Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **45**, 1–12; https://doi.org/10.14232/ejqtde.2019.1.45 www.math.u-szeged.hu/ejqtde/

Existence and uniqueness of positive even homoclinic solutions for second order differential equations

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Received 12 February 2019, appeared 28 June 2019
Communicated by Petru Jebelean

Abstract. This paper is concerned with the existence of positive even homoclinic solutions for the p-Laplacian equation

$$(|u'|^{p-2}u')' - a(t)|u|^{p-2}u + f(t,u) = 0, \qquad t \in \mathbb{R},$$

where $p \geq 2$ and the functions a and f satisfy some reasonable conditions. Using the Mountain Pass Theorem, we obtain the existence of a positive even homoclinic solution. In case $p = 2$, the solution obtained is unique under a condition of monotonicity on the function $u \longmapsto \frac{f(t,u)}{u}$. Some known results in the literature are generalized and significantly improved.

Keywords: homoclinic solution, the (PS)-condition, Mountain Pass Theorem, p-Laplacian equation, uniqueness.

2010 Mathematics Subject Classification: 34C37, 35A15, 37J45.

1 Introduction

In this paper, we study the existence of positive even homoclinic solutions for the p-Laplacian equation

$$(|u'|^{p-2}u')' - a(t)|u|^{p-2}u + f(t,u) = 0, \qquad t \in \mathbb{R}, \tag{1.1}$$

where $p \geq 2$. We assume that

(H0) $a \in C^1(\mathbb{R},\mathbb{R}), f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ is continuously differentiable with respect to the first variable and there exist constants a_0, A such that $0 < a_0 \leq a(t) \leq A$. Moreover, $a(-t) = a(t)$, $f(-t,u) = f(t,u)$ and $ta'(t) > 0$, $tf_t(t,u) < 0$ for $t \neq 0$, $u > 0$.

By a solution of (1.1), we mean a function $u \in C^1(\mathbb{R},\mathbb{R})$ such that $(|u'|^{p-2}u')' \in C(\mathbb{R},\mathbb{R})$ and equation (1.1) holds for every $t \in \mathbb{R}$. We say that a solution u of (1.1) is a nontrivial homoclinic solution (to 0) if $u \not\equiv 0$, $u(t) \to 0$ and $u'(t) \to 0$ as $|t| \to \infty$.

When $p = 2$, equation (1.1) reduces to the second order differential equation

$$u'' - a(t)u + f(t,u) = 0, \qquad t \in \mathbb{R}, \tag{1.2}$$

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which is a generalization of

$$u'' - a(t)u + b(t)u^2 + c(t)u^3 = 0, \qquad t \in \mathbb{R}. \tag{1.3}$$

The existence of a nontrivial positive homoclinic solution of equation (1.3) follows from [7], where the coefficients are either even or periodic. In the case of evenness and under the following conditions mainly

$$0 < a < a(t), \qquad 0 \le b \le b(t) \le B, \qquad 0 < c \le c(t) \le C, \quad \text{for all } t \in \mathbb{R}, \tag{1.4}$$

with a, b, c, B, C are real constants and

$$ta'(t) > 0, \qquad tb'(t) \le 0, \qquad tc'(t) < 0 \quad \text{for all } t \neq 0,$$

the authors proved the existence of a unique nontrivial even positive homoclinic solution by using variational approach. Their result extends the existence theorem established earlier by Korman and Lazer in [10], where $b(t)$ is identically zero. It is well known that equation (1.3) plays a key role in biomathematics models suggested by Austin [1] and Cronin [3] to describe an aneurysm of the circle of Willis. Also, equation (1.1) was considered, recently in [17], in the special case where $f(t,u) = \lambda b(t)|u|^{q-2}u$, with $2 \le p < q$, $\lambda > 0$ and the functions a and b are strictly positive and even.

