)d\tau \\
&+ 2D^2\delta\|\xi\|d(\psi_1,\psi_2)e^{-a(t-s)}\sum_{k=1}^{r(t,s)-1} e^{-(a+\varepsilon)(t_{j+k}-s)} + D\delta(1+L)\sum_{k=1}^{r(t,s)-1} e^{-at}\bar{\phi}(t_{j+k}) \\
&+ 2D^2\delta\|\xi\|d(\psi_1,\psi_2)e^{-a(t-s)}e^{-(a+\varepsilon)(t_{j+r(t,s)}-s)} + D\delta(1+L)e^{-at}\bar{\phi}(t_{j+r(t,s)}) \\
\leq &\, 2D^2\delta\|\xi\|d(\psi_1,\psi_2)e^{-a(t-s)}\left(\int_s^t e^{-\varepsilon(\tau-s)}d\tau + R_j^{a+\varepsilon}\right) \\
&+ D\delta(1+L)\int_s^t e^{-a(t-\tau)-\varepsilon\tau}\bar{\phi}(\tau)d\tau + D\delta(1+L)\sum_{k=1}^{r(t,s)} e^{-a(t-t_{j+k})}\bar{\phi}(t_{j+k}),
\end{aligned}
$$

where we use $\|P(t)\| \leq De^{\varepsilon t}$.

Setting $\varpi_2(t) = e^{a(t-s)}\bar{\phi}(t)$, we obtain

$$
\begin{aligned}
\varpi_2(t) &\leq 2D^2\delta\|\xi\|d(\psi_1,\psi_2)\left(\int_s^t e^{-\varepsilon(\tau-s)}d\tau + R_j^{a+\varepsilon}\right) \\
&\quad + D\delta(1+L)\int_s^t e^{a(\tau-s)}e^{-\varepsilon\tau}\bar{\phi}(\tau)d\tau + D\delta(1+L)\sum_{k=1}^{r(t,s)} e^{a(t_{j+k}-s)}\bar{\phi}(t_{j+k}) \\
&\leq 2D^2\delta\|\xi\|d(\psi_1,\psi_2)\left(\frac{1}{\varepsilon} + R_j^{a+\varepsilon}\right) + D\delta(1+L)\left(\int_s^t e^{-\varepsilon\tau}\varpi_2(\tau)d\tau + \sum_{k=1}^{r(t,s)}\varpi_2(t_{j+k})\right).
\end{aligned}
$$

Using Lemma 3.2, we have

$$
\begin{aligned}
\varpi_2(t) &\leq 2D^2\delta\|\xi\|d(\psi_1,\psi_2)\left(\frac{1}{\varepsilon} + R_j^{a+\varepsilon}\right)\prod_{k=1}^{r(t,s)}(1+D\delta(1+L))\exp\left(\int_s^t D\delta(1+L)e^{-\varepsilon\tau}d\tau\right) \\
&\leq 2D^2\delta\|\xi\|e^{\frac{D\delta(1+L)}{\varepsilon}}d(\psi_1,\psi_2)\left(\frac{1}{\varepsilon} + R_j^{a+\varepsilon}\right)(1+D\delta(1+L))^{r(t,s)}.
\end{aligned}
$$

Using (2.1) and taking δ sufficiently small so that $e^{\frac{D\delta(1+L)}{\varepsilon}} \leq 2$, we obtain

$$\varpi_2(t) \leq 2\beta e^{\rho \ln(1+D\delta(1+L))(t-s)} \|\xi\| d(\psi_1, \psi_2),$$

where $\beta = 2D^2\delta\left(\frac{1}{\varepsilon} + R_j^{a+\varepsilon}\right)$. The proof is complete. □

Now we will rewrite (3.5), (3.7) in an equivalent form.

Lemma 3.6. *Assume that* (1.1) *admits a nonuniform exponential dichotomy. Given $\delta > 0$ sufficiently small, $a + b > \varepsilon$ and $\psi \in \mathcal{Z}$, the following properties hold:*

1. For each $(s, \xi) \in \mathbb{R}_0^+ \times E(s)$ and $t \geq s$ and if

$$\begin{aligned} \psi(t, u(t) = {} & W(t,s)\psi(s,\xi) + \int_s^{t_{j+1}} W(t,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau \\ & + \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau \\ & + \int_{s_{j+r(t,s)}}^{t} W(t,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau \\ & + \sum_{k=1}^{r(t,s)} W(t, s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k}, u(t_{j+k}), \psi(t_{j+k}, u(t_{j+k}))), s_{j+r(t,s)} < t \leq t_{j+r(t,s)+1}, \end{aligned} \tag{3.11}$$

or

$$\begin{aligned} \psi(t, u(t)) = {} & W(t,s)\psi(s,\xi) + \int_s^{t_{j+1}} W(t,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau \\ & + \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau \\ & + \sum_{k=1}^{r(t,s)-1} W(t, s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k}, u(t_{j+k}), \psi(t_{j+k}, u(t_{j+k}))) \\ & + Q(t)g_{j+r(t,s)}(t, u(t_{j+r(t,s)}), \psi(t_{j+r(t,s)}, u(t_{j+r(t,s)}))), t_{j+r(t,s)} < t \leq s_{j+r(t,s)}, \end{aligned} \tag{3.12}$$

then

$$\begin{aligned} \psi(s,\xi) = {} & -\int_s^{t_{j+1}} W(s,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau \\ & - \sum_{k=1}^{\infty} \int_{s_{j+k}}^{t_{j+k+1}} W(s,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau \\ & - \sum_{k=1}^{\infty} W(s, s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k}, u(t_{j+k}), \psi(t_{j+k}, u(t_{j+k}))). \end{aligned} \tag{3.13}$$

2. If (3.13) *holds for each $(s, \xi) \in \mathbb{R}_0^+ \times E(s)$ with $t \geq s$, then* (3.11) *and* (3.12) *hold for each $(s, \xi) \in \mathbb{R}_0^+ \times E(s)$ and $t \geq s$.*

Proof. We first show that the integral and the series in (3.13) are well-defined for each $(s, \xi) \in \mathbb{R}_0^+ \times E(s)$. For each $\tau \geq s$, using Lemma 3.3 and (2.8), we have

$$\begin{aligned} \|f(\tau, u(\tau), \psi(\tau, u(\tau)))\| & \leq (1+L)\delta e^{-2\varepsilon\tau} \|u(\tau)\| \\ & \leq 2D\delta(1+L)e^{-2\varepsilon\tau} e^{-a(\tau-s)+\varepsilon s} \|\xi\|, \end{aligned} \tag{3.14}$$

and

$$\begin{cases} \|g_i(s_i, u(t_i), \psi(t_i, u(t_i)))\| \\ \qquad \le 2D\delta(1+L)e^{-(a+2\varepsilon)s_i - a(t_i - s) + \varepsilon s}\|\xi\|, \quad i = j+k, k = 1, 2, \dots, r(t,s), \\ \|g_i(t, u(t_i), \psi(t_i, u(t_i)))\| \\ \qquad \le 2D\delta(1+L)e^{-(a+2\varepsilon)t - a(t_i - s) + \varepsilon s}\|\xi\|, \quad i = j + r(t,s). \end{cases} \tag{3.15}$$

Using (2.3), (3.14) and (3.15), we have

$$\begin{aligned} &\int_s^{t_{j+1}} \|W(s,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau\| \\ &\qquad + \sum_{k=1}^{\infty} \int_{s_{j+k}}^{t_{j+k+1}} \|W(s,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau\| \\ &\quad \le 2D^2\delta(1+L)\|\xi\| \int_s^{\infty} e^{(-b-a-\varepsilon)(\tau - s)} d\tau \le \frac{2D^2\delta(1+L)\|\xi\|}{b+a+\varepsilon} < \infty, \end{aligned}$$

and

$$\begin{aligned} &\sum_{k=1}^{\infty} \|W(s, s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k}, u(t_{j+k}), \psi(t_{j+k}, u(t_{j+k})))\| \\ &\quad \le 2D^2\delta(1+L)\|\xi\| \sum_{k=1}^{\infty} e^{(-b-a-\varepsilon)(s_{j+k}-s) - a(t_{j+k}-s)} \\ &\quad \le 2D^2\delta(1+L)\|\xi\| \sum_{k=1}^{\infty} e^{(-b-a-\varepsilon)(t_{j+k}-s) - a(t_{j+k}-s)} \\ &\quad \le 2D^2\delta(1+L)\|\xi\| R_j^{b+2a+\varepsilon} < \infty. \end{aligned}$$

This implies that the right-hand side of (3.13) is well-defined.

