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Remark on local boundary regularity condition of a suitable weak solution to the 3D MHD equations

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Abstract. We study a local regularity condition for a suitable weak solution of the magnetohydrodynamics equations in a half space $\mathbb{R}^3_+$. More precisely, we prove that a suitable weak solution is Hölder continuous near boundary provided that the quantity

$$\limsup_{r\to 0} \frac{1}{\sqrt{r}} \Big\| \|u\|_{L^2(B^+_{x,r})} \Big\|_{L^\infty(t-r^2,t)}$$

is sufficiently small near the boundary. Furthermore, we briefly add some global regularity criteria of weak solutions to this system.

Keywords: magnetohydrodynamics equations, suitable weak solutions, local regularity condition.

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1 Introduction

We study the regularity problem for a suitable weak solution $(u, b, \pi) : Q_T \to \mathbb{R}^3_+ \times \mathbb{R}^3_+ \times \mathbb{R}$ of the three-dimensional incompressible 3D magnetohydrodynamic (MHD) equations

$$\begin{cases} u_t - \triangle u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \pi = 0 \\ b_t - \triangle b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0 \\ \text{div } u = 0 \quad \text{and} \quad \text{div } b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x) \end{cases} \quad \text{in } \ Q_T := \mathbb{R}^3_+ \times [0, T). \tag{1.1}$$

Here, u is the fluid flow vector, b is the magnetic vector and $\pi = p + \frac{|b|^2}{2}$ is the total pressure. We consider the initial value problem of (1.1), which requires initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad b(x,0) = b_0(x), \qquad x \in \mathbb{R}^3_+. \tag{1.2}$$

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We assume that the initial data $u_0(x)$, $b_0(x) \in L^2(\mathbb{R}^3)$ hold the incompressibility, i.e. $\operatorname{div} u_0(x) = 0$ and $\operatorname{div} b_0(x) = 0$, respectively. The boundary conditions of u and b are given as slip and slip conditions, respectively, namely

$$u \cdot \nu = 0, \ (\nabla \times u) \times \nu = 0, \quad \text{and} \quad b \cdot \nu = 0, \ (\nabla \times b) \times \nu = 0, \qquad \text{on } \partial\mathbb{R}^3_+, \tag{1.3}$$

where $\nu = (0,0,-1)$ is the outward unit normal vector along boundary $\partial\mathbb{R}^3_+$. Suitable weak solutions mean solutions that solve MHD equations in the sense of distribution and satisfy the local energy inequality (see Definition 2.1 in section 2 for details).

Let $x = (x_1, x_2, 0) \in \partial\mathbb{R}^3_+$. For a point $z = (x,t) \in \partial\mathbb{R}^3_+ \times (0,T)$, we denote

$$B_{x,r} := \{y \in \mathbb{R}^3_+ : |y - x| < r\}, \qquad B^+_{x,r} := \{y \in B_{x,r} : y_3 > 0\},$$
$$Q_{z,r} := B_{x,r} \times (t - r^2, t), \qquad Q^+_{z,r} := \{(y,t) \in Q_{z,r} : y_3 > 0\}, \qquad r < \sqrt{t}.$$

We say that solutions u and b are regular at $z \in \overline{\mathbb{R}^3_+} \times (0,T)$ if u and b are Hölder continuous for some $Q^+_{z,r}$, $r > 0$. Otherwise, it is said that u and b are singular at z.

For the existence of weak solutions for 3D MHD equations, it is well known that it is globally in time and moreover, in the two-dimensional case, it become regular in [4]. On the other hand, the existence of weak solution for MHD equations with boundary condition (1.3) in dimension three is proved in [13] and it is shown in [16] that if weak solutions become regular under some conditions. However, in dimension three, a regularity question remains open not yet as in Navier–Stokes equations.

We review some of known results in this direction related to our concerns, in particular, we focus on the boundary regularity.

In [21], authors proved that a suitable weak solution (u,b) to the 3D MHD equations become regular near a boundary point z if the following condition is satisfied: There exists $\epsilon > 0$ such that for $1 \le \frac{3}{p} + \frac{2}{q} \le 2$, $1 \le q \le \infty$ and $(p,q) \ne (\infty, 1)$,

$$\limsup_{r \to 0} r^{-(\frac{3}{p}+\frac{2}{q}-1)} \left\| \|u\|_{L^p(B^+_{x,r})} \right\|_{L^q(t-r^2,t)} < \epsilon.$$

(cf. [11, 19, 20] for the dimension three or [7] for the dimension four). Recently, the author in [12] proved a suitable weak solution (u,b) are Hölder continuous near the boundary, provided, on a parabolic cylinder, the scaled $L^{p,q}_{x,t}$-norm of the velocity with $\frac{3}{p} + \frac{2}{q} \le 2$, $2 < q \le \infty$ is sufficiently small near a boundary point $x = (x_1, x_2, 0)$. Here we highlight that additional condition are imposed on only velocity vector field.