During the last decades the study of homoclinic solutions for the p-Laplacian equation (1.1) and the more general Hamiltonian system

$$\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) - a(t)|u|^{p-2}u + \nabla V(t,u(t)) = 0, \qquad t \in \mathbb{R},$$

where $p > 1, V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, has been investigated by many authors with various nonlinearities (see [4, 9, 12, 16, 17] and references therein). Whereas, the existence results for even homoclinics are scarce. Moreover, the question of uniqueness is treated only in limited cases (see [2, 18]) and frequently remains open.

Motivated by the above works mainly, in this paper, we study the existence of positive even homoclinic solution for the p-Laplacian equation (1.1). This will be done under assumptions less restrictive than the so-called Ambrosetti–Rabinowitz superquadraticity condition. In particular, the nonlinearity f may vanish and change sign. Also, the inequalities in (1.4) may be dropped. On the other hand, since our approach is based on critical point theory, more efforts have to be paid to guarantee the uniqueness of the solution. In this direction, we establish some criteria to ensure the uniqueness of the homoclinic solution obtained for (1.2). To the best knowledge of the authors it is the first time where uniqueness of even homoclinic solutions for second order differential equations with general nonlinearity is considered.

Our main results are the following.

Theorem 1.1. *Under the assumptions $(H0)$ and*

(H1) $f(t,u) = o(|u|^{p-1})$ *as* $|u| \to 0$ *uniformly in* t,

(H2) there exists $\mu > p$ *such that*

$$\mu F(t,u) \le f(t,u)u, \qquad \forall\, t \in \mathbb{R}, \ \ u \ge 0,$$

where $F(t,u) = \int_0^u f(t,s)ds$,

(H3) $F(t_0, u_0) > 0$ *for some* $t_0 \in \mathbb{R}$ *and* $u_0 > 0$,

the equation (1.1) *has at least one positive nontrivial homoclinic solution. Moreover this solution is an even function with* $u'(t) < 0$ *for* $t > 0$.

Example 1.2. Let

$$f(t,u) = (e^{-t^2} - 1)u^2 + u^3, \qquad \forall\,(t,u) \in \mathbb{R}^2.$$

It is easy to see that the function f satisfies all the assumptions of Theorem (1.1) with $p = 2$ and $\mu = 3$ but does not satisfy neither the (AR)-condition nor the condition (1.4) above. Hence Theorem (1.1) extends the results in [7, 10, 17] mainly.

In case $p = 2$, we have the following result.

Theorem 1.3. *Under the assumptions* $(H0)$–$(H3)$ *and*

(H4) *for a.e.* $t \in \mathbb{R}$, *the function* $u \mapsto \frac{f(t,u)}{u}$ *is increasing on* $]0, +\infty[$,

the homoclinic solution obtained above for equation (1.2) *is unique.*

2 Preliminary results

We shall obtain a solution of (1.1) as the limit as $T \to \infty$ of the solutions of

$$\begin{cases} (|u'|^{p-2}u')' - a(t)|u|^{p-2}u + f(t,u) = 0, & t \in (-T,T) \\ u(-T) = u(T) = 0. \end{cases} \tag{2.1}$$

For each $T \geq 1$, we define the Sobolev space

$$E_T = \left\{ u \in W^{1,p}((-T,T),\mathbb{R}) : u(-T) = u(T) = 0 \right\},$$

endowed with the norm

$$\|u\| = \left(\int_{-T}^{T} (|u'(t)|^p + |u(t)|^p) dt \right)^{\frac{1}{p}}.$$

To prove our theorems we need the following theorem introduced in [14]:

Theorem 2.1 (Mountain Pass Theorem)**.** *Let* E *be a real Banach space and* $I \in \mathcal{C}^1(E,\mathbb{R})$ *satisfy (PS)-condition. Suppose that* I *satisfies the following conditions:*

(i) $I(0) = 0$;

(ii) *there exists* $\rho,\ \alpha > 0$ *such that* $I|\partial B_\rho(0) \geq \alpha$;

