



Alternative iterative technique

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Received 19 February 2019, appeared 5 August 2019

Communicated by Paul Eloe

Abstract. The standard methods of applying iterative techniques do not apply when the nonlinear term is neither monotonic (corresponding to an increasing or decreasing operator) nor Lipschitz (corresponding to a condensing operator). However, by applying the Layered Compression–Expansion Theorem in conjunction with an alternative inversion technique, we show how one can apply monotonicity techniques to a right focal boundary value problem.

Keywords: iteration, alternate inversion, layered, sum of operators, right focal problem.

2010 Mathematics Subject Classification: 47H10, 34B15.

1 Introduction

In the Layered Compression–Expansion Fixed Point Theorem [1] it was shown that, if the operator $T = R + S$ and one could find r^* and s^* such that

$$R(r^* + s^*) = r^* \quad \text{and} \quad S(r^* + s^*) = s^*,$$

then $x^* = r^* + s^*$ is a fixed point of T , since

$$Tx^* = Rx^* + Sx^* = R(r^* + s^*) + S(r^* + s^*) = r^* + s^* = x^*.$$

To find a solution of a right focal boundary value problem, we will use this fixed point result, in conjunction with a variation of an alternative inversion technique presented by Avery–Peterson [2,3] and Burton–Zhang [4,5] which converts fixed point problems of the form

$$x(t) = \int_0^1 G(t,s) f(x(s)) ds$$

into fixed point problems of the form

$$u(t) = f \left(\int_0^1 G(t,s) u(s) ds \right).$$

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In particular, we will apply these modern techniques to the boundary value problem

$$\begin{aligned}x''(t) + f_+(x(t)) + f_-(x(t)) &= 0, & t \in (0, 1), \\x(0) = 0 = x'(1),\end{aligned}$$

where $f_+ : \mathbb{R} \rightarrow [0, \infty)$ is a strictly increasing, differentiable function and $f_- : \mathbb{R} \rightarrow [0, \infty)$ is a strictly decreasing, differentiable function to arrive at a fixed point problem of the form

$$u(t) = f_+ \left(\int_0^1 G(t, s) \left(u(s) + f_- \left(f_+^{-1}(u(s)) \right) \right) ds \right)$$

which one can solve by iteration under suitable conditions. In our main results section we show how to convert the differential equation into a fixed point problem of this form and conclude with an example.

2 Main results

Let $f_+ : \mathbb{R} \rightarrow [0, \infty)$ be a strictly increasing, differentiable function and $f_- : \mathbb{R} \rightarrow [0, \infty)$ be a strictly decreasing, differentiable function and consider the right focal boundary value problem

$$x''(t) + f_+(x(t)) + f_-(x(t)) = 0, \quad t \in (0, 1), \quad (2.1)$$

$$x(0) = 0 = x'(1). \quad (2.2)$$

If we let $x = r + s$ then the right focal boundary value problem (2.1), (2.2) becomes the equivalent boundary value problem

$$r''(t) + f_+(r(t) + s(t)) + s''(t) + f_-(r(t) + s(t)) = 0, \quad t \in (0, 1), \quad (2.3)$$

$$r(0) + s(0) = 0 = r'(1) + s'(1). \quad (2.4)$$

Any solution of the system of boundary value problems

$$r''(t) + f_+(r(t) + s(t)) = 0, \quad s''(t) + f_-(r(t) + s(t)) = 0, \quad t \in (0, 1), \quad (2.5)$$

$$r(0) = 0 = r'(1), \quad s(0) = 0 = s'(1) \quad (2.6)$$

corresponds to a solution of (2.3), (2.4) which corresponds to a solution of the original focal boundary value problem (2.1), (2.2). If we let

$$u(t) = -r''(t) \quad \text{and} \quad v(t) = -s''(t),$$

then

$$r(t) = \int_0^1 G(t, \tau) u(\tau) d\tau \quad (2.7)$$

and

$$s(t) = \int_0^1 G(t, \tau) v(\tau) d\tau; \quad (2.8)$$

here,

$$G(t, \tau) = \min\{t, \tau\}, \quad (t, \tau) \in [0, 1] \times [0, 1], \quad (2.9)$$

is the Green's function for the right focal boundary value problem. Substituting for r, r'', s and s'' in (2.5), (2.6) we arrive at the equivalent fixed point formulation given by

$$u(t) = f_+ \left(\int_0^1 G(t, \tau)(u(\tau) + v(\tau))d\tau \right), \quad t \in [0, 1] \quad (2.10)$$

$$v(t) = f_- \left(\int_0^1 G(t, \tau)(u(\tau) + v(\tau))d\tau \right), \quad t \in [0, 1]. \quad (2.11)$$