The motivation of our study is to establish new local regularity condition to 3D MHD equations in the bounded domains with slip boundary conditions (1.3). The local regularity problem with Dirichlet boundary conditions is proved in [11, Theorem 1.1]. Its proof is also applied to the problem with the slip boundary condition (1.3) for a fluid vector field. Considering the slip boundary condition (1.3), to estimate the local pressure quantity, we use Calderón–Zygmund estimate with the reflection method. For this, we give a definition of a suitable weak solution for the magnetohydrodynamic equation with the slip boundary condition (1.3) in [12, Appendix]. Moreover, It is known that for a suitable weak solution the set of singular points in space-time has one-dimensional parabolic Hausdorff measure zero (see e.g. [18] or [19]).

The organization of the present paper is as follows. In Section 2, we introduce some notation and state our main theorems. Section 3 is devoted to prove the main theorem. In Section 4, we briefly add some global regularity criteria of weak solutions to this system.

2 Main results: local boundary regularity

In this section, we introduce some scaling invariant functionals and a suitable weak solution.

We first start with some notations used in the paper. Let Ω be a domain in $\mathbb{R}^3_+$ and I be a finite time interval. For $1 \le q \le \infty$, we denote the usual Sobolev spaces by $W^{k,q}(\Omega) = \{u \in L^q(\Omega) : D^\alpha u \in L^q(\Omega), 0 \le |\alpha| \le k\}$. As usual, $W^{k,q}_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ in the $W^{k,q}(\Omega)$ norm. We also denote by $W^{-k,q'}(\Omega)$ the dual space of $W^{k,q}_0(\Omega)$, where q and q' are Hölder conjugates. We write the average of f on E as $⨍_E f$, that is $⨍_E f = \int_E f / |E|$. For a function $f(x,t)$, we denote

$$\|f\|_{L^{p,q}_{x,t}(\Omega\times I)} = \|f\|_{L^q_t(I;L^p_x(\Omega))} = \Big\| \|f\|_{L^p_x(\Omega)} \Big\|_{L^q_t(I)}.$$

For vector fields u, v we write $(u_i v_j)_{i,j=1,2,3}$ as $u \otimes v$. We denote by $C = C(\alpha, \beta, \dots)$ a generic constant, which may change from line to line.

We introduce scaling invariant quantities near boundary. Let $z = (x,t) \in \partial\mathbb{R}^3_+ \times I$ and we set

$$A_u(r) := \sup_{t-r^2 \le s < t} \frac{1}{r} \int_{B^+_{x,r}} |u(y,s)|^2 dy, \qquad E_u(r) := \frac{1}{r} \int_{Q^+_{z,r}} |\nabla u(y,s)|^2 dyds,$$
$$A_b(r) := \sup_{t-r^2 \le s < t} \frac{1}{r} \int_{B^+_{x,r}} |b(y,s)|^2 dy, \qquad E_b(r) := \frac{1}{r} \int_{Q^+_{z,r}} |\nabla b(y,s)|^2 dyds,$$
$$M_u(r) := \frac{1}{r^2} \int_{Q^+_{z,r}} |u(y,s)|^3 dyds, \qquad (g)_r(s) := ⨍_{B^+_{x,r}} g(\cdot, s) dy$$

Next we recall a suitable weak solution for the 3D MHD equations.

Definition 2.1. A triple of (u, b, π) is a suitable weak solution to (1.1)–(1.3) if the following conditions are satisfied.

(a) The functions $u, b : Q_T \to \mathbb{R}^3$ and $\pi : Q_T \to \mathbb{R}$ satisfy

$$u, b \in L^\infty\big(I; L^2(B^+_{x,r})\big) \cap L^2\big(I; W^{1,2}(B^+_{x,r})\big), \qquad \pi \in L^{\frac{3}{2}}\big(I; L^{\frac{3}{2}}(B^+_{x,r})\big).$$

(b) (u, b, π) solves the MHD equations in Q_T in the sense of distributions and u and b satisfy the boundary conditions (1.3) in the sense of traces.