(iii) *there exists* $e \in E \backslash \overline{B}_\rho$ *such that* $I(e) \leq 0$, *where* $B_\rho(0)$ *is an open ball in* E *of radius* ρ *centered at* 0;

then I *possesses a critical value* $c \geq \alpha$. *Moreover,* c *can be characterized as*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1],E) : \gamma(0) = 0,\ \gamma(1) = e\}.$$

Proposition 2.2 ([17])**.** *Let $u \in W^{1,p}_{loc}(\mathbb{R})$. Then:*

(1) *If $T \geq 1$, for $t \in [T-\frac{1}{2}, T+\frac{1}{2}]$,*

$$\max_{t\in[T-\frac{1}{2},T+\frac{1}{2}]} |u(t)| \leq 2^{\frac{p-1}{p}} \left(\int_{T-\frac{1}{2}}^{T+\frac{1}{2}} |u'(s)|^p + |u(s)|^p ds \right)^{\frac{1}{p}}. \tag{2.2}$$

(2) *For every $u \in W^{1,p}_0(-T,T)$,*

$$\|u\|_{L^\infty(-T,T)} \leq 2\|u\|. \tag{2.3}$$

Lemma 2.3. *Let $p \geq 2$, $u \in C^1(\mathbb{R})$ and $(|u'|^{p-2}u')' \in C(\mathbb{R})$. Then*

$$(|u'(t)|^p)' = \frac{p}{p-1}(|u'(t)|^{p-2}u'(t))'u'(t). \tag{2.4}$$

Proof. Let

$$(|u'(t)|^p)' = p|u'(t)|^{p-2}u'(t)u''(t), \tag{2.5}$$

on the other hand, one has

$$(|u'(t)|^p)' = (|u'(t)|^{p-2}u'(t)u'(t))' = (|u'(t)|^{p-2}u'(t))'u'(t) + (|u'(t)|^{p-2}u'(t))u''(t). \tag{2.6}$$

Combining (2.5) with (2.6), we establish (2.4). □

Let us consider the problem

$$\begin{cases} (|u'|^{p-2}u')' + g(t,u) = 0, & t \in (-T,T) \\ u(-T) = u(T) = 0, \end{cases} \tag{2.7}$$

where $g \in C^1([-T,T] \times \mathbb{R}^+)$ and satisfies

$$\begin{aligned} g(-t,u) &= g(t,u), \quad t \in (-T,T),\ u > 0, \\ g(t,0) &= 0, \quad t \in (-T,T) \\ tg_t(t,u) &< 0, \quad t \in (-T,T) \setminus \{0\},\ u > 0. \end{aligned} \tag{2.8}$$

The following lemma is an extension of Lemma 1 of [11] for p-Laplacian nonlinear equations.

Lemma 2.4 ([17])**.** *Assume that $g \in C^1([-T,T] \times \mathbb{R}^+)$ satisfies (2.8). Then any positive solution of (2.7) is an even function such that $\max\{u(t), -T \leq t \leq T\} = u(0)$ and $u'(t) < 0$ for $t \in (0,T)$. Moreover, any two positive solutions of (2.7) do not intersect on $(-T,T)$ (and hence they are strictly ordered on $(-T,T)$).*

Proposition 2.5. *Under the assumptions $(H0)$–$(H3)$, the problem (2.1) possesses a nontrivial positive solution u_T for any $T \geq 1$. Moreover, there exist constants $K, c > 0$, such that*

(i)

$$\int_{-T}^{T} (|u_T'(t)|^p + |u_T(t)|^p)dt \leq K, \qquad \forall\ T \geq 1, \tag{2.9}$$