Since the functions f_- and f_+ are assumed to be strictly monotonic, the system (2.10), (2.11) is equivalent to

$$u(t) = f_+ \left(\int_0^1 G(t, \tau) \left(u(\tau) + f_-(f_+^{-1}(u(\tau))) \right) d\tau \right), \quad t \in [0, 1] \quad (2.12)$$

$$v(t) = f_-(f_+^{-1}(u(t))), \quad t \in [0, 1]. \quad (2.13)$$

Let $E = C[0, 1]$, P be the subset of nonnegative elements of E , and for $u \in E$ define

$$\|u\| = \max_{t \in [0, 1]} |u(t)|.$$

Thus P is a cone in the Banach space E and the norm is a monotonic norm on P . The proof of our main result hinges on the existence of an equivalent norm which is monotone on P which is a normality condition on our cone. See Theorem 1.1.1 of *Nonlinear problems in abstract cones* [6] for a thorough discussion of normality as well as Theorem 2.1 for its application in monotone iterative techniques.

Let $u \in P$, and define the operator

$$A : E \rightarrow E$$

by

$$(Au)(t) = f_+ \left(\int_0^1 G(t, \tau) \left(u(\tau) + f_-(f_+^{-1}(u(\tau))) \right) d\tau \right), \quad t \in [0, 1]. \quad (2.14)$$

We are now ready to state and prove our main result which provides a new perspective to the monotone iterative technique presented in Chapter 2 of the *Monotone iterative techniques for nonlinear differential equations* [7] monograph due to the form of the nonlinear term ($f = f_+ + f_-$).

Theorem 2.1. *Let G and A be given by (2.9) and (2.14), respectively. Suppose $d > 0$ with*

$$f_+ \left(\frac{d + f_-(f_+^{-1}(d))}{2} \right) \leq d,$$

and

$$\left(f_-(f_+^{-1}) \right)'(z) \leq -1$$

for all $z \in (0, d)$. Also assume that

$$f_+ : [0, d] \rightarrow [0, \infty)$$

is a strictly increasing, differentiable function and

$$f_- : [0, d] \rightarrow [0, \infty)$$

is a strictly decreasing, differentiable function. Then the sequence u_n defined recursively by

$$u_0(t) \equiv d \quad \text{for all } t \in [0, 1]$$

and

$$u_{n+1} = Au_n, \quad \text{for } n \geq 0$$

converges to a fixed point u^* of the operator A and for $v^* = f_-(f_+^{-1}(u^*))$ we have that

$$x^* = r^* + s^*,$$

where

$$r^*(t) = \int_0^1 G(t, \tau) u^*(\tau) d\tau \quad \text{and} \quad s^*(t) = \int_0^1 G(t, \tau) v^*(\tau) d\tau,$$

is a solution of (2.1), (2.2).

Proof. Let $d > 0$ and suppose that

$$\begin{aligned} f_+ \left(\frac{d + f_-(f_+^{-1}(d))}{2} \right) &\leq d, \\ \left(f_-(f_+^{-1}) \right)'(z) &\leq -1, \end{aligned}$$

for all $z \in (0, d)$ with

$$f_+ : [0, d] \rightarrow [0, \infty)$$

being a strictly increasing, differentiable function and

$$f_- : [0, d] \rightarrow [0, \infty)$$

being a strictly decreasing, differentiable function. It is a standard exercise by the Arzelà–Ascoli Theorem to show that A in (2.14) is a completely continuous operator, using the properties of G in (2.9) and the continuity of f_+ , f_- and f_+^{-1} . Let

$$P_d = \{x \in P : \|x\| \leq d\}.$$

Claim: u_n is a decreasing sequence and $u_n \in P_d$.

We proceed by induction. Since f_+ is an increasing function, for all $t \in [0, 1]$, we have that

$$\begin{aligned} u_1(t) &= Au_0(t) \\ &= f_+ \left(\int_0^1 G(t, \tau) (d + f_-(f_+^{-1}(d))) d\tau \right) \\ &= f_+ \left(\frac{t(2-t)(d + f_-(f_+^{-1}(d)))}{2} \right) \\ &\leq f_+ \left(\frac{d + f_-(f_+^{-1}(d))}{2} \right) \\ &\leq d = u_0(t). \end{aligned}$$

Hence $u_1 \leq u_0$, and since the norm is monotonic we have that $\|u_1\| \leq \|u_0\| = d$; consequently $u_1 \in P_d$.