(c) u, b and π satisfy the local energy inequality

$$\begin{aligned} &\int_{B^+_{x,r}} (|u(x,t)|^2 + |b(x,t)|^2)\phi(x,t)dx + 2\int_{t_0}^t \int_{B^+_{x,r}} (\left|\nabla u(x,t')\right|^2 + \left|\nabla b(x,t')\right|^2)\phi(x,t')dxdt' \\ &\quad \le \int_{t_0}^t \int_{B^+_{x,r}} (|u|^2 + |b|^2)(\partial_t \phi + \Delta\phi)dxdt' + \int_{t_0}^t \int_{B^+_{x,r}} \left(|u|^2 + |b|^2 + 2\pi\right) u \cdot \nabla\phi dxdt' \\ &\quad - 2\int_{t_0}^t \int_{B^+_{x,r}} (b \cdot u)(b \cdot \nabla\phi)dxdt'. \end{aligned}$$

for all $t \in I = (0,T)$ and for all nonnegative function $\phi \in C^\infty_0(\mathbb{R}^3 \times \mathbb{R})$.

Following the argument in [2,7], we denote $Q^+_{z_0,r}$ by Q^+_r and let ξ be a cutoff function, which vanishes outside of Q^+_ρ and equals 1 in $Q^+_{\frac{\rho}{2}}$, and satisfies $|\nabla\xi| \le C_0\rho^{-1}$, and $|\xi_t|, |\Delta\xi| \le C_0\rho^{-2}$.

Define the backward heat kernel as $\Gamma(x,t) = \frac{1}{4\pi(r^2-t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4(r^2-t)}}$. Note that $\Gamma_t + \Delta\Gamma = 0$ Taking the test function $\phi = \Gamma\xi$ in the local energy inequality, we obtain

$$\begin{aligned}
&\int_{B_r^+} (|u(x,t)|^2 + |b(x,t)|^2)\phi(x,t)dx + 2\int_{t_0}^{t}\int_{B_r^+} (\left|\nabla u(x,t')\right|^2 + \left|\nabla b(x,t')\right|^2)\phi(x,t')dxdt' \\
&\quad \le \int_{t_0}^{t}\int_{B_r^+} (|u|^2 + |b|^2)(\Gamma\Delta\xi + \Gamma\partial_t\xi + 2\nabla\Gamma\nabla\xi)dxdt' \\
&\quad\quad + \int_{t_0}^{t}\int_{B_r^+} \left(|u|^2 + |b|^2 + 2\pi\right) u\cdot\nabla\phi dxdt' - 2\int_{t_0}^{t}\int_{B_r^+} (b\cdot u)(b\cdot\nabla\phi)dxdt'. \qquad (2.1)
\end{aligned}$$

By straightforward computations, it is easy to verify that

$$\begin{aligned}
\Gamma(x,t) &\ge C_0^{-1} r^{-3}, \\
|\nabla\phi|\,|\nabla\Gamma|\,\xi + |\nabla\xi|\,\Gamma &\le C_0 r^{-4}, \\
|\Gamma\Delta\xi| + |\Gamma\partial_t\xi| + 2\,|\nabla\xi\nabla\Gamma| &\le C_0 r^{-5}.
\end{aligned}$$

Using the property of a test function, the local energy inequality (2.1) becomes to

$$\begin{aligned}
&\int_{B_r^+} (|u(x,t)|^2 + |b(x,t)|^2)\phi(x,t)dx + 2\int_{t_0}^{t}\int_{B_r^+} (\left|\nabla u(x,t')\right|^2 + \left|\nabla b(x,t')\right|^2)\phi(x,t')dxdt' \\
&\quad \le C_0\left(\frac{r}{\rho}\right)^2 \frac{1}{r^3}\int_{B_\rho^+} (\left|u(x,t')\right|^2 + \left|b(x,t')\right|^2 dxdt' + C_0\left(\frac{\rho}{r}\right)^2\frac{1}{\rho^2}\int_{B_\rho^+} (|u|^3 + |u|\,|b|^2 + |u|\,|\pi|)dxdt
\end{aligned}$$

(see e.g., [21] or [7, Lemma 3.8])

Note that due to the local energy estimate (2), we do not need to deal with the square norm of b. For this reason, the analysis become simple and concise.

Now we are ready to state our result.