(ii)

$$u_T(0) > c, \qquad \forall\ T \geq 1. \tag{2.10}$$

Proof. Consider the modified problem

$$\begin{cases} (|u'|^{p-2}u')' - a(t)|u|^{p-2}u + f(t,u^+) = 0, & t \in (-T,T) \\ u(-T) = u(T) = 0, \end{cases} \tag{2.11}$$

where $u^+ = \max(u,0)$. By (H1), we have $f(t,0) = 0$ for all $t \in \mathbb{R}$. So, analogously to [5,17], it is easy to see that solutions of (2.11) are positive solutions of (2.1).
To prove the existence of a solution to (2.11), we consider the functional I_T defined on E_T by

$$I_T(u) = \frac{1}{p}\int_{-T}^{T}\Big(|u'(t)|^p + a(t)|u(t)|^p\Big)dt - \int_{-T}^{T} F(t,u^+(t))dt, \tag{2.12}$$

for all $u \in E_T$. It is well known that under the assumptions of Theorem (1.1), $I_T \in C^1(E_T,\mathbb{R})$ and

$$I_T'(u).v = \int_{-T}^{T}\Big(|u'(t)|^{p-2}u'(t)v'(t) + a(t)|u(t)|^{p-2}u(t)v(t)\Big)dt - \int_{-T}^{T} f(t,u^+(t))v(t)dt, \tag{2.13}$$

for all $u,v \in E_T$.

Step 1: The functional I_T satisfies the (PS)-condition.

Let $\{u_j\} \subset E_T$ be such that $I_T(u_j)$ is bounded and $I_T'(u_j) \to 0$ as $j \to +\infty$. Then, by (H2), (2.12) and (2.13), there exists a constant $M_T > 0$ such that

$$\begin{aligned} M_T + \|u_j\| &\geq \mu I_T(u_j) - I_T'(u_j)u_j \\ &= (\frac{\mu}{p} - 1)\int_{-T}^{T}\Big(|u_j'(t)|^p + a(t)|u_j(t)|^p\Big)dt + \int_{-T}^{T}\Big(f(t,u_j^+)u_j^+ - \mu F(t,u_j^+)\Big)dt \\ &\geq \hat{a}\frac{\mu - p}{p}\|u_j\|^p, \end{aligned}$$

where $\hat{a} = \min\{1, a_0\}$. Since $\mu > p$, then the sequence $\{u_j\}$ is bounded in E_T. By the compact imbedding $E_T \subset C[-T,T]$, there exists $u \in E_T$ and a subsequence of $\{u_j\}$, still denoted by $\{u_j\}$ such that

$$u_j \rightharpoonup u \quad \text{in } E_T, \tag{2.14}$$
$$u_j \to u \quad \text{in } C[-T,T]. \tag{2.15}$$

From equation (2.13), one has

$$\begin{aligned} (I_T'(u_j) - I_T'(u)).(u_j - u) = &\int_{-T}^{T}\Big(|u_j'(t)|^{p-2}u_j'(t) - |u'(t)|^{p-2}u'(t)\Big)(u_j' - u')dt \\ &+ \int_{-T}^{T} a(t)\Big(|u_j(t)|^{p-2}u_j(t) - |u(t)|^{p-2}u(t)\Big)(u_j - u)dt \\ &- \int_{-T}^{T}\Big(f(t,u_j^+) - f(t,u^+)\Big)(u_j - u)dt. \end{aligned} \tag{2.16}$$

Since $I_T'(u_j) \to 0$ as $j \to +\infty$, we have

$$\lim_{j\to+\infty}(I_T'(u_j) - I_T'(u)).(u_j - u) = 0 \tag{2.17}$$

and by continuity of f and (2.15), we have

$$\lim_{j\to+\infty}\int_{-T}^{T}\Big(f(t,u_j^+) - f(t,u^+)\Big)(u_j - u)dt = 0. \tag{2.18}$$

For any $\xi, \eta \in \mathbb{R}$; we have the following inequality (see Remark 3.2 in [15])

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq \frac{2}{p}\frac{|\xi-\eta|^p}{2^{p-1}-1}, \qquad p \geq 2.$$