Assume that (3.11) and (3.12) hold for each $(s, \xi) \in \mathbb{R}_0^+ \times E(s)$ and $t \ge s$. Therefore, we will consider the following two cases:

Case **I**. Let $s_{j+r(t,s)} < t \le t_{j+r(t,s)+1}$, the identity (3.11) can be written in the form

$$\begin{aligned} \psi(s,\xi) = {} & W(s,t)Q(t)\psi(t, u(t)) - \int_s^{t_{j+1}} W(s,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau \\ & - \sum_{k=1}^{r(t,s)-1} \int_{s_{j+k}}^{t_{j+k+1}} W(s,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau \\ & - \int_{s_{j+r(t,s)}}^{t} W(s,\tau)Q(\tau)f(\tau, u(\tau), \psi(\tau, u(\tau)))d\tau \\ & - \sum_{k=1}^{r(t,s)} W(s, s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k}, u(t_{j+k}), \psi(t_{j+k}, u(t_{j+k}))), \end{aligned} \tag{3.16}$$

where we use $Q^2(t) = Q(t)$. From Lemma 3.3 we have

$$\begin{aligned} \|W(s,t)Q(t)\psi(t, u(t))\| &\le LDe^{-b(t-s)+\varepsilon t}\|u(t)\| \\ &\le 2LD^2\|\xi\|e^{(-b-a+\varepsilon)(t-s)+2\varepsilon s}. \end{aligned} \tag{3.17}$$

We note that since $-b-a+\varepsilon < 0$, the right-hand side of (3.17) tends to zero as $t \to +\infty$. Thus, (3.16) yields (3.13) by letting $t \to +\infty$.

Case **II**. Let $t_{j+r(t,s)} < t \leq s_{j+r(t,s)}$, the identity (3.12) can be written in the form

$$\begin{aligned}\psi(s,\xi) = W(s,t)Q(t)\psi(t,u(t)) - \int_s^{t_{j+1}} W(s,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ - \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(s,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ - \sum_{k=1}^{r(t,s)-1} W(s,s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\psi(t_{j+k},u(t_{j+k}))) \\ - W(s,t)Q(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),\psi(t_{j+r(t,s)},u(t_{j+r(t,s)}))).\end{aligned} \tag{3.18}$$

Using Lemma 3.3 and (3.15), we have

$$\begin{aligned}&\|W(s,t)Q(t)g_{j+r(t,s)}(t,u(t_{j+r(t,s)}),\psi(t_{j+r(t,s)},u(t_{j+r(t,s)})))\| \\ &\leq 2D^2(1+L)\|\xi\|e^{(-b-a-\varepsilon)(t-s)}.\end{aligned} \tag{3.19}$$

Therefore, the right-hand side of (3.19) tends to zero as $t \to +\infty$. Thus, (3.18) yields (3.13) by letting $t \to +\infty$.

Assume that (3.13) holds for each $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$ and $t \geq s$.

Let $s_{j+r(t,s)} < t \leq t_{j+r(t,s)+1}$, we have

$$\begin{aligned}W(t,s)\psi(s,\xi) = &- \int_s^{t_{j+1}} W(t,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ &- \sum_{k=1}^{r(t,s)-1}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ &- \int_{s_{j+r(t,s)}}^{t} W(t,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ &- \sum_{k=1}^{r(t,s)} W(t,s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\psi(t_{j+k},u(t_{j+k}))) \\ &- \int_t^{t_{j+r(t,s)+1}} W(t,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ &- \sum_{r(t,s)+1}^{\infty}\int_{s_{j+k}}^{t_{j+k+1}} W(t,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ &- \sum_{r(t,s)+1}^{\infty} W(t,s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\psi(t_{j+k},u(t_{j+k}))),\end{aligned}$$

which implies that identity (3.11) holds for each $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$ and $t \geq s$.

Let $t_{j+r(t,s)} < t \leq s_{j+r(t,s)}$, the identity in (3.12) can be obtained by a similar idea to that in (3.11). The proof is complete. □

Lemma 3.7. *Assume that* (1.1) *admits a nonuniform exponential dichotomy. Given $\delta > 0$ sufficiently small, and $a + b > \rho \ln(1 + D\delta(1+L))$, there exists a unique function $\psi \in \mathcal{Z}$ such that* (3.13) *holds for every $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$.*

Proof. For each $\psi \in \mathcal{Z}$ and $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$, we define the operator $\mathcal{J}$ by

$$\begin{aligned}(\mathcal{J}\psi)(s,\xi) = & -\int_s^{t_{j+1}} W(s,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ & -\sum_{k=1}^{\infty}\int_{s_{j+k}}^{t_{j+k+1}} W(s,\tau)Q(\tau)f(\tau,u(\tau),\psi(\tau,u(\tau)))d\tau \\ & -\sum_{k=1}^{\infty} W(s,s_{j+k})Q(s_{j+k})g_{j+k}(s_{j+k},u(t_{j+k}),\psi(t_{j+k},u(t_{j+k}))),\end{aligned} \tag{3.20}$$

where $u = u_\psi$ is the unique function given in Lemma 3.3 such that $u_\psi(s) = \xi$. Since $f(t,0)= g_i(t,0) = 0$, we have $(\mathcal{J}\psi)(s,0) = 0$ for every $s > 0$. Next, let $u = u_\psi$ and $\bar{u} = \bar{u}_\psi$ be the unique functions given by Lemma 3.3 such that $u(s) = \xi$ and $\bar{u}(s) = \bar{\xi}$. Using Lemma 3.4, we obtain

$$\begin{aligned}&\|f(\tau,u(\tau),\psi(\tau,u(\tau))) - f(\tau,\bar{u}(\tau),\psi(\tau,\bar{u}(\tau)))\| \\ &\quad \le 2D\delta(1+L)e^{(-a+\rho\ln(1+D\delta(1+L)))(\tau-s)+\varepsilon s-2\varepsilon\tau}\|\xi-\bar{\xi}\|,\end{aligned}$$

and

$$\begin{aligned}&\|g_i(s_i,u(t_i),\psi(t_i,u(t_i))) - g_i(s_i,\bar{u}(t_i),\psi(t_i,\bar{u}(t_i)))\| \\ &\quad \le 2D\delta(1+L)e^{(-a+\rho\ln(1+D\delta(1+L)))(t_i-s)+\varepsilon s-(a+2\varepsilon)s_i}\|\xi-\bar{\xi}\|, \qquad i=j+k, k=1,\dots,r(t,s).\end{aligned}$$

From (2.3) and (3.20), we obtain

$$\begin{aligned}&\|(\mathcal{J}\psi)(s,\xi) - (\mathcal{J}\psi)(s,\bar{\xi})\| \\ &\quad \le \int_s^\infty \|W(s,\tau)Q(\tau)\|\|f(\tau,u(\tau),\psi(\tau,u(\tau))) - f(\tau,\bar{u}(\tau),\psi(\tau,\bar{u}(\tau)))\|d\tau \\ &\qquad + \sum_{k=1}^\infty \|W(s,s_{j+k})Q(s_{j+k})\| \\ &\qquad \times \|g_{j+k}(s_{j+k},u(t_{j+k}),\psi(t_{j+k},u(t_{j+k}))) - g_{j+k}(s_{j+k},\bar{u}(t_{j+k}),\psi(t_{j+k},\bar{u}(t_{j+k})))\| \\ &\quad \le 2\delta D^2(1+L)\|\xi-\bar{\xi}\|\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}e^{(-a+\rho\ln(1+D\delta(1+L)))(\tau-s)+\varepsilon s-2\varepsilon\tau}d\tau \\ &\qquad + 2\delta D^2(1+L)\|\xi-\bar{\xi}\|\sum_{k=1}^\infty e^{-b(s_{j+k}-s)}e^{(-a+\rho\ln(1+D\delta(1+L)))(t_{j+k}-s)+\varepsilon s-(a+\varepsilon)s_{j+k}} \\ &\quad \le \left(\frac{2\delta D^2}{-\beta_1} + 2\delta D^2 R_j^{\beta_1}\right)(1+L)\|\xi-\bar{\xi}\|,\end{aligned}$$

where $\beta_1 = -b-a-\varepsilon+\rho\ln(1+D\delta(1+L))$. Taking $\delta > 0$ sufficiently small, we obtain

$$\|(\mathcal{J}\psi)(s,\xi) - (\mathcal{J}\psi)(s,\bar{\xi})\| \le L\|\xi-\bar{\xi}\|$$

for every $s \ge 0$ and $\xi,\bar{\xi} \in E(s)$, so $\mathcal{J}(\mathcal{Z}) \subset \mathcal{Z}$.