Theorem 2.2. *Let (u,b,π) be a suitable weak solution of the MHD equations (1.1)–(1.3) according to Definition 2.1. There exists $\epsilon > 0$ such that for some point $z = (x,t) \in \mathbb{R}^3_+ \times (0,T)$ u is locally in $L^{2,\infty}_{x,t}$ near z and*

$$\limsup_{r\to 0} \frac{1}{\sqrt{r}} \left\| \|u\|_{L^2(B^+_{x,r})} \right\|_{L^\infty(t-r^2,t)} < \epsilon. \qquad (2.2)$$

Then, u and b are regular at z.

3 Proof of Theorem 2.2

Next we prove a local ϵ-regularity condition near boundary for the MHD equations, which is a key role for our proof (see [9, 18]). In fact, the proof in [9, 18] also hold for our case due to the pressure estimate (3.9) below.

Proposition 3.1. *Let (u,b,π) be a suitable weak solution of (1.1)–(1.3). Then there exists a positive number ε_* with the following property. Assume that for a point $z_0 = (x_0,t_0) \in Q_T$ the inequality*

$$\limsup_{r\to 0} \frac{1}{r^2}\int_{Q^+_{z_0,r}} |u|^3 + |b|^3 + |\pi|^{\frac{3}{2}} < \varepsilon_*$$

holds. Then z_0 is a regular point of (u,b).

3.1 Boundary interior estimates

In this section, we prove a local regularity criterion for the 3D MHD equations. For simplicity, we write $\Psi(r) := A_u(r) + A_b(r) + E_u(r) + E_b(r)$. Let $z = (x,t) \in \partial\mathbb{R}^3_+ \times I$ and from now on, without loss of generality, we assume $x = 0$ by translation. We first recall that the local energy estimate.

$$\Psi\Big(\frac{r}{2}\Big) \le C\bigg(M_u^{\frac{2}{3}}(r) + M_u(r) + \frac{1}{r^2}\int_{Q^+_{z,r}} |u|\,|b|^2\,dz + \frac{1}{r^2}\int_{Q^+_{z,r}} |u|\,|\pi|\,dz\bigg).$$

Next lemma is estimates of the scaled integral of cubic term of u and multiple of u and square of b.

Lemma 3.2. *Let $z \in \partial\mathbb{R}^3_+ \times I$. Under the assumption above, for $0 < 4r < \rho$,*

$$M_u(r) \le C\Big(\frac{\rho}{r}\Big)\Psi(\rho)\epsilon, \tag{3.1}$$

and

$$\frac{1}{r^2}\int_{Q_{z,r}} |u|\,|b|^2\,dz \le C\Big(\frac{\rho}{r}\Big)\Psi(\rho)\epsilon. \tag{3.2}$$

Proof. It is sufficient to show estimate (3.2) because (3.1) can be proved in the same way as (3.2). We note first that via Hölder's inequality

$$\frac{1}{r^2}\int_{Q^+_{z,r}} |u|\,|b|^2\,dxds \le \frac{1}{r^{1/2}}\,\|u\|_{L^{2,\infty}_{x,t}(Q^+_{z,r})}\,\frac{1}{r^{3/2}}\,\|b\|^2_{L^{4,2}_{x,t}(Q^+_{z,r})}\,. \tag{3.3}$$

So, we see that

$$\begin{aligned}
\|b\|_{L^4_x(B^+_{x,r})} &\le \|b\|^{\frac14}_{L^2_x(B^+_{x,r})}\,\|b-(b)_r\|^{\frac34}_{L^6_x(B^+_{x,r})} + \|b\|^{\frac14}_{L^2_x(B^+_{x,r})}\,\|(b)_r\|^{\frac34}_{L^6_x(B^+_{x,r})}\\
&\le C\,\|b\|^{\frac14}_{L^2_x(B^+_{x,r})}\,\|\nabla b\|^{\frac34}_{L^2_x(B^+_{x,r})} + \|b\|_{L^2_x(B^+_{x,r})}\,r^{-\frac68},
\end{aligned}$$