By the last inequality, one has

$$\begin{aligned}
&\Big(|u_j'(t)|^{p-2}u_j'(t) - |u'(t)|^{p-2}u'(t)\Big)(u_j'(t) - u'(t)) \\
&\quad + a(t)\Big(|u_j(t)|^{p-2}u_j(t) - |u(t)|^{p-2}u(t)\Big)(u_j(t) - u(t)) \\
&\geq \frac{2}{p(2^{p-1}-1)}|u_j'(t) - u'(t)|^p + a\frac{2}{p(2^{p-1}-1)}|u_j(t) - u(t)|^p \\
&\geq \frac{2\hat{a}}{p(2^{p-1}-1)}\Big(|u_j'(t) - u'(t)|^p + |u_j(t) - u(t)|^p\Big).
\end{aligned}$$

This coupled with (2.16)–(2.18), implies

$$\lim_{j\to+\infty} \|u_j - u\|^p \leq 0.$$

So $u_j \to u$ in E_T.

Step 2: Obviously $I_T(0) = 0$. Furthermore, in view of (H1), we see that,

$$F(t,u) = o(|u|^p) \quad \text{as } |u| \to 0, \text{uniformly in } t \in \mathbb{R},$$

that is, there exists $\delta \in (0,1)$ such that

$$F(t,u) \leq \frac{a_0}{2p}|u|^p, \ \text{for } |u| \leq \delta. \tag{2.19}$$

Letting $\rho := \frac{\delta}{2}$ and $u \in E_T$, such that $\|u\| = \rho$, then $0 < \|u\|_\infty \leq \delta$.

By (2.19), we have

$$\begin{aligned}
I_T(u) &= \int_{-T}^{T} \frac{1}{p}\Big(|u'(t)|^p + a(t)|u(t)|^p\Big)dt - \int_{-T}^{T} F(t,u^+)dt \\
&\geq \frac{1}{p}\int_{-T}^{T} |u'(t)|^p dt + \frac{a_0}{2p}\int_{-T}^{T} |u(t)|^p dt \\
&\geq \frac{\hat{a}}{2p}\|u\|^p = \frac{\hat{a}}{2p}\rho^p =: \alpha > 0.
\end{aligned}$$

Hence, the functional I_T satisfies the condition (ii) of the Mountain Pass Theorem.

Step 3: Firstly, without loss of generality, we may assume $u_0 = 1$ in (H3). Then, by the continuity of F, there exist constants $c_1 > 0, \eta > 0$ such that

$$F(t,1) \geq c_1, \qquad \forall t \in [t_0 - \eta, t_0 + \eta]. \tag{2.20}$$

On the other hand, by (H2), it's easy to check that

$$F(t,u) \geq F(t,1)u^\mu, \qquad \forall t \in \mathbb{R}, u \geq 1. \tag{2.21}$$

Combining (2.20) and (2.21), one obtains

$$F(t,u) \geq c_1 u^\mu - c_2, \qquad \forall t \in [t_0 - \eta, t_0 + \eta], \ u \geq 0, \tag{2.22}$$

where $c_2 = \max\{|F(t,u) - c_1 u^\mu|; 0 \le u \le 1, |t - t_0| \le \eta\}$.

Now, let $\hat{u} \in E$ be given by

$$\hat{u}(t) = \begin{cases} \cos[\frac{\pi}{2\eta}(t - t_0)], & \text{if } t \in [t_0 - \eta, t_0 + \eta]; \\ 0, & \text{if } t \in [-T, T] \backslash [t_0 - \eta, t_0 + \eta]. \end{cases} \tag{2.23}$$

Then, for all $s > 0$ we have by (2.22),

$$\begin{aligned} I(s\hat{u}) &= \frac{s^p}{p}\|\hat{u}\|^p - \int_{t_0-\eta}^{t_0+\eta} F(t, s\hat{u})dt \\ &\le \frac{s^p}{p}\|\hat{u}\|^p - c_1 s^\mu \int_{t_0-\eta}^{t_0+\eta} \hat{u}^\mu(t)dt + 2c_2 t_0. \end{aligned}$$