Next, we show that operator $\mathcal{J}$ is a contraction. Given $\psi_1,\psi_2 \in \mathcal{Z}$ and $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$, let u_{ψ_1} and u_{ψ_2} be the unique functions given by Lemma 3.3 such that $u_{\psi_1}(s) = u_{\psi_2}(s) = \xi$. From Lemma 3.5, we obtain

$$\begin{aligned}&\|f(\tau,u_{\psi_1}(\tau),\psi_1(\tau,u_{\psi_1}(\tau))) - f(\tau,u_{\psi_2}(\tau),\psi_2(\tau,u_{\psi_2}(\tau)))\| \\ &\quad \le \delta e^{-2\varepsilon\tau}(\|u_{\psi_1}(\tau)\|d(\psi_1,\psi_2) + (1+L)\|u_{\psi_1}(\tau)-u_{\psi_2}(\tau)\|) \\ &\quad \le 2\delta(D+(1+L)\beta)e^{(-a+\rho\ln(1+D\delta(1+L)))(\tau-s)+\varepsilon s-2\varepsilon\tau}\|\xi\|d(\psi_1,\psi_2),\end{aligned}$$

and

$$\begin{aligned}
&\|g_i(s_i,u_{\psi_1}(t_i),\psi_1(t_i,u_{\psi_1}(t_i))) - g_i(s_i,u_{\psi_2}(t_i),\psi_2(t_i,u_{\psi_2}(t_i)))\| \\
&\quad \le \delta e^{-(a+2\varepsilon)s_i}(\|u_{\psi_1}(t_i)\|d(\psi_1,\psi_2) + (1+L)\|u_{\psi_1}(t_i) - u_{\psi_2}(t_i)\|) \\
&\quad \le 2\delta(D+(1+L)\beta)e^{(-a+\rho\ln(1+D\delta(1+L)))(t_i-s)+\varepsilon s-(a+2\varepsilon)s_i}\|\xi\|d(\psi_1,\psi_2), \\
&\qquad i = j+k,\ k = 1,2,\ldots,r(t,s).
\end{aligned}$$

From (2.3) and (3.20), we obtain

$$\begin{aligned}
&\|(\mathcal{J}\psi_1)(s,\xi) - (\mathcal{J}\psi_2)(s,\xi)\| \\
&\quad \le \int_s^\infty \|W(s,\tau)Q(\tau)\|\|f(\tau,u_{\psi_1}(\tau),\psi_1(\tau,u_{\psi_1}(\tau))) - f(\tau,u_{\psi_2}(\tau),\psi_2(\tau,u_{\psi_2}(\tau)))\|d\tau \\
&\qquad + \sum_{k=1}^{\infty} \|W(s,s_{j+k})Q(s_{j+k})\|\|g_i(s_i,u_{\psi_1}(t_i),\psi_1(t_i,u_{\psi_1}(t_i))) - g_i(s_i,u_{\psi_2}(t_i),\psi_2(t_i,u_{\psi_2}(t_i)))\| \\
&\quad \le 2D\delta(D+(1+L)\beta)\|\xi\|d(\psi_1,\psi_2)\int_s^\infty e^{\beta_1(\tau-s)}d\tau \\
&\qquad + 2D\delta(D+(1+L)\beta)\|\xi\|d(\psi_1,\psi_2)\sum_{k=1}^{\infty} e^{\beta_1(t_{j+k}-s)} \\
&\quad \le 2D\delta(D+(1+L)\beta)\left(\frac{1}{-\beta_1} + R_j^{\beta_1}\right)\|\xi\|d(\psi_1,\psi_2).
\end{aligned}$$

Now taking $\delta > 0$ sufficiently small, then the operator $\mathcal{J}$ is a contraction in the complete metric space $\mathcal{Z}$. Hence, there exists a unique function $\psi \in \mathcal{Z}$ such that $\mathcal{J}\psi = \psi$, for every $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$. The proof is complete. □

The following stable manifold theorem is in the sense that we have the unique graph of the form $\mathcal{W}_\psi^s$ (for some function $\psi \in \mathcal{Z}$) which is invariant under the semiflow.

Theorem 3.8. *Assume that* (1.1) *admits a nonuniform exponential dichotomy. If* $a+b > \max\{\varepsilon, \rho\ln(1+D\delta(1+L))\}$*, then provided that* $\delta > 0$ *is sufficiently small, there exists a unique function* $\psi \in \mathcal{Z}$ *such that*

$$\Psi_t(s,\xi,\psi(s,\xi)) \in \mathcal{W}_\psi^s, \quad \text{for every } t \ge 0. \tag{3.21}$$

Furthermore, for every $s \ge 0$, $\xi,\bar{\xi} \in E(s)$ *and* $t \ge s$, *we have*

$$\|\Psi_{t-s}(s,\xi,\psi(s,\xi)) - \Psi_{t-s}(s,\bar{\xi},\psi(s,\bar{\xi}))\| \le 2D(1+L)e^{(-a+\rho\ln(1+D\delta(1+L)))(t-s)+\varepsilon s}\|\xi-\bar{\xi}\|.$$

Proof. From Lemma 3.3, for each $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$ and $\psi \in \mathcal{Z}$, there exists a unique function $u_\psi \in \Omega$. Using Lemmas 3.6 and 3.7, for each $(s,\xi) \in \mathbb{R}_0^+ \times E(s)$, there exists a unique function $\psi \in \mathcal{Z}$ such that (3.11) and (3.12) hold. This shows that (3.21) holds, for any sufficiently small δ.

It remains to establish the inequality in the theorem. We denote again by u_ψ and $\bar{u}_\psi$ the unique functions given by Lemma 3.3 such that $u_\psi(s) = \xi$ and $\bar{u}_\psi(s) = \bar{\xi}$. From Lemma 3.4, we have

$$\begin{aligned}
&\|\Psi_{t-s}(s,\xi,\psi(s,\xi)) - \Psi_{t-s}(s,\bar{\xi},\psi(s,\bar{\xi}))\| \\
&\quad = \|(t,u_\psi(t),\psi(t,u_\psi(t))) - (t,\bar{u}_\psi(t),\psi(t,\bar{u}_\psi(t)))\| \\
&\quad \le (1+L)\|u_\psi(t) - \bar{u}_\psi(t)\| \\
&\quad \le 2D(1+L)e^{(-a+\rho\ln(1+D\delta(1+L)))(t-s)+\varepsilon s}\|\xi-\bar{\xi}\|.
\end{aligned}$$