where we used the Poincaré inequality and the following estimate

$$\begin{aligned}
\|(b)_r\|^{\frac34}_{L^6_x(B^+_{x,r})} &= \left\|\frac{1}{r^3}\int_{B^+_{x,r}} bdx\right\|^{\frac34}_{L^6_x(B_{x,r})} = \left\|\frac{1}{r^3}\|b\|_{L^2(B^+_{x,r})}\|1\|_{L^2(B_{x,r})}\right\|^{\frac34}_{L^6_x(B^+_{x,r})}\\
&= \Big(\int_{B_{x,r}}\Big|\frac{1}{r^3}\|b\|_{L^2(B^+_{x,r})}\|1\|_{L^2(B^+_{x,r})}\Big|^6 dx\Big)^{\frac18}\\
&= \|b\|^{\frac34}_{L^2(B^+_{x,r})}\Big(\int_{B^+_{x,r}}\Big|\frac{1}{r^3}\|1\|_{L^2(B^+_{x,r})}\Big|^6 dx\Big)^{\frac18} = \|b\|^{\frac34}_{L^2(B^+_{x,r})}\Big(\int_{B^+_{x,r}}\Big|\frac{1}{r^3}r^{\frac32}\Big|^6 dx\Big)^{\frac18}\\
&= \|b\|^{\frac34}_{L^2(B^+_{x,r})}\Big(r^{-\frac32 6}r^3\Big)^{\frac18} = \|b\|^{\frac34}_{L^2(B^+_{x,r})}r^{-\frac68}.
\end{aligned}$$

Taking L^2 norm in temporal variable and using Young's inequality,

$$\|b\|^2_{L^{4,2}_{x,t}(Q^+_{z,r})} \le r^{\frac12}\,\|b\|^2_{L^{2,\infty}_{x,t}(Q_{z,r})} + r^{\frac12}\,\|\nabla b\|^2_{L^{2,2}_{x,t}(Q^+_{z,r})}\,,$$

due to the estimate

$$\begin{aligned}
\|b\|^2_{L^{4,2}_{x,t}(Q^+_{z,r})} &\le \int_{-r^2}^0 r^{-3/8}\,\|b\|^{\frac12}_{L^2_x(B^+_{x,r})}\,r^{3/8}\,\|\nabla b\|^{\frac32}_{L^2_x(B^+_{x,r})}\,dt + \int_{-r^2}^0 \|b\|^2_{L^2_x(B^+_{x,r})}\,r^{-\frac32}dt\\
&\le \int_{-r^2}^0 r^{-3/2}\,\|b\|^2_{L^2_x(B^+_{x,r})} + r^{1/2}\,\|\nabla b\|^2_{L^2_x(B^+_{x,r})}\,dt + \int_{-r^2}^0 \|b\|^2_{L^2_x(B^+_{x,r})}\,r^{-\frac32}dt\\
&\le r^{1/2}\,\|b\|^2_{L^{2,\infty}_{x,t}(B^+_{x,r})} + r^{1/2}\,\|\nabla b\|^2_{L^{2,2}_{x,t}(B^+_{x,r})} + \|b\|^2_{L^{2,\infty}_{x,t}(B^+_{x,r})}\,r^{2-\frac32}.
\end{aligned}$$

Recalling (3.3), we can have

$$\begin{aligned}\frac{1}{r^2}\int_{Q^+_{z,r}}|u|\,|b|^2\,dxds &\le \frac{1}{r^{1/2}}\|u\|_{L^{2,\infty}_{x,t}(Q^+_{z,r})}\frac{1}{r^{3/2}}\,\|b\|^2_{L^{4,2}_{x,t}(Q^+_{z,r})}\\ &\le C\epsilon\Big((r^{\frac{1}{2}})\frac{1}{r^{3/2}}\|b\|^2_{L^{2,\infty}_{x,t}(Q^+_{z,r})}+(r^{\frac{1}{2}})\frac{1}{r^{3/2}}\,\|\nabla b\|^2_{L^{2,2}_{x,t}(Q^+_{z,r})}\Big)\\ &\le C\epsilon\Big(\frac{1}{r}\|b\|^2_{L^{2,\infty}_{x,t}(Q^+_{z,r})}+\frac{1}{r}\,\|\nabla b\|^2_{L^{2,2}_{x,t}(Q^+_{z,r})}\Big)\le C\Psi(r)\epsilon\le C(\frac{\rho}{r})\Psi(\rho)\epsilon.\end{aligned}$$

This completes the proof. □

For an estimate for the scaled pressure quantity, we need the following pressure representation. Its proof is similar to that in [1, Theorem 2.1] and we only give a sketch proof.