Since $\mu > p$ then $I(s\hat{u}) < 0 = I(0)$ for some $s > 0$ such that $\|s\hat{u}\| > \rho$, where ρ is defined in Step 2. So, the functional I_T satisfies all the conditions of the Mountain Pass Theorem and therefore there exists a solution $u_T \in E_T$ such that

$$c_T = I_T(u_T) = \inf_{w \in \Gamma_T} \max_{\xi \in [0,1]} I_T(w(\xi)), \qquad I'_T(u_T) = 0, \tag{2.24}$$

where

$$\Gamma_T = \{w \in C([0,1], E_T) : w(0) = 0,\ w(1) = s\hat{u}\}.$$

Using the variational characterization (2.24), we have

$$c_T \ge \frac{\hat{a}}{p}\rho^p > 0.$$

Hence, u_T is a nontrivial positive solution of (2.1). Moreover, by Lemma (2.4), one gets

$$\max_{-T \le t \le T} u_T(t) = u_T(0) \quad \text{and} \quad u'_T(t) < 0, \qquad \forall\ t \in (0, T).$$

Step 4: Uniform estimates.

Let $T_1 \ge T \ge 1$. By continuation with zero of a function $u \in E_T$ to $[-T_1, T_1]$, we have $E_T \subset E_{T_1}$ and $\Gamma_T \subset \Gamma_{T_1}$. Using the variational characterization (2.24), we infer that $c_{T_1} \le c_T \le c_1$ and then

$$\int_{-T}^{T} \Big(\frac{1}{p}(|u'_T(t)|^p + a(t)|u_T(t)|^p) - F(t, u_T)\Big)dt \le c_1,$$

therefore, by (H2)

$$\begin{aligned} \int_{-T}^{T} \frac{1}{p}\Big(|u'_T(t)|^p + a(t)|u_T(t)|^p\Big)dt &\le \int_{-T}^{T} F(t, u_T)dt + c_1, \\ &\le \frac{1}{\mu}\int_{-T}^{T} f(t, u_T)u_T dt + c_1. \end{aligned} \tag{2.25}$$

Multiplying the equation (2.1) by u_T and integrating by parts, we get

$$\int_{-T}^{T} (|u'_T(t)|^p + a(t)|u_T(t)|^p)dt = \int_{-T}^{T} f(t, u_T)u_T dt. \tag{2.26}$$

Using (2.26) in (2.25), we obtain

$$c_1 \ge \left(\frac{1}{p} - \frac{1}{\mu}\right)\int_{-T}^{T} (|u'_T(t)|^p + a(t)|u_T(t)|^p)dt \ge \frac{\hat{a}(\mu - p)}{\mu p}\|u_T\|^p, \tag{2.27}$$

which gives (2.9) with $K = \frac{c_1 \mu p}{\hat{a}(\mu - p)}$.

Step 5: It remains to show that there is a constant $c > 0$ such that

$$u_T(0) > c \quad \text{uniformly in } T. \tag{2.28}$$

With this aim, we introduce the "energy function" for $t \geq 0$ (where $u_T(t) \geq 0$), by

$$E(t) = \frac{p-1}{p}|u_T'(t)|^p - \frac{a(t)}{p}|u_T(t)|^p + F(t, u_T(t)).$$

Differentiating $E(t)$ and using (2.11), (2.4) and (H0), we obtain

$$E'(t) = -\frac{1}{p}a'(t)|u_T(t)|^p + F_t(t, u_T(t)) \leq 0 \quad \text{for all } 0 \leq t \leq T.$$