The proof is complete. □

4 C^1 regularity

Without loss of generality, we assume that (1.1) admits a nonuniform exponential dichotomy. In this section (using ideas from [7]) we establish the C^1 regularity of the sections $\mathcal{W}^s_\psi \bigcap(\{s\} \times \mathbb{R}^n)$ for each $s_i < t \leq t_{i+1}$ and $s_0 \leq s \leq t_1$ with $i \in \mathbb{N}$, where $\mathcal{W}^s_\psi$ is the stable manifold in Theorem 3.8. Let $j = 0$ and $s_0 \leq s \leq t_1$. Let

$$R_0^c = \sup_{t>s} \sum_{k=1}^{r(t,s)} e^{-c(t_k-s)} < \infty.$$

Now, we recall the Fiber contraction principle [7]. Given metric spaces $X = (X, d_X)$ and $Y = (Y, d_Y)$, we define a distance in $X \times Y$ by

$$d((x,y),(\bar{x},\bar{y})) = d_X(x,\bar{x}) + d_Y(y,\bar{y}).$$

We consider transformations $\mathcal{S} : X \times Y \to X \times Y$ of the following form

$$\mathcal{S}(x,y) = (\mathcal{J}(x), \mathcal{A}(x,y)),$$

for some functions $\mathcal{J} : X \to X$ and $\mathcal{A} : X \times Y \to Y$. We say that $\mathcal{S}$ is a Fiber contraction if there exists $\lambda \in (0,1)$ such that

$$d_Y(\mathcal{A}(x,y), \mathcal{A}(x,\bar{y})) \leq \lambda d_Y(y,\bar{y})$$

for every $x \in X$ and $y, \bar{y} \in Y$. For each $x \in X$ we define a transformations $\mathcal{A}_x : Y \to Y$ by $\mathcal{A}_x(y) = \mathcal{A}(x,y)$. We also say that a fixed point $x_0 \in X$ of $\mathcal{J}$ is attracting if $\mathcal{J}^n(x) \to x_0$ when $n \to \infty$, for every $x \in X$.

Next, we need the following assumptions (for the maps below):

*H*1. $\mathcal{A}|S$ is of class C^1;

*H*2. $\mathcal{A}(0^+)$, $\mathcal{A}(t^+)$ and $\mathcal{A}(t^-)$ are well-defined for every $t > 0$;

*H*3. $\mathcal{A}'(0^+)$, $\mathcal{A}'(t^+)$ and $\mathcal{A}'(t^-)$ computed with respect to $\mathcal{A}(0^+)$, $\mathcal{A}(t^+)$ and $\mathcal{A}(t^-)$ are well-defined for every $t > 0$;

*H*4. $f|(S \times X)$ is of class C^1, and $f(t,0) = f(t,u) = 0$ for each $t \geq 0$ and $u \in X$ with $\|u\| \geq c$, for some constant $c > 0$;

*H*5. $f(0^+,x)$, $f(t^+,x)$ and $f(t^-,x)$ are well-defined for every $t > 0$;

*H*6. $\frac{\partial f}{\partial t}(0^+,x)$, $\frac{\partial f}{\partial t}(t^+,x)$ and $\frac{\partial f}{\partial t}(t^-,x)$ computed with respect to $f(0^+,x)$, $f(t^+,x)$ and $f(t^-,x)$ are well-defined for every $t > 0$;

*H*7. g_i is of class C^1, and $g_i(t,0) = g_i(t,u) = 0$ for each $i \in \mathbb{N}$ and $u \in X$ with $\|u\| \geq c$, for some constant $c > 0$.

Under these assumptions we will consider the following C^1 regularity of the section of the stable manifold. In order to consider the C^1 regularity of the stable manifold, we will give some definitions and lemmas.

Definition 4.1 (see [7, Lemma 9]). If $\mathcal{S}$ is a continuous Fiber contraction principle, $x_0 \in X$ is an attracting fixed point of $\mathcal{J}$, and $y_0 \in Y$ is an fixed point of $\mathcal{A}_{x_0}$, then (x_0, y_0) is an attracting fixed point of $\mathcal{S}$.

Lemma 4.2 (see [14, Section 1.2]). *Let $\bar{x} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a piecewise continuous function at most with discontinuities of the first kind at the points t_i. If*

$$\bar{x}(t) \leq \bar{\alpha} + \int_s^t (\varrho + \gamma \bar{x}(\tau)) d\tau + \sum_{s \leq t_i < t} (\varrho + \gamma \bar{x}(t_i)), \qquad t \geq s$$

for some constants $\bar{\alpha} \geq 0, \varrho \geq 0, \gamma > 0$, then the following estimate holds

$$\bar{x}(t) \leq \left(\bar{\alpha} + \frac{\varrho}{\gamma} \right) (1+\gamma)^{r(t,s)} e^{\gamma(t-s)} - \frac{\varrho}{\gamma}.$$

Set $\mathcal{K} = \mathbb{R}^+ \setminus \{s_i : i \in \mathbb{N}\}$. We consider the space $\mathcal{F}$ of continuous functions

$$\Phi : \{(s,\xi) \in \mathcal{K} \times E(s)\} \to \coprod_{s \in \mathbb{R}_0^+} L(s),$$

where $L(s)$ is the family of linear transformations from $E(s)$ to $F(s)$, such that $\Phi(s,\xi) \in L(s)$ for every $s \geq 0$ and $\xi \in E(s)$, with the norm

$$\|\Phi\| := \sup\{\|\Phi(s,\xi)\| : (s,\xi) \in \mathcal{K} \times E(s)\} \leq L, \tag{4.1}$$

having an extension to $\mathbb{R}_0^+ \times X$ with at most discontinuities of the first kind in the first variable. We will consider the subset $\mathcal{F}_0 \subset \mathcal{F}$ consisting of functions $\Phi \in \mathcal{F}$ such that $\Phi(s,0) = 0$ for every $s \geq 0$. One can easily verify that $\mathcal{F}_0$ and $\mathcal{F}$ are complete metric spaces with the distance in (4.1).

Given δ as in Theorem 3.8 and $\psi \in \mathcal{Z}$, we consider the unique solution $u_\psi(t,\xi)$ given by Lemma 3.3 for each $\xi \in E(s)$ with $t > s \geq 0$. Due to the continuous dependence of the solutions of an impulsive differential equation on the initial conditions (see [14, Section 1.2]) and Lemma 3.7, the solution $(t,\psi,s,\xi) \mapsto u_\psi(t,\xi)$ is continuous on $\mathcal{K} \times \mathcal{Z} \times \{(s,\xi) \in \mathcal{K} \times E(s)\}$. We let

$$y_\psi(t) = (t, u_\psi(t,\xi), \psi(t, u_\psi(t,\xi))), \qquad \bar{y}_\psi(s_i,t_i) = (s_i, u_\psi(t_i,\xi), \psi(t_i, u_\psi(t_i,\xi))),$$

and $z_\psi(t) = (t, u_\psi(t,\xi))$. We define a linear transformation $\mathcal{A}(\psi,\Phi)$ for each $(\psi,\Phi) \in \mathcal{Z} \times \mathcal{F}$ by

$$\begin{aligned}
\mathcal{A}(\psi,\Phi)(s,\xi) = &-\int_s^{t_1} W(s,\tau)Q(\tau) \left(\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T(\tau) + \frac{\partial f}{\partial \psi}(y_\psi(\tau))\Phi(z_\psi(\tau)) \right) d\tau \\
&- \sum_{k=1}^{\infty} \int_{s_k}^{t_{k+1}} W(s,\tau)Q(\tau) \left(\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T(\tau) + \frac{\partial f}{\partial \psi}(y_\psi(\tau))\Phi(z_\psi(\tau)) \right) d\tau \\
&- \sum_{k=1}^{\infty} W(s,s_k)Q(s_k) \left(\frac{\partial g_k}{\partial u_\psi}(\bar{y}_\psi(s_k,t_k))T(t_k) + \frac{\partial g_k}{\partial \psi}(\bar{y}_\psi(s_k,t_k))\Phi(z_\psi(t_k)) \right)
\end{aligned} \tag{4.2}$$

for $s \in \mathcal{K}$ and $\xi \in E(s)$, and where the function $T = T_{\psi,\Phi,\xi}$ is uniquely determined by