Lemma 3.3. *Suppose u, b and π is measurable functions and a distribution, respectively, satisfying (1.1)–(1.3) in the sense of distributions. Then π has the following representation: for almost all time $t\in(0,T)$*

$$\pi(x,t)=\frac{-\delta_{ij}}{3}(u^*_iu^*_j-b^*_ib^*_j)+\frac{3}{4\pi}\int_{\mathbb{R}^3_+}\left(\frac{\partial^2}{\partial y_i\partial y_j}\frac{1}{|x-y|}\right)(u^*_iu^*_j-b^*_ib^*_j)(y,t)dy$$

in the sense of distributions, where δ_{ij} is the Kronecker delta function. Here, $u^*(y)=u(y)$ and $b^*(y)=b(y)$ for $y_3>0$, and

$$\begin{aligned}&u^*_1(y,t)=u_1(y^*,t), && u^*_2(y,t)=u_2(y^*,t), && u^*_3(y,t)=-u_3(y^*,t),\\ &b^*_1(y,t)=b_1(y^*,t), && b^*_2(y,t)=b_2(y^*,t), && b^*_3(y,t)=-b_3(y^*,t)\end{aligned}$$

for $y_3<0$, and $y^*=(y_1,y_2,-y_3)$.

Proof. Set $g(x,t)=-[(u\cdot\nabla)u](x,t)+[(b\cdot\nabla)b](x,t)$ for $x_3\ge 0$. Define $g^*=(g^*_1,g^*_2,g^*_3)$ by

$$g^*_1(x,t)=\begin{cases}g_1(x,t), & \text{if } x_3\ge 0,\\ g_1(x^*,t), & \text{if } x_3<0,\end{cases}\tag{3.4}$$

$$g^*_2(x,t)=\begin{cases}g_2(x,t), & \text{if } x_3\ge 0,\\ g_2(x^*,t), & \text{if } x_3<0,\end{cases}\tag{3.5}$$

$$g^*_3(x,t)=\begin{cases}g_3(x,t), & \text{if } x_3\ge 0,\\ -g_3(x^*,t), & \text{if } x_3<0.\end{cases}\tag{3.6}$$

We also consider g^* as the even-even-odd extension. Since $u_3=0$ and $b_3=0$, it is easy to see $g_3=0$.

By observing $\partial_3u_{0,1}=\partial_3u_{0,2}=u_{0,3}=0$ and $\partial_3b_{0,1}=\partial_3b_{0,2}=b_{0,3}=0$ on $x_3=0$, it follows that $u^*_0\in C^1(\mathbb{R}^3)$, and $g^*\in C(\mathbb{R}^3)$. Observe that for $j=1,2,3$,

$$g^*_j(x,t)=-[(u^*\cdot\nabla)u^*_j](x,t)+[(b^*\cdot\nabla)b^*_j](x,t)\quad\text{for } x_3<0.$$

Hence,

$$g^*(x,t)=\begin{cases}-\nabla\cdot(u\otimes u)(x,t)+\nabla\cdot(b\otimes b)(x,t), & \text{if } x_3\ge 0,\\ -\nabla\cdot(u^*\otimes u^*)(x,t)+\nabla\cdot(b^*\otimes b^*)(x,t), & \text{if } x_3\le 0.\end{cases}\tag{3.7}$$

Since $\partial_3 u_1 = \partial_3 u_2 = u_3 = 0$ and $\partial_3 b_1 = \partial_3 b_2 = b_3 = 0$ on $x_3 = 0$, it follows that

$$g^*(x,t) = -\nabla \cdot (u^* \otimes u^*)(x,t) + \nabla \cdot (b^* \otimes b^*)(x,t)$$

in the sense of distributions. Now we construct (v,q) a solution of the Stokes system in $\mathbb{R}^3$:

$$\begin{aligned} v_t - \Delta v + \nabla p = f, &\qquad \operatorname{div} v = 0, \\ h_t - \Delta h = \tilde{f}, &\qquad \operatorname{div} h = 0 \qquad \text{in } \mathbb{R}^3 \times (0,T) \end{aligned} \tag{3.8}$$

with initial data $v(x,0) = u_0^*(x)$, $h(x,0) = b_0^*(x)$ and infinity conditions $v(x,t) \to 0$ and $h(x,t) \to 0$ as $|x| \to \infty$. Then, q satisfies the Laplace equation $q(x,t) = \operatorname{div} g^*(x,t)$ in $R^3 \times (0,T)$. We try to find q integrable. By integral representation, q is expressed by

$$\begin{aligned} q(x,t) &= -\frac{3}{4\pi}\int_{R^3} \frac{1}{|x-y|}\partial_j g_j^*(y,t)dy. \\ &= \frac{-\delta_{ij}}{3}(u_i^* u_j^* - b_i^* b_j^*) + \frac{3}{4\pi}\int_{\mathbb{R}^3_+}\left(\frac{\partial^2}{\partial y_i \partial y_j}\frac{1}{|x-y|}\right)(u_i^* u_j^* - b_i^* b_j^*)(y,t)dy. \end{aligned}$$