Hence

$$E(0) \geq E(T) = \frac{1}{p}|u_T'(T)|^p \geq 0.$$

Since $u_T(t)$ is even, $u_T'(0) = 0$, then

$$E(0) = -\frac{a(0)}{p}|u_T(0)|^p + F(0, u_T(0)) \geq 0,$$

which implies

$$F(0, u_T(0)) \geq \frac{a(0)}{p}|u_T(0)|^p,$$

and consequently

$$\frac{F(0, u_T(0))}{|u_T(0)|^p} \geq \frac{a(0)}{p}. \tag{2.29}$$

On the other hand, by (H1), one gets

$$\frac{F(t,u)}{|u|^p} \to 0 \quad \text{as } |u| \to 0, \textit{uniformly in } t. \tag{2.30}$$

Comparing (2.29) with (2.30), we obtain the estimate (2.28). □

3 Proof of Theorem 1.1

Take $T_n \to \infty$ and consider the problem (2.11) on the interval $(-T_n, T_n)$,

$$\begin{cases} (|u'|^{p-2}u')' - a(t)|u|^{p-2}u + f(t, u^+) = 0, & t \in (-T_n, T_n) \\ u(-T_n) = u(T_n) = 0. \end{cases} \tag{3.1}$$

Let u_n be the solution of (3.1) given by Proposition 2.5 and extended by zero outside the interval $[-T_n, T_n]$.

Claim 1: Arguing as in [17], we see that the sequence $(u_n)_n$ admits a subsequence, still denoted by $(u_n)_n$, that converges to a certain function u in $C^1_{loc}(\mathbb{R})$. Hence, we can pass to the limit in the equation (3.1), and we conclude that $u(t)$ solves (1.1). Moreover, we have

$$\int_{-\infty}^{+\infty} (|u'(t)|^p + |u(t)|^p)dt < \infty. \tag{3.2}$$

Since by Lemma 2.4 the functions $u_n(t)$ are even, with the only maximum at $t = 0$, the same is true for their limit $u(t)$. That $u'(t) < 0$ for $t > 0$ is easily seen by differentiating (1.1) (a similar argument can be found in [11]).

Claim 2: We will prove that $u(t)$ is nonzero and $u(\pm\infty) = u'(\pm\infty) = 0$.

Firstly, by (2.28), there is a constant $c > 0$ such that

$$u_n(0) > c \quad \text{uniformly in } n \in \mathbb{N}. \tag{3.3}$$

By passing to the limit as $n \to \infty$ in (3.3), we obtain

$$u(0) \geq c > 0,$$

which implies that u is not identically zero. Moreover, from (3.2) and Proposition 2.2, it follows

$$\lim_{T_n \to \pm\infty} \max_{t \in [T_n - \frac{1}{2}, T_n + \frac{1}{2}]} |u(t)| \leq \lim_{T_n \to \pm\infty} 2^{\frac{p-1}{p}} \Big(\int_{T_n - \frac{1}{2}}^{T_n + \frac{1}{2}} |u'(t)|^p + |u(t)|^p dt \Big)^{\frac{1}{p}} = 0, \tag{3.4}$$

so $u(\pm\infty) = 0$.

Next we prove that $u'(+\infty) = 0$ (the arguments for $u'(-\infty) = 0$ are similar). By the assumptions (H0), (H1) and equation (1.1) there exists $M > 0$ such that

$$\Big| \Big(|u'(t)|^{p-2} u'(t) \Big)' \Big| \leq M, \qquad \forall\, t \in \mathbb{R}.$$

If $u'(+\infty) \neq 0$, there exist $\epsilon_1 > 0$ and a monotone increasing sequence $t_k \longrightarrow +\infty$ such that $|u'(t_k)| \geq (2\epsilon_1)$. Then for $t \in [t_k, t_k + \frac{\epsilon_1}{M}]$, one has

$$\begin{aligned} |u'(t)|^{p-1} &= \Big| |u'(t_k)|^{p-2} u'(t_k) + \int_{t_k}^{t} \Big(|u'(s)|^{p-2} u'(s) \Big)' ds \Big| \\ &\geq\ |u'(t_k)|^{p-1} - \int_{t_k}^{t_k + \frac{\epsilon_1}{M}} \Big| \Big(|u'(s)|^{p-2} u'(s) \Big)' \Big| ds \\ &\geq\ 2\epsilon_1 - \frac{\epsilon_1}{M} M = \epsilon_1, \end{aligned}$$

which is in contradiction with (3.4).

