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Almost sure stability with general decay rate of neutral stochastic pantograph equations with Markovian switching

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Abstract. This paper focuses on the general decay stability of nonlinear neutral stochastic pantograph equations with Markovian switching (NSPEwMSs). Under the local Lipschitz condition and non-linear growth condition, the existence and almost sure stability with general decay of the solution for NSPEwMSs are investigated. By means of M-matrix theory, some sufficient conditions on the general decay stability are also established for NSPEwMSs.

Keywords: neutral stochastic pantograph equations, Markovian switching, existence and uniqueness results, general decay stability.

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1 Introduction

In this paper, we are concerned with the asymptotic stability of neutral stochastic pantograph equations with Markovian switching (NSPEwMSs)

$$d[x(t) - D(x(qt), t, r(t))] = f(x(t), x(qt), t, r(t))dt + g(x(t), x(qt), t, r(t))dw(t), \quad t \geq t_0, \tag{1.1}$$

where $0 < q < 1$, the coefficients $f : R^n \times R^n \times [t_0, \infty) \times S \to R^n$ and $g : R^n \times R^n \times [t_0, \infty) \times S \to R^{n\times m}$ are Borel-measurable, $D : R^n \times [t_0, \infty) \times S \to R^n$ is the neutral term and $w(t)$ is an m-dimensional Brownian motion. Actually, Eq. (1.1) can be regarded as a perturbed system of the deterministic pantograph equations

$$\frac{d[x(t) - D(x(qt), t, r(t))]}{dt} = f(x(t), x(qt), t, r(t)). \tag{1.2}$$

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Since Eq. (1.2) possess a wide range of applications in applied mathematics and engineering, the asymptotic properties and numerical analysis of the solution have been widely studied, for example, Hale [11], Iserles et al. [5,15–17].

As a class of special stochastic delay systems, stochastic pantograph equations (SPEs) with unbounded delay have been investigated by many scholars, we can refer to Baker and Buckwar [3], Appleby and Buckwar [2], Fan et al. [8,9], Milosevic [20], Xiao et al. [35], Guo and Li [10]. In the study of stochastic delay systems, the stability is one of the important issues and has many important applications in practice. There are many results on the exponential stability theorem for stochastic differential delay equations (SDDEs) and SDDEs with Markovian switching. We mention here [4,14,21–25,37] among others. On the other hand, some of the exponential stability criteria related to the moment exponential stability of solutions to neutral SDDEs and neutral SDDEs with Markovian switching were considered in [13,18,19,26,27,34] and the references therein. Motivated by above mentioned works, some scholars began to study the exponential stability of SPEs with Markovian switching (SPEwMSs). For example, Zhou and Xue [38] investigated the exponential stability of a class of SPEwMSs, where the coefficients were dominated by polynomials with high orders. You et al. [36] discussed the robust exponential stability of highly nonlinear SPEwMSs, in virtue of M-matrix theory, they established exponential stability criterion for SPEwMSs. Similarly, Shen et al. [31] considered a class of nonlinear NSPEwMSs and established the exponential stability criteria for NSPEwMSs without the linear growth condition.

In fact, not all stochastic differential systems are exponentially stable, there are also a lot of stochastic systems which are stable but subject to a lower decay rate other than exponential decay. Therefore, much literatures focuses on the polynomial stability of stochastic differential systems. We mention here only [7,29]. These two kinds of stability show that the speed which the solution decays to zero is different. Then these stability concepts are extended to general decay stability (see [1,6,30,32,33]). To the best of our knowledge, there is no existing result on almost sure stability with general decay rate for NSPEwMSs (1.1). By applying the Itô formula and the non-negative semi-martingale convergence theorem, we study the almost sure decay stability of Eq.(1.1) and give the upper bound of general decay rate. Meanwhile, we impose some conditions on f,g and establish the sufficient criteria on general decay stability in terms of M-matrix.

The paper is organized as follows. In Section 2, we introduce some hypotheses concerning Eq. (1.1) and we establish the existence and uniqueness of solutions to NSPEwMSs under the local Lipschitz condition and nonlinear growth condition; In Section 3, by applying the Itô formula and stochastic inequality, we study the almost sure stability with general decay rate for NSPEwMSs (1.1); By means of M-matrix, we establish some sufficient criteria on general decay stability; Finally, we give two examples to illustrate our theory.