$$\begin{aligned}
T(t) = W(t,s)P(s) &+ \int_s^{t_1} W(t,\tau)P(\tau) \left(\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T(\tau) + \frac{\partial f}{\partial \psi}(y_\psi(\tau))\Phi(z_\psi(\tau)) \right) d\tau \\
&+ \sum_{k=1}^{r(t,s)-1} \int_{s_k}^{t_{k+1}} W(t,\tau)P(\tau) \left(\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T(\tau) + \frac{\partial f}{\partial \psi}(y_\psi(\tau))\Phi(z_\psi(\tau)) \right) d\tau \\
&+ \int_{s_{r(t,s)}}^{t} W(t,\tau)P(\tau) \left(\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T(\tau) + \frac{\partial f}{\partial \psi}(y_\psi(\tau))\Phi(z_\psi(\tau)) \right) d\tau \\
&+ \sum_{k=1}^{r(t,s)} W(t,s_k)P(s_k) \left(\frac{\partial g_k}{\partial u_\psi}(\bar{y}_\psi(s_k,t_k))T(t_k) + \frac{\partial g_k}{\partial \psi}(\bar{y}_\psi(s_k,t_k))\Phi(z_\psi(t_k)) \right)
\end{aligned} \tag{4.3}$$

for $t \geq s$. We note that $T(t)$ is a linear transformation from $E(s) \to E(t)$ with $T(s) = \mathrm{Id}_{E(s)}$. From the continuity of the functions $(t, \psi, s, \xi) \mapsto u_\psi(t, \xi)$, ψ and Φ, the function $(t, \psi, s, \xi) \mapsto T_{\psi,\Phi,\xi}$ is also continuous on $\mathcal{K} \times \mathcal{Z} \times \{(s, \xi) \in \mathcal{K} \times E(s)\}$.

Lemma 4.3. *Let $\delta > 0$ be sufficiently small. If $b + \varepsilon > D\delta + \rho \ln(1 + D\delta)$, then the operator $\mathcal{A}$ is well-defined, and $\mathcal{A}(\mathcal{Z} \times \mathcal{F}) \subset \mathcal{F}$.*

Proof. In order to show that operator $\mathcal{A}$ is well-defined, we let Y be equal to $\|\mathcal{A}(\psi, \Phi)(s, \xi)\|$, that is

$$\begin{aligned} \mathrm{Y} \leq & \int_s^{t_1} \left\| W(s,\tau)Q(\tau) \left(\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T(\tau) + \frac{\partial f}{\partial \psi}(y_\psi(\tau))\Phi(z_\psi(\tau)) \right) \right\| d\tau \\ &+ \sum_{k=1}^{\infty} \int_{s_k}^{t_{k+1}} \left\| W(s,\tau)Q(\tau) \left(\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T(\tau) + \frac{\partial f}{\partial \psi}(y_\psi(\tau))\Phi(z_\psi(\tau)) \right) \right\| d\tau \\ &+ \sum_{k=1}^{\infty} \left\| W(s,s_k)Q(s_k) \left(\frac{\partial g_k}{\partial u_\psi}(\bar{y}_\psi(s_k,t_k))T(t_k) + \frac{\partial g_k}{\partial \psi}(\bar{y}_\psi(s_k,t_k))\Phi(z_\psi(t_k)) \right) \right\|. \end{aligned}$$

For every $t > 0$, $k \in \mathbb{N}$, and $u \in X$. From (2.8) and as $x \to y$, we have

$$\left\| \frac{\partial f}{\partial u}(t,u) \right\| \leq \delta e^{-2\varepsilon t} \quad \text{and} \quad \left\| \frac{\partial g_k}{\partial u}(t,u) \right\| \leq \delta e^{-(a+2\varepsilon)t}. \tag{4.4}$$

From (2.3) and (4.4) we have that

$$\begin{aligned} \mathrm{Y} \leq & D\delta \Big(\int_s^{t_1} e^{-b(\tau-s)-\varepsilon\tau}(\|T(\tau)\| + L)d\tau + \sum_{k=1}^{\infty} \int_{s_k}^{t_{k+1}} e^{-b(\tau-s)-\varepsilon\tau}(\|T(\tau)\| + L)d\tau \Big) \\ &+ D\delta \sum_{k=1}^{\infty} e^{-b(s_k-s)-as_k-\varepsilon s_k}(\|T(t_k)\| + L) \\ \leq & D\delta \int_s^{\infty} e^{-b(\tau-s)-\varepsilon\tau}(\|T(\tau)\| + L)d\tau + D\delta \sum_{k=1}^{\infty} e^{-b(s_k-s)-as_k-\varepsilon s_k}(\|T(t_k)\| + L) \\ \leq & D\delta \int_s^{\infty} e^{-b(\tau-s)-\varepsilon\tau}(\|T(\tau)\| + L)d\tau + D\delta \sum_{k=1}^{\infty} e^{(-a-b-\varepsilon)(s_k-s)}(\|T(t_k)\| + L) \\ \leq & D\delta \int_s^{\infty} e^{-b(\tau-s)-\varepsilon\tau}(\|T(\tau)\| + L)d\tau + D\delta \sum_{k=1}^{\infty} e^{(-a-b-\varepsilon)(t_k-s)}(\|T(t_k)\| + L). \end{aligned} \tag{4.5}$$

Using (2.2), (4.3) and (4.4), we have

$$\begin{aligned} \|T(t)\| \leq & De^{-a(t-s)+\varepsilon s} + D\delta \int_s^t e^{-a(t-\tau)-\varepsilon\tau}(\|T(\tau)\| + L)d\tau \\ &+ D\delta \sum_{k=1}^{r(t,s)} e^{-at-\varepsilon s_k}(\|T(t_k)\| + L) \\ \leq & De^{\varepsilon s} + \int_s^t (D\delta L + D\delta\|T(\tau)\|)d\tau + \sum_{k=1}^{r(t,s)} (D\delta L + D\delta\|T(t_k)\|). \end{aligned}$$

It follows from Lemma 4.2 that

$$\begin{aligned} \|T(t)\| &\leq (De^{\varepsilon s} + L)(1 + D\delta)^{r(t,s)} e^{D\delta(t-s)} - L \\ &\leq (De^{\varepsilon s} + L)e^{(D\delta+\rho\ln(1+\delta D))(t-s)} - L \\ &\leq (De^{\varepsilon t_1} + L)e^{(D\delta+\rho\ln(1+\delta D))(t-s)} - L. \end{aligned} \tag{4.6}$$

Put (4.6) into (4.5) and we have

$$\begin{aligned} Y &\leq \delta D(De^{\varepsilon t_1} + L)\left(\int_s^\infty e^{\bar{c}(\tau - s)} d\tau + \sum_{k=1}^{\infty} e^{(-a+\bar{c})(t_k - s)} \right) \\ &\leq \delta D\left(\frac{1}{-\bar{c}} + R_0^{a-\bar{c}} \right)(De^{\varepsilon t_1} + L), \end{aligned}$$

where

$$\bar{c} = -b + \rho \ln(1 + D\delta) + D\delta - \varepsilon; \tag{4.7}$$

note $\bar{c} < 0$. Let $\delta > 0$ be sufficiently small. Then, $\mathcal{A}(\psi, \Phi)$ is well-defined and $\mathcal{A}(\mathcal{Z} \times \mathcal{F}) \subset \mathcal{F}$. The proof is complete. □

Next, we consider the transformation $\mathcal{S} : \mathcal{Z} \times \mathcal{F} \to \mathcal{Z} \times \mathcal{F}$ given by

$$\mathcal{S}(\psi, \Phi) = (\mathcal{J}(\psi), \mathcal{A}(\psi, \Phi)),$$

where the operator $\mathcal{J}$ is defined in (3.20).