4 Proof of Theorem 1.3

Let v be another positive solution of (1.2) (which is also an even function with the only maximum at $t = 0$). Multiplying both sides of (1.2) by v and integrating by parts on $\mathbb{R}$ we get

$$\int_{-\infty}^{+\infty} \Big[-u'v' - a(t)uv + f(t,u)v \Big] dt = 0. \tag{4.1}$$

Also, we have

$$\int_{-\infty}^{+\infty} \Big[-u'v' - a(t)uv + f(t,v)u \Big] dt = 0. \tag{4.2}$$

Subtracting (4.2) from (4.1), we get

$$\int_{-\infty}^{+\infty} \Big[\frac{f(t,u)}{u} - \frac{f(t,v)}{v} \Big] uv dt = 0.$$

It follows from (H4) that u and v cannot be ordered, and so they have to intersect. By the existence-uniqueness theorem for initial value problems, two cases are possible: either u and v have at least two positive points of intersection, or only one positive point of intersection.

Assume first $\xi_1 > 0$ is the smallest positive point of intersection and $\xi_2 > \xi_1$ the next one, and $u(t) < v(t)$ on (ξ_1, ξ_2). Multiply the equation (1.2) by u' and integrate from ξ_1 to ξ_2. Denoting by $t = t_1(u)$ the inverse function of $u(t)$ on (ξ_1, ξ_2). Also, denoting by $g(t,u) = -a(t)u + f(t,u)$, and $u_1 = u(\xi_1) = v(\xi_1)$, $u_2 = u(\xi_2) = v(\xi_2)$, we get

$$\frac{1}{2}u'^2(\xi_2) - \frac{1}{2}u'^2(\xi_1) + \int_{u_1}^{u_2} g(t_1(u), u)du = 0, \tag{4.3}$$

Doing the same for $v(t)$, and denoting its inverse on (ξ_1, ξ_2) by $t = t_2(v)$, we obtain

$$\frac{1}{2}v'^2(\xi_2) - \frac{1}{2}v'^2(\xi_1) + \int_{u_1}^{u_2} g(t_2(v), v)dv = 0, \tag{4.4}$$

Subtracting (4.4) from (4.3), we get

$$\frac{1}{2}\Big(u'^2(\xi_2) - v'^2(\xi_2)\Big) + \frac{1}{2}\Big(v'^2(\xi_1) - u'^2(\xi_1)\Big) + \int_{u_2}^{u_1} \Big[g(t_1(u), u) - g(t_2(u), u)\Big]du = 0, \tag{4.5}$$

Note that $u_2 < u_1$ and $t_2(u) > t_1(u)$ for all $u \in (u_2, u_1)$. Since $g(t,u)$ is decreasing in t, then

$$\int_{u_2}^{u_1} \Big[g(t_1(u), u) - g(t_2(u), u)\Big]du \leq 0. \tag{4.6}$$

On the other hand , it is easy to see that

$$u'(\xi_1) \leq v'(\xi_1) \leq 0, \quad v'(\xi_2) \leq u'(\xi_2) \leq 0,$$

which imply

$$\frac{1}{2}\Big(u'^2(\xi_2) - v'^2(\xi_2)\Big) + \frac{1}{2}\Big(v'^2(\xi_1) - u'^2(\xi_1)\Big) < 0. \tag{4.7}$$

Combining (4.6), (4.7) with (4.5) we obtain a contradiction, which rules out the case of two positive intersection points. If ξ_1 is the only intersection point, we integrate from ξ_1 to ∞, obtaining a similar contradiction. Uniqueness of the solution follows.

Acknowledgements

The authors are grateful to the anonymous referee for comments that greatly improved the manuscript.

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