2 Preliminaries and the global solution

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions. Let $w(t)$ be an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $t \geq t_0 > 0$ and $\mathbb{C}([qt,t];R^n)$ denote the family of the continuous functions φ from $[qt,t] \to R^n$ with the norm $\|\varphi\| = \sup_{qt\leq\theta\leq t}|\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by $A^\top$. If A is a matrix, its norm $\|A\|$ is defined by $\|A\| = \sup\{|Ax| : |x| = 1\}$. $\mathcal{L}^p_{\mathcal{F}_t}([qt,t];R^n)$ denote the family of all $(\mathcal{F}_t)$-measurable, $\mathbb{C}([qt;t], R^n)$-valued random variables $\varphi = \{\varphi(\theta) : qt \le \theta \le t\}$ such that $E\|\varphi\|^2 < \infty$.

Let $r(t), t \ge t_0$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $S = \{1, 2, \dots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by:

$$P(r(t+\Delta) = j \mid r(t) = i) = \begin{cases} \gamma_{ij}\Delta + \circ(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ij}\Delta + \circ(\Delta), & \text{if } i = j. \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j, $i \neq j$, While $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. Let us consider the nonlinear NSPEwMSs

$$d[x(t) - D(x(qt), t, r(t))] = f(x(t), x(qt), t, r(t))dt + g(x(t), x(qt), t, r(t))dw(t), \quad t \ge t_0 \quad (2.1)$$

with initial data $\{x(t) : qt_0 \le t \le t_0\} = \xi \in \mathcal{L}^2_{\mathcal{F}_{t_0}}([qt_0, t_0]; R^n)$.

In this paper, the following hypotheses are imposed on the coefficients f, g and D.

Assumption 2.1. For each integer $d \ge 1$, there exist a positive constant k_d such that

$$|f(x, y, t, i) - f(\bar{x}, \bar{y}, t, i)|^2 \vee |g(x, y, t, i) - g(\bar{x}, \bar{y}, t, i)|^2 \le k_d(|x - \bar{x}|^2 + |y - \bar{y}|^2), \quad (2.2)$$

for those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le d$ and $(t, i) \in [t_0, \infty) \times S$.

Assumption 2.2. For all $u, v \in R^n$ and $(t, i) \in [t_0, \infty) \times S$, there exists a constant $k_i \in (0, 1)$ such that

$$|D(u, t, i) - D(v, t, i)|^2 \le k_i |u - v|^2. \quad (2.3)$$

Let $k_0 = \max_{i\in S} k_i$ and $D(0, t, i) = 0$.

It is known that Assumptions 2.1 and 2.2 only guarantee that Eq. (2.1) has a unique maximal solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition in terms of Lyapunov functions.

Let $C(R^n \times [t_0, \infty) \times S; R_+)$ denote the family of continuous functions from $R^n \times [t_0, \infty) \times KS$ to R_+. Also denote by $C^2(R^n \times [t_0, \infty) \times S; R_+)$ the family of all continuous non-negative functions $V(x, t, i)$ defined on $R^n \times [t_0, \infty) \times S$ such that for each $i \in S$, they are continuously twice differentiable in x. Given $V \in C^2(R^n \times [t_0, \infty) \times S; R_+)$, we define the function $LV : R^n \times R^n \times [t_0, \infty) \times S \to R$ by

$$\begin{aligned} LV(x, y, t, i) = {} & V_t(x - D(y, t, i), t, i) + V_x(x - D(y, t, i), t, i) f(x, y, t, i) \\ & + \frac{1}{2}\text{trace}[g^\top(x, y, t, i) V_{xx}(x - D(y, t, i), t, i) g(x, y, t, i)] \\ & + \sum_{j=1}^{N} \gamma_{ij} V(x - D(y, t, i), t, j), \end{aligned} \quad (2.4)$$

where

$$V_t(x, t, i) = \frac{\partial V(x, t, i)}{\partial t}, \qquad V_x(x, t, i) = \left(\frac{\partial V(x, t, i)}{\partial x_1}, \dots, \frac{\partial V(x, t, i)}{\partial x_n}\right),$$