Lemma 4.4. *Let $\delta > 0$ be sufficiently small. If $b + \varepsilon > D\delta + \rho \ln(1 + D\delta)$, then the operator $\mathcal{S}$ is a Fiber contraction.*

Proof. Let $\xi \in E(s)$, $\psi \in \mathcal{Z}$ and $\Phi, \bar{\Phi} \in \mathcal{F}$. Let T_Φ and $T_{\bar{\Phi}}$ satisfy the following

$$T_\Phi = T_{\psi,\Phi,\xi} \quad \text{and} \quad T_{\bar{\Phi}} = T_{\psi,\bar{\Phi},\xi}.$$

Using (4.2) and (4.4), we have

$$\begin{aligned} &\|\mathcal{A}(\psi, \Phi)(s, \xi) - \mathcal{A}(\psi, \bar{\Phi})(s, \xi)\| \\ &\leq D \int_s^\infty e^{-b(\tau - s) + \varepsilon\tau} \left\| \frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T_\Phi(\tau) + \frac{\partial f}{\partial \psi}(y_\psi(\tau))\Phi(z_\psi(\tau)) - \frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T_{\bar{\Phi}}(\tau) \right. \\ &\quad \left. - \frac{\partial f}{\partial \psi}(y_\psi(\tau))\bar{\Phi}(z_\psi(\tau)) \right\| d\tau + D \sum_{k=1}^{\infty} e^{-b(s_k - s) + \varepsilon s_k} \left\| \frac{\partial g_k}{\partial u_\psi}(\bar{y}_\psi(s_k, t_k))T_\Phi(t_k) \right. \\ &\quad \left. + \frac{\partial g_k}{\partial \psi}(\bar{y}_\psi(s_k, t_k))\Phi(z_\psi(t_k)) - \frac{\partial g_k}{\partial u_\psi}(\bar{y}_\psi(s_k, t_k))T_{\bar{\Phi}}(t_k) - \frac{\partial g_k}{\partial \psi}(\bar{y}_\psi(s_k, t_k))\bar{\Phi}(z_\psi(t_k)) \right\| \\ &\leq \delta D \int_s^\infty e^{-b(\tau - s) - \varepsilon\tau} (\|T_\Phi(\tau) - T_{\bar{\Phi}}(\tau)\| + \|\Phi(z_\psi(\tau)) - \bar{\Phi}(z_\psi(\tau))\|) d\tau \\ &\quad + \delta D \sum_{k=1}^{\infty} e^{-b(s_k - s) - (a+\varepsilon)s_k} (\|T_\Phi(t_k) - T_{\bar{\Phi}}(t_k)\| + \|\Phi(z_\psi(t_k)) - \bar{\Phi}(z_\psi(t_k))\|) \\ &\leq \delta D \int_s^\infty e^{-b(\tau - s) - \varepsilon\tau} (\|T_\Phi(\tau) - T_{\bar{\Phi}}(\tau)\| + \|\Phi(z_\psi(\tau)) - \bar{\Phi}(z_\psi(\tau))\|) d\tau \\ &\quad + \delta D \sum_{k=1}^{\infty} e^{(-a-b-\varepsilon)(s_k - s)} (\|T_\Phi(t_k) - T_{\bar{\Phi}}(t_k)\| + \|\Phi(z_\psi(t_k)) - \bar{\Phi}(z_\psi(t_k))\|) \\ &\leq \delta D \int_s^\infty e^{-b(\tau - s) - \varepsilon\tau} (\|T_\Phi(\tau) - T_{\bar{\Phi}}(\tau)\| + \|\Phi(z_\psi(\tau)) - \bar{\Phi}(z_\psi(\tau))\|) d\tau \\ &\quad + \delta D \sum_{k=1}^{\infty} e^{(-a-b-\varepsilon)(t_k - s)} (\|T_\Phi(t_k) - T_{\bar{\Phi}}(t_k)\| + \|\Phi(z_\psi(t_k)) - \bar{\Phi}(z_\psi(t_k))\|). \end{aligned} \tag{4.8}$$

Using (4.3) and (4.4), we have

$$
\begin{aligned}
&\|T_\Phi(t) - T_{\bar{\Phi}}(t)\| \\
&\quad \le \delta D \int_s^t e^{-a(t-\tau)-\varepsilon\tau} \|T_\Phi(\tau) - T_{\bar{\Phi}}(\tau)\| d\tau + \delta D \|\Phi - \bar{\Phi}\| \int_s^t e^{-a(t-\tau)-\varepsilon\tau} d\tau \\
&\qquad + \delta D \sum_{k=1}^{r(t,s)} e^{-a(t-s_k)-(a+\varepsilon)s_k} \|T_\Phi(t_k) - T_{\bar{\Phi}}(t_k)\| + \delta D \|\Phi - \bar{\Phi}\| \sum_{k=1}^{r(t,s)} e^{-at-\varepsilon s_k} \\
&\quad \le \delta D \|\Phi - \bar{\Phi}\| \left(e^{-at} \int_s^t e^{(a-\varepsilon)\tau} d\tau + R_0^\varepsilon \right) \\
&\qquad + \delta D \int_s^t e^{-a(t-\tau)-\varepsilon\tau} \|T_\Phi(\tau) - T_{\bar{\Phi}}(\tau)\| d\tau + \delta D \sum_{k=1}^{r(t,s)} e^{-at} \|T_\Phi(t_k) - T_{\bar{\Phi}}(t_k)\| \\
&\quad \le \delta D \|\Phi - \bar{\Phi}\| \left(\frac{1}{|a-\varepsilon|} + R_0^\varepsilon \right) \\
&\qquad + \delta D \bigg(\int_s^t \|T_\Phi(\tau) - T_{\bar{\Phi}}(\tau)\| d\tau + \sum_{k=1}^{r(t,s)} \|T_\Phi(t_k) - T_{\bar{\Phi}}(t_k)\| \bigg).
\end{aligned}
$$

Setting $K = \delta D\big(\frac{1}{|a-\varepsilon|} + R_0^\varepsilon\big)$, it follows from Lemma 3.2 that

$$
\begin{aligned}
\|T_\Phi(t) - T_{\bar{\Phi}}(t)\| &\le \delta D \|\Phi - \bar{\Phi}\| \left(\frac{1}{|a-\varepsilon|} + R_0^\varepsilon \right) (1 + D\delta)^{r(t,s)} e^{D\delta(t-s)} \\
&\le K \|\Phi - \bar{\Phi}\| e^{(D\delta + \rho \ln(1+D\delta))(t-s)}.
\end{aligned} \tag{4.9}
$$

Put (4.9) into (4.8) and we have

$$
\begin{aligned}
&\|\mathcal{A}(\psi, \Phi)(s, \xi) - \mathcal{A}(\psi, \bar{\Phi})(s, \xi)\| \\
&\quad \le K\delta D \|\Phi - \bar{\Phi}\| \int_s^\infty e^{\bar{c}(\tau - s)} d\tau + \delta D e^{bs} \|\Phi - \bar{\Phi}\| \int_s^\infty e^{(-b-\varepsilon)\tau} d\tau \\
&\qquad + K\delta D \|\Phi - \bar{\Phi}\| \sum_{k=1}^{\infty} e^{(-a+\bar{c})(t_k - s)} + \delta D \|\Phi - \bar{\Phi}\| \sum_{k=1}^{\infty} e^{(-a-b-\varepsilon)(t_k - s)} \\
&\quad \le \delta D \left(\frac{K}{-\bar{c}} + \frac{e^{bt_1}}{b+\varepsilon} + K R_0^{a-\bar{c}} + R_0^{a+b+\varepsilon} \right) \|\Phi - \bar{\Phi}\|.
\end{aligned}
$$

Let $\delta > 0$ be sufficiently small, and then the operator $\mathcal{S}$ is a fiber contraction. The proof is complete. □

To apply Definition 4.1 it remains to verify that $\mathcal{S}$ is continuous.