Lastly, it remains to check $u \equiv v$, $b \equiv h$ and $\pi \equiv q + c_0$ for a constant c_0 in $R^3_+ \times (0,T)$. Thus, the proof of this parts is almost same to that the arguments in [1, Theorem 2.1]. This complete the proof. □

Lemma 3.3 implies that

$$\|\pi\|_{L^p(\mathbb{R}^3_+)} \le C\left(\|u\|^2_{L^{2p}(\mathbb{R}^3_+))} + \|b\|^2_{L^{2p}(\mathbb{R}^3_+))}\right), \qquad 1 < p < \infty. \tag{3.9}$$

Following in [21, Lemma 3.3], we also obtain the following pressure estimate.

Lemma 3.4. *For $0 < 4r < \rho$, we have*

$$\begin{aligned} \frac{1}{r^{3/2}}\|\pi - (\pi)_r\|_{L^{2,1}_{x,t}(Q^+_{z,r})}(r) &\le C\left(\frac{\rho}{r}\right)\left(\|u-(u)_\rho\|^2_{L^{4,2}_{x,t}(Q^+_{z,r})}(\rho) + \|b-(b)_\rho\|^2_{L^{4,2}_{x,t}(Q^+_{z,r})}(\rho)\right) \\ &\quad + C\left(\frac{r}{\rho}\right)^{\frac{3}{2}}\|\pi - (\pi)_r\|_{L^{2,1}_{x,t}(Q^+_{z,r})}(\rho). \end{aligned} \tag{3.10}$$

Under the hypothesis (2.2) and Lemma 3.2, we note first that for $4r < \rho$

$$M_u(r) + \frac{1}{r^2}\int_{Q_{z,r}} |u|\,|b|^2\,dz \le C\epsilon\left(\frac{\rho}{r}\right)\Psi(\rho). \tag{3.11}$$

Next, due to the pressure estimate (3.10), we obtain

$$\begin{aligned} \frac{1}{r^2}\int_{Q^+_{z,r}} |u|\,|\pi|\,dz &\le \frac{1}{r^{1/2}}\|u\|_{L^{2,\infty}_{x,t}(B^+_{z,r})}\frac{1}{r^{3/2}}\,\|\pi\|_{L^{2,1}_{x,t}(Q^+_{z,r})} \\ &\le C\epsilon\left(\frac{\rho}{r}\right)\Psi(\rho) + C\epsilon\left(\frac{r}{\rho}\right)^{\frac{3}{2}}\|\pi - (\pi)_r\|_{L^{2,1}_{x,t}(Q^+_{z,r})}(\rho) \end{aligned} \tag{3.12}$$

Proof of Theorem 2.2. Following [11] or [21], we prove Theorem 2.2. Combining estimates (3.11) and (3.12), we have via the local energy inequality

$$\Psi\left(\frac{r}{2}\right) \le C\epsilon\left(\frac{\rho}{r}\right)\Psi(\rho) + C\epsilon\left(\frac{r}{\rho}\right)^{\frac{3}{2}}\|\pi - (\pi)_r\|_{L^{2,1}_{x,t}(Q^+_{z,r})}(\rho) \tag{3.13}$$

Set $P(\rho) = \|\pi - (\pi)_r\|_{L^{2,1}_{x,t}(Q^+_{z,r})}(\rho)$. Let ϵ_3 be a small positive number, which will be specified later. Now via estimates (3.10) and (3.13) we consider

$$\Psi\left(\frac{r}{2}\right) + \epsilon_3 P\left(\frac{r}{2}\right) \leq C(\epsilon + \epsilon_3)\left(\frac{\rho}{r}\right)\Psi(\rho) + C(\epsilon + \epsilon_3)\left(\frac{r}{\rho}\right)^{\frac{3}{2}} P(\rho).$$