$$V_{xx}(x, t, i) = \left(\frac{\partial^2 V(x, t, i)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Assumption 2.3. There exist a function $V \in C^{2,1}(R^n \times [t_0, \infty) \times S; R_+)$ and some positive constants $c_1, c_2, \alpha_i, (i = 0, 1, 2, 3, 4), \gamma > 2$, such that for all $(x, y, t, i) \in R^n \times R^n \times [t_0, \infty) \times S$

$$c_1|x|^2 \le V(x, t, i) \le c_2|x|^2, \quad \forall (x, t, i) \in R^n \times [t_0, \infty) \times S \tag{2.5}$$

and

$$LV(x, y, t, i) \le \alpha_0 - \alpha_1|x|^2 + \alpha_2|y|^2 - \alpha_3|x|^\gamma + \alpha_4|y|^\gamma. \tag{2.6}$$

Lemma 2.4 (see [28]). *Let $p \ge 1$ and $a, b \in R^n$. Then, for any $\delta \in (0, 1)$,*

$$|a + b|^p \le \frac{|a|^p}{(1-\delta)^{p-1}} + \frac{|b|^p}{\delta^{p-1}}.$$

Lemma 2.5 (see [28]). *Let $A(t), U(t)$ be two $\mathcal{F}_t$-adapted increasing processes on $t \ge 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued local martingale with $M(0) = 0$ a.s. Let ζ be a nonnegative $\mathcal{F}_0$-measurable random variable. Assume that $x(t)$ is nonnegative and*

$$x(t) = \zeta + A(t) - U(t) + M(t) \quad \textit{for } t \ge 0.$$

If $\lim_{t\to\infty} A(t) < \infty$ a.s. then for almost all $\omega \in \Omega$, $\lim_{t\to\infty} x(t) < \infty$ and $\lim_{t\to\infty} U(t) < \infty$, that is, both $x(t)$ and $U(t)$ converge to finite random variables.

Theorem 2.6. *Let Assumptions 2.1–2.3 hold. Then for any given initial data ξ, there is a unique global solution $x(t)$ to Eq. (2.1) on $t \in [t_0, \infty)$. Moreover, the solution has the properties that*

$$E|x(t)|^2 \le C \tag{2.7}$$

for any $t \ge t_0$.

Proof. Since the coefficients of Eq. (2.1) are locally Lipschitz continuous, for any given initial data ξ, there is a maximal local solution $x(t)$ on $t \in [t_0, \sigma_\infty)$, where σ_∞ is the explosion time. Let $\bar{k}_0 > 0$ be sufficiently large for $\|\xi\| < \bar{k}_0$. For each integer $k \ge \bar{k}_0$, define the stopping time

$$\tau_k = \inf\{t \in [t_0, \sigma_e) : |x(t)| \ge k\}.$$

Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \le \sigma_\infty$ a.s. Note if we can show that $\tau_\infty = \infty$ a.s., then $\sigma_\infty = \infty$ a.s. So we just need to show that $\tau_\infty = \infty$ a.s.

We shall first show that $\tau_\infty > \frac{t_0}{q}$ a.s. By the generalised Itô formula (see e.g. [25]) and condition (2.6), we can show that, for any $k \ge \bar{k}_0$ and $t_1 \ge t_0$,

$$\begin{aligned} &EV(z(\tau_k \wedge t_1), \tau_k \wedge t_1, r(\tau_k \wedge t_1)) \\ &\quad \le EV(z(t_0), t_0, r(t_0)) + E\int_{t_0}^{\tau_k \wedge t_1} \Big(\alpha_0 - \alpha_1|x(t)|^2 + \alpha_2|x(qt)|^2 - \alpha_3|x(t)|^\gamma + \alpha_4|x(qt)|^\gamma\Big)dt, \end{aligned}$$

where $z(t) = x(t) - D(x(qt), t, r(t))$. Let us now restrict $t_1 \in [t_0, \frac{t_0}{q}]$. By condition (2.5), we then get

$$c_1E|z(\tau_k \wedge t_1)|^2 \le H_1 - \alpha_1 E\int_{t_0}^{\tau_k\wedge t_1} |x(t)|^2 dt - \alpha_3 E\int_{t_0}^{\tau_k \wedge t_1} |x(t)|^\gamma)dt, \tag{2.8}$$

where

$$\begin{aligned} H_1 &= c_2E|z(t_0)|^2 + \int_{t_0}^{\frac{t_0}{q}} \Big(\alpha_0 + \alpha_2|x(qt)|^2 + \alpha_4|x(qt)|^\gamma\Big)dt \\ &\le 2c_2(1 + k_0)E\|\xi\|^2 + \frac{1}{q}E\int_{qt_0}^{t_0} \Big(\alpha_0 + \alpha_2|x(t)|^2 + \alpha_4|x(t)|^\gamma\Big)dt < \infty. \end{aligned}$$