For induction purposes, assume that $u_m \leq u_{m-1}$ and $u_m \in P_d$ for all $m \leq n$. For every $\tau \in [0, 1]$ since $u_n(\tau), u_{n-1}(\tau) \in [0, d]$, by the Mean Value Theorem there is a $z \in (0, d)$ such that

$$f_-\left(f_+^{-1}(u_n(\tau))\right) - f_-\left(f_+^{-1}(u_{n-1}(\tau))\right) = \left(f_-\left(f_+^{-1}\right)\right)'(z)(u_n(\tau) - u_{n-1}(\tau)).$$

Since

$$\left(f_-\left(f_+^{-1}\right)\right)'(z) \leq -1$$

for all $z \in (0, d)$, we have that

$$f_-\left(f_+^{-1}(u_n(\tau))\right) - f_-\left(f_+^{-1}(u_{n-1}(\tau))\right) \leq -1(u_n(\tau) - u_{n-1}(\tau)) = u_{n-1}(\tau) - u_n(\tau).$$

It follows that

$$u_n(\tau) + f_-\left(f_+^{-1}(u_n(\tau))\right) \leq u_{n-1}(\tau) + f_-\left(f_+^{-1}(u_{n-1}(\tau))\right),$$

and therefore

$$\begin{aligned} & \int_0^1 G(t, \tau) \left(u_n(\tau) + f_-\left(f_+^{-1}(u_n(\tau))\right)\right) d\tau \\ & \leq \int_0^1 G(t, \tau) \left(u_{n-1}(\tau) + f_-\left(f_+^{-1}(u_{n-1}(\tau))\right)\right) d\tau. \end{aligned}$$

Since f_+ is an increasing function, we have

$$\begin{aligned} & f_+\left(\int_0^1 G(t, \tau) \left(u_n(\tau) + f_-\left(f_+^{-1}(u_n(\tau))\right)\right) d\tau\right) \\ & \leq f_+\left(\int_0^1 G(t, \tau) \left(u_{n-1}(\tau) + f_-\left(f_+^{-1}(u_{n-1}(\tau))\right)\right) d\tau\right); \end{aligned}$$

that is,

$$u_{n+1} \leq u_n,$$

and by the monotonicity of the norm we have that

$$\|u_{n+1}\| \leq \|u_n\| \leq d.$$

Hence $u_{n+1} \in P_d$. Therefore we have that

$$\{u_n\}_{n=1}^\infty$$

is a monotonic sequence in the bounded, closed subset P_d of the cone P .

Thus, since A in (2.14) is completely continuous with

$$u_{n+1} = A(u_n),$$

there is a subsequence $\{u_{n_k}\}_{k=1}^\infty$ with

$$u_{n_k} \rightarrow u^* \in P_d.$$

The norm is monotonic, thus we have that

$$u_n \rightarrow u^* \in P_d,$$

since for any $k \in \mathbb{N}$, when $j \geq n_k$ we have

$$\|u_j - u^*\| \leq \|u_{n_k} - u^*\| \quad \text{since } u_j - u^* \leq u_{n_k} - u^*.$$

Therefore u^* is a fixed point of A , and for

$$v^* = f_-(f_+^{-1}(u^*))$$

we have a solution (u^*, v^*) for (2.10), (2.11). Hence, we have, using (2.7), (2.8), that

$$x^* = r^* + s^*,$$

where

$$r^*(t) = \int_0^1 G(t, \tau) u^*(\tau) d\tau \quad \text{and} \quad s^*(t) = \int_0^1 G(t, \tau) v^*(\tau) d\tau,$$

is a solution of our original focal boundary value problem (2.1), (2.2). □

Example 2.2. The function $f(x) = \frac{x}{2} + \sqrt{8}e^{\sqrt{8}-\sqrt{x}}$ has

$$f_+(x) = \frac{x}{2}$$

which is a strictly increasing, differentiable function and

$$f_-(x) = \sqrt{8}e^{\sqrt{8}-\sqrt{x}}$$

which is a strictly decreasing, differentiable function on $(0, 4)$. Also

$$f_+^{-1}(x) = 2x,$$

and thus (for $d = 4$)

$$f_+ \left(\frac{4 + f_-(f_+^{-1}(4))}{2} \right) = \left(\frac{1}{2} \right) \left(\frac{4 + \sqrt{8}e^{\sqrt{8}-\sqrt{4}}}{2} \right) \leq 4$$

and

$$\left(f_-(f_+^{-1}) \right)'(x) = \left(\frac{-\sqrt{8}e^{\sqrt{8}-\sqrt{2x}}}{\sqrt{2x}} \right) < -1$$

for all $x \in (0, 4)$. Therefore, the sequence u_n , defined recursively by

$$u_0(t) \equiv 4 \quad \text{for all } t \in [0, 1]$$

and

$$u_{n+1} = Au_n, \quad \text{for } n \geq 0,$$

converges to a fixed point u^* of the operator A and for

$$v^* = f_-(f_+^{-1}(u^*)) = \sqrt{8}e^{\sqrt{8}-\sqrt{2u^*}},$$

we have that

$$x^* = r^* + s^*,$$

where

$$r^*(t) = \int_0^1 G(t, \tau) u^*(\tau) d\tau \quad \text{and} \quad s(t) = \int_0^1 G(t, \tau) v^*(\tau) d\tau,$$

is a solution of (2.1), (2.2).

Remark 2.3. While other methods can be applied easily and are straightforward to obtain existence of solutions for Example 2.2, we chose it as an example illustrating the iteration of the paper, but for which standard iteration techniques do not apply. Our goal has been to expand the application of iteration for which computers can play a critical role.

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