We choose ϵ_3 and ϵ such that

$$0 < \epsilon_3 < \min\left\{\frac{\theta}{8C}\right\}, \qquad 0 < \epsilon < \min\left\{\frac{\epsilon^*}{8}, \frac{\epsilon_3}{8C} \frac{\theta}{8C}\right\},$$

where ϵ^* is the number introduced in Proposition 3.1. Take $r = \theta\rho$ with $0 < \theta < \frac{1}{8}$. We then obtain

$$\Psi(\theta r) + \epsilon_3 S(\theta r) \leq \frac{\epsilon^*}{4} + \frac{1}{4}\Big(\Psi(r) + \epsilon_3 S(r)\Big).$$

Usual method of iteration implies that there exists a sufficiently small $r_0 > 0$ such that for all $r < r_0$

$$\Psi(r) + \epsilon_3 S(r) \leq \frac{\epsilon^*}{2}.$$

This completes the proof. $\square$

4 Comment

4.1 Global boundary regularity criteria of weak solutions

In this section, comparison to the previous section, we see some global boundary regularity criteria of weak solutions to the equations (1.1)–(1.3). And we add a related model which is applied for out analysis.

It is well known the regularity criteria for weak solutions to the 3D MHD equations with respect to the velocity vector [8, 10] or the total pressure [3, 17, 23] (also comparison to [5] and [6]).

Theorem 4.1. *For the initial data in $H^s(\mathbb{R}^3_+)$, $s \geq 3$, if the velocity vector u, the magnetic vector b and the total pressure π, associated with smooth solutions of the equations (1.1)–(1.3) satisfy one of the following conditions:*

1. $u \in L^{\frac{2r}{r-3}}(0, T; L^r(\mathbb{R}^3_+))$, *with* $3 < r \leq \infty$,
2. $\nabla u \in L^{\frac{2r}{2r-3}}(0, T; L^r(\mathbb{R}^3_+))$, *with* $\frac{3}{2} < r \leq \infty$,
3. $\pi \in L^{\frac{2r}{2r-3}}(0, T; L^r(\mathbb{R}^3_+))$, *with* $\frac{3}{2} < r \leq \infty$,
4. $\nabla \pi \in L^{\frac{2r}{3r-3}}(0, T; L^r(\mathbb{R}^3_+))$, *with* $1 < r \leq \infty$,

then (u, b) can be extended smoothly beyond $t = T$.

Remark 4.2. Note that these quantities in Theorem 4.1 are scale invariant.

Remark 4.3. It is used Lemma 3.3 for the proof of Theorem 4.1. For the regularity criteria for the total pressure (or regularity criteria for the velocity vector), the proof is almost same to that in [3] (or [22]). Indeed, we note first that in case $\mathbb{R}^3_+$, the slip boundary conditions are rewritten in terms of components of vectors as

$$u_{1,x_3} = u_{2,x_3} = u_3 = 0, \qquad b_{1,x_3} = b_{2,x_3} = b_3 = 0 \quad \text{on} \ \{x_3 = 0\}. \tag{4.1}$$

Thus, the boundary term which is appeared by the integration by parts, vanishes due to the boundary condition (4.1) and $n = (0, 0, -1)$ on $\{x_3 = 0\}$ (see e.g. [10]). And owing to Lemma 3.3, we deal with pressure terms appropriately according to the argument in [3] or [22]. For these reasons, we omit the detailed proof.

4.2 Viscoelastic model with damping

The authors of [14] introduced the viscoelastic model with damping:

$$\begin{cases} u_t - \triangle u + (u\cdot\nabla)u + \nabla P = \nabla(FF^{\perp}) \\ F_t - \mu\triangle F + (u\cdot\nabla)F = \nabla u F \\ \operatorname{div} u = 0, \\ u(x,0) = u_0(x), \end{cases} \qquad \text{in} \ \ Q_T := \mathbb{R}^3 \times [0, T), \tag{4.2}$$

for a parameter $\mu > 0$. Here, $u = u(x,t) \in \mathbb{R}^3$ represents the fluid's velocity, $P = P(x,t) \in \mathbb{R}$ represents the fluid's pressure, and $F = F(x,t) \in \mathbb{R}^3 \times \mathbb{R}^3$ represents the local deformation tensor of the fluid. We denote $(\nabla \cdot F)_i = \frac{\partial F_{ij}}{\partial_{x_j}}$ for a matrix F, in the (i,j)-th entries, where we use the Einstein summation convention.

Thanks to Hynd's local analysis result of a suitable weak solution to this system [15], Theorem 2.2 holds replacing b by $F := F_k$ in the proof of Theorem 2.2.

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