Lemma 4.5. *Let $\delta > 0$ be sufficiently small. If $b + \varepsilon > D\delta + \rho \ln(1 + D\delta)$, then the operator $\mathcal{S}$ is continuous.*

Proof. Let $\xi \in E(s)$, $\psi, \varphi \in \mathcal{Z}$ and $\Phi \in \mathcal{F}$. Let T_ψ and T_φ satisfy the following

$$T_\psi = T_{\psi,\Phi,\xi} \quad \text{and} \quad T_\varphi = T_{\varphi,\Phi,\xi}.$$

Using (4.2) and (4.4), we have

$$
\begin{aligned}
&\|\mathcal{A}(\psi,\Phi)(s,\xi)-\mathcal{A}(\varphi,\Phi)(s,\xi)\| \\
&\quad \le D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\left\|\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T_\psi(\tau)+\frac{\partial f}{\partial \psi}(y_\psi(\tau))\Phi(z_\psi(\tau))-\frac{\partial f}{\partial u_\varphi}(y_\varphi(\tau))T_\varphi(\tau)\right. \\
&\qquad \left.-\frac{\partial f}{\partial \varphi}(y_\varphi(\tau))\Phi(z_\varphi(\tau))\right\|d\tau+D\sum_{k=1}^\infty e^{-b(s_k-s)+\varepsilon s_k}\left\|\frac{\partial g_k}{\partial u_\psi}(\bar{y}_\psi(s_k,t_k))T_\psi(t_k)\right. \\
&\qquad \left.+\frac{\partial g_k}{\partial \psi}(\bar{y}_\psi(s_k,t_k))\Phi(z_\psi(t_k))-\frac{\partial g_k}{\partial u_\varphi}(\bar{y}_\varphi(s_k,t_k))T_\varphi(t_k)-\frac{\partial g_k}{\partial \varphi}(\bar{y}_\varphi(s_k,t_k))\Phi(z_\varphi(t_k))\right\| \\
&\quad \le D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\left\|\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))-\frac{\partial f}{\partial u_\varphi}(y_\varphi(\tau))\right\|\|T_\psi(\tau)\|d\tau \\
&\qquad +D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\left\|\frac{\partial f}{\partial u_\varphi}(y_\varphi(\tau))\right\|\|T_\psi(\tau)-T_\varphi(\tau)\|d\tau \\
&\qquad +D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\left\|\frac{\partial f}{\partial \psi}(y_\psi(\tau))-\frac{\partial f}{\partial \varphi}(y_\varphi(\tau))\right\|\|\Phi(z_\psi(\tau))\|d\tau \\
&\qquad +D\int_s^\infty e^{-b(\tau-s)+\varepsilon\tau}\left\|\frac{\partial f}{\partial \varphi}(y_\varphi(\tau))\right\|\|\Phi(z_\psi(\tau))-\Phi(z_\varphi(\tau))\|d\tau \\
&\qquad +D\sum_{k=1}^\infty e^{-b(s_k-s)+\varepsilon s_k}\left\|\frac{\partial g_k}{\partial u_\psi}(\bar{y}_\psi(s_k,t_k))-\frac{\partial g_k}{\partial u_\varphi}(\bar{y}_\varphi(s_k,t_k))\right\|\|T_\psi(t_k)\| \\
&\qquad +D\sum_{k=1}^\infty e^{-b(s_k-s)+\varepsilon s_k}\left\|\frac{\partial g_k}{\partial u_\varphi}(\bar{y}_\varphi(s_k,t_k))\right\|\|T_\psi(t_k)-T_\varphi(t_k)\| \\
&\qquad +D\sum_{k=1}^\infty e^{-b(s_k-s)+\varepsilon s_k}\left\|\frac{\partial g_k}{\partial \psi}(\bar{y}_\psi(s_k,t_k))-\frac{\partial g_k}{\partial \varphi}(\bar{y}_\varphi(s_k,t_k))\right\|\|\Phi(z_\psi(t_k))\| \\
&\qquad +D\sum_{k=1}^\infty e^{-b(s_k-s)+\varepsilon s_k}\left\|\frac{\partial g_k}{\partial \varphi}(\bar{y}_\varphi(s_k,t_k))\right\|\|\Phi(z_\psi(t_k))-\Phi(z_\varphi(t_k))\|. \qquad (4.10)
\end{aligned}
$$

Let $\bar{K}=(De^{\varepsilon t_1}+L)$. From (4.6) and (4.7), we have

$$
\begin{aligned}
&\|\mathcal{A}(\psi,\Phi)(s,\xi)-\mathcal{A}(\varphi,\Phi)(s,\xi)\| \\
&\quad \le 4\delta D\bar{K}e^{-\varepsilon s}\int_s^\infty e^{(-b+D\delta+\rho\ln(1+\delta D)-\varepsilon)(\tau-s)}d\tau+4\delta DL\int_s^\infty e^{-b(\tau-s)-\varepsilon\tau}d\tau \\
&\qquad +4\delta D\sum_{k=1}^\infty e^{(-a-b-\varepsilon)(s_k-s)+(D\delta+\rho\ln(1+\delta D))(t_k-s)}+4\delta DL\sum_{k=1}^\infty e^{-b(s_k-s)-(a+\varepsilon)s_k} \\
&\quad \le 4\delta D\bar{K}e^{-\varepsilon s}\int_s^\infty e^{\bar{c}(\tau-s)}d\tau+4\delta DL\int_s^\infty e^{-b(\tau-s)-\varepsilon\tau}d\tau \\
&\qquad +4\delta D\sum_{k=1}^\infty e^{(-a+\bar{c})(t_k-s)}+4\delta DL\sum_{k=1}^\infty e^{(-a-b-\varepsilon)(s_k-s)} \\
&\quad \le 4\delta D\bar{K}\int_s^\infty e^{\bar{c}(\tau-s)}d\tau+4\delta DLe^{bt_1}\int_s^\infty e^{(-b-\varepsilon)(\tau-s)}d\tau \\
&\qquad +4\delta D\sum_{k=1}^\infty e^{(-a+\bar{c})(t_k-s)}+4\delta DL\sum_{k=1}^\infty e^{(-a-b-\varepsilon)(s_k-s)}.
\end{aligned}
$$

Now given arbitrarily $\epsilon > 0$, there exists $\sigma > 0$ (independent of s and ξ) such that

$$\begin{aligned}4\delta D\bar{K}\int_{s+\sigma}^{\infty} e^{\bar{c}(\tau-s)}d\tau + 4\delta D L e^{bt_1}\int_{s+\sigma}^{\infty} e^{(-b-\varepsilon)(\tau-s)}d\tau \\ \leq \frac{4\delta D\bar{K}}{-\bar{c}}e^{\bar{c}\sigma} + \frac{4\delta D L e^{bt_1}}{b+\varepsilon}e^{(-b-\varepsilon)\sigma} \leq 2\epsilon, \quad (\text{as } \sigma \to \infty).\end{aligned}$$

and

$$4\delta D\sum_{t_k \geq s+\sigma}^{\infty} e^{(-a+\bar{c})(t_k-s)} + 4\delta D L\sum_{s_k \geq s+\sigma}^{\infty} e^{(-a-b-\varepsilon)(s_k-s)} \leq 2\epsilon, \quad (\text{as } \sigma \to \infty).$$

Next, we consider the integrals and series from s to $s+\sigma$. Define the functions

$$\begin{aligned}\mathcal{B}_1(v,\psi)(s,\xi) &= D\bar{K}e^{\varepsilon s}e^{(-b+D\delta+\rho\ln(1+\delta D)+\varepsilon)v}\frac{\partial f}{\partial u_\psi}(y_\psi(v+s)),\\ \mathcal{B}_2(v,\psi)(s,\xi) &= D\delta e^{-bv-\varepsilon(v+s)}T_\psi(v+s),\\ \mathcal{B}_3(v,\psi)(s,\xi) &= DLe^{-bv+\varepsilon(v+s)}\frac{\partial f}{\partial u_\psi}(y_\psi(v+s)),\\ \mathcal{B}_4(v,\psi)(s,\xi) &= D\delta e^{-bv-\varepsilon(v+s)}\Phi(z_\psi(v+s)),\end{aligned}$$

for each $v \in [0,\sigma]$ and $\psi \in \mathcal{Z}$. We write

$$\begin{aligned}D\bar{K}e^{\varepsilon s}&\int_s^{s+\sigma} e^{(-b+D\delta+\rho\ln(1+\delta D)+\varepsilon)(\tau-s)}\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))d\tau\\ &+ D\delta\int_s^{s+\sigma} e^{-b(\tau-s)-\varepsilon\tau}T_\psi(\tau)d\tau + DL\int_s^{s+\sigma} e^{-b(\tau-s)+\varepsilon\tau}\frac{\partial f}{\partial \psi}(y_\psi(\tau))d\tau\\ &+ D\delta\int_s^{s+\sigma} e^{-b(\tau-s)-\varepsilon\tau}\Phi(z_\psi(\tau))d\tau\\ =& \int_0^{\sigma}(\mathcal{B}_1(v,\psi)+\mathcal{B}_2(v,\psi)+\mathcal{B}_3(v,\psi)+\mathcal{B}_4(v,\psi))(s,\xi)dv.\end{aligned}$$

Therefore, from (4.10), it is sufficient to show that the maps

$$\psi \mapsto \int_0^{\sigma}(\mathcal{B}_1(v,\psi)+\mathcal{B}_2(v,\psi)+\mathcal{B}_3(v,\psi)+\mathcal{B}_4(v,\psi))(s,\xi)dv \tag{4.11}$$

and

$$\begin{aligned}\psi \mapsto D\bar{K}&\sum_{s\leq t_k<s+\sigma} e^{-b(s_k-s)+\varepsilon s_k+(D\delta+\rho\ln(1+\delta D))(t_k-s)}\frac{\partial g_k}{\partial u_\psi}(\bar{y}_\psi(s_k,t_k))\\ &+ D\sigma\sum_{s\leq t_k<s+\sigma} e^{-b(s_k-s)-(a+\varepsilon)s_k}T_\psi(t_k) + DL\sum_{s\leq s_k<s+\sigma} e^{-b(s_k-s)+\varepsilon s_k}\frac{\partial g_k}{\partial \psi}(\bar{y}_\psi(s_k,t_k))\\ &+ D\sigma\sum_{s\leq s_k<s+\sigma} e^{-b(s_k-s)-(a+\varepsilon)s_k}\Phi(z_\psi(t_k))\end{aligned} \tag{4.12}$$

are continuous. Since the functions

$$(t,\psi,s,\xi) \mapsto u_\psi(t,\xi) \quad \text{and} \quad (t,\psi,s,\xi) \mapsto T_\psi(\psi,\Phi,\xi)(t)$$

are continuous, the integral in (4.11) and the sum in (4.12) are continuous as functions of (t,ψ,s,ξ).

Finally, repeat the procedure in the proof of [7, Lemma 12] and we obtain that the map $\psi \mapsto \mathcal{A}(\psi,\Phi)$ is continuous, and the operator $\mathcal{J}$ in (3.20) is continuous. Thus the Fiber contraction $\mathcal{S}$ ia also continuous. The proof is complete. □

To establish the C^1 regularity, we establish the following.

Lemma 4.6. *If ψ is of class C^1 in ξ, then $\mathcal{J}\psi$ is of class C^1 in ξ, and*

$$\frac{\partial(\mathcal{J}\psi)}{\partial\xi} = \mathcal{A}\left(\psi, \frac{\partial\psi}{\partial\xi}\right), \tag{4.13}$$

(where we use $\frac{\partial\psi}{\partial\xi} = \frac{\partial\psi}{\partial u_\psi} \cdot \frac{\partial u_\psi}{\partial\xi}$).

Proof. If ψ is of class C^1 in ξ, then the solution $u_\psi(t,\xi)$ is also of class C^1 in ξ for $t \in \mathcal{K}$. Furthermore, denoting $\Phi = \frac{\partial\psi}{\partial\xi}$ the solution of (4.3) is given by $T(t) = \frac{\partial u_\psi}{\partial\xi}$. Therefore, using Lemma 3.7 and (4.2), we have

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\Big(W(s,\tau)Q(\tau)f(\tau,u_\psi(\tau),\psi(\tau,u_\psi(\tau)))\Big)\\
&\quad = W(s,\tau)Q(\tau)\left(\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))\frac{\partial u_\psi}{\partial\xi}(\tau,\xi) + \frac{\partial f}{\partial\psi}(y_\psi(\tau))\frac{\partial\psi}{\partial u_\psi}(\tau,u_\psi(\tau))\frac{\partial u_\psi}{\partial\xi}(\tau,\xi)\right)\\
&\quad = W(s,\tau)Q(\tau)\left(\frac{\partial f}{\partial u_\psi}(y_\psi(\tau))T(\tau) + \frac{\partial f}{\partial\psi}(y_\psi(\tau))\Phi(z_\psi(\tau))\right),
\end{aligned}$$

and similarly

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\left(W(s,s_k)Q(s_k)g_k(s_k,u(t_k),\psi(t_k,u(t_k)))\right)\\
&\quad = W(s,s_k)Q(s_k)\left(\frac{\partial g_k}{\partial u_\psi}(\bar{y}_\psi(s_k,t_k))\frac{\partial u_\psi}{\partial\xi}(t_k,\xi) + \frac{\partial g_k}{\partial\psi}(\bar{y}_\psi(s_k,t_k))\frac{\partial\psi}{\partial u_\psi}(t_k,u(t_k))\frac{\partial u_\psi}{\partial\xi}(t_k,\xi)\right)\\
&\quad = W(s,s_{j+k})Q(s_k)\left(\frac{\partial g_k}{\partial u_\psi}(\bar{y}_\psi(s_k,t_k))T(t_k) + \frac{\partial g_k}{\partial\psi}(\bar{y}_\psi(s_k,t_k))\Phi(z_\psi(t_k))\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathcal{A}\left(\psi,\frac{\partial\psi}{\partial\xi}\right) = &-\int_s^{t_1}\frac{\partial}{\partial\xi}\Big(W(s,\tau)Q(\tau)f(\tau,u_\psi(\tau),\psi(\tau,u_\psi(\tau)))\Big)d\tau\\
&-\sum_{k=1}^{\infty}\int_{s_k}^{t_{k+1}}\frac{\partial}{\partial\xi}\Big(W(s,\tau)Q(\tau)f(\tau,u_\psi(\tau),\psi(\tau,u_\psi(\tau)))\Big)d\tau\\
&-\sum_{k=1}^{\infty}\frac{\partial}{\partial\xi}\left(W(s,s_k)Q(s_k)g_k(s_k,u(t_k),\psi(t_k,u(t_k)))\right).
\end{aligned}$$

From Lemma 4.3, we can conclude that $\mathcal{J}\psi$ is of class C^1 in ξ, and (4.13) holds. The proof is complete. □

Theorem 4.7. *Assume that (1.1) admits a nonuniform exponential dichotomy, and (2.8) holds with $\delta > 0$ sufficiently small. Then for the unique function ψ in Theorem 3.8, the map $\xi \mapsto \psi(s,\xi)$ is of class C^1 for each $s \geq 0$.*

Proof. We consider the pair $(\psi_1,\Phi_1) = (0,0) \in \mathcal{Z}\times\mathcal{F}$. From Lemma 4.6, we obtain $\Phi_1 = \frac{\partial\psi_1}{\partial\xi}$. From Lemmas 4.4 and 4.5, the operator $\mathcal{S}$ is a continuous Fiber contraction. Therefore, we can define recursively a sequence $(\psi_n,\Phi_n) \in \mathcal{Z}\times\mathcal{F}$ by

$$(\psi_{n+1},\Phi_{n+1}) = \mathcal{S}(\psi_n,\Phi_n) = (\mathcal{J}\psi_n,\mathcal{A}(\psi_n,\Phi_n)).$$

One can verify that each function ψ_n is of class C^1 in ξ using Lemma 3.7. Therefore, we assume that ψ_n is of class C^1 in ξ with $\Phi_n = \frac{\partial \psi_n}{\partial \xi}$, and it follows from Lemma 4.6 that ψ_{n+1} is of class C^1 in ξ with

$$\frac{\partial \psi_{n+1}}{\partial \xi} = \frac{\partial (\mathcal{J}\psi_n)}{\partial \xi} = \mathcal{A}(\psi_n, \Phi_n) = \Phi_{n+1}. \tag{4.14}$$

Now let ψ_0 be the unique fixed point of $\mathcal{J}$, and let Φ_0 be unique fixed point $\Phi \mapsto \mathcal{A}(\psi_0, \Phi)$. It follows from Definition 4.1 that the sequences ψ_n and Φ_n uniformly converge respectively to ψ_0 and Φ_0 on bounded subsets. Next, we recall that if a functions sequence h_n of C^1 uniformly converges, and the sequence h_n' of its derivatives also uniformly converges, then the limits of h_n is of class C^1, and its derivative is the limit of h_n'. Therefore, it follows from (4.14) that the function ψ_0 is of class of C^1 in ξ, and that

$$\frac{\partial \psi_0}{\partial \xi} = \Phi_0.$$

The proof is complete. □

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