It then follows that

$$E|z(\tau_k \wedge t_1)|^2 \le \frac{H_1}{c_1}, \quad t_0 \le t_1 \le \frac{t_0}{q} \tag{2.9}$$

for any $k \ge \bar{k}_0$. By Lemma 2.4, we get

$$\begin{aligned} E|x(\tau_k \wedge t_1)|^2 &\le \frac{E|z(\tau_k \wedge t_1)|^2}{1-\delta} + \frac{E|D(x(q(\tau_k \wedge t_1)), \tau_k \wedge t_1, r(\tau_k \wedge t_1))|^2}{\delta} \\ &\le \frac{E|z(\tau_k \wedge t_1)|^2}{1-\delta} + \frac{k_0 E|x(q(\tau_k \wedge t_1))|^2}{\delta}. \end{aligned}$$

This implies

$$\sup_{t_0 \le t_1 \le \frac{t_0}{q}} E|x(\tau_k \wedge t_1)|^2 \le \frac{\sup_{t_0 \le t_1 \le \frac{t_0}{q}} E|z(\tau_k \wedge t_1)|^2}{1-\delta} + \frac{k_0 \sup_{t_0 \le t_1 \le \frac{t_0}{q}} E|x(q(\tau_k \wedge t_1))|^2}{\delta}. \tag{2.10}$$

Letting $\delta = \sqrt{k_0}$, it follows from (2.9) that

$$\sup_{t_0 \le t_1 \le \frac{t_0}{q}} E|x(\tau_k \wedge t_1)|^2 \le \frac{H_1}{c_1(1-\sqrt{k_0})} + \sqrt{k_0} E\|\xi\|^2 + \sqrt{k_0} \sup_{t_0 \le t_1 \le \frac{t_0}{q}} E|x(\tau_k \wedge t_1)|^2. \tag{2.11}$$

Hence, we have

$$E|x(\tau_k \wedge t_1)|^2 \le \frac{H_1}{c_1(1-\sqrt{k_0})} + \frac{\sqrt{k_0}}{1-\sqrt{k_0}} E\|\xi\|^2, \quad t_0 \le t_1 \le \frac{t_0}{q}. \tag{2.12}$$

In particular, $E|x(\tau_k \wedge \frac{t_0}{q})|^2 \le \frac{H_1}{c_1(1-\sqrt{k_0})} + \frac{\sqrt{k_0}}{1-\sqrt{k_0}} E\|\xi\|^2$, $\forall k \ge \bar{k}_0$. This implies $k^2 P(\tau_k \le \frac{t_0}{q}) \le \frac{H_1}{c_1(1-\sqrt{k_0})} + \frac{\sqrt{k_0}}{1-\sqrt{k_0}} E\|\xi\|^2$. Letting $k \to \infty$, we hence obtain that $P\big(\tau_\infty \le \frac{t_0}{q}\big) = 0$, namely

$$P\left(\tau_\infty > \frac{t_0}{q}\right) = 1. \tag{2.13}$$

Letting $k \to \infty$ in (2.13) yields

$$E|x(t_1)|^2 \le \frac{H_1}{c_1(1-\sqrt{k_0})} + \frac{\sqrt{k_0}}{1-\sqrt{k_0}} E\|\xi\|^2, \quad t_0 \le t_1 \le \frac{t_0}{q}. \tag{2.14}$$

Let us now proceed to prove $\tau_\infty > \frac{t_0}{q^2}$ a.s. given that we have shown (2.12)–(2.14). For any $k \ge \bar{k}_0$ and $t_1 \in [t_0, \frac{t_0}{q^2}]$, it follows from (2.6) that

$$c_1 E|z(\tau_k \wedge t_1)|^2 \le H_2 - \alpha_1 E \int_{t_0}^{\tau_k \wedge t_1} |x(t)|^2 dt - \alpha_3 E \int_{t_0}^{\tau_k \wedge t_1} |x(t)|^\gamma) dt, \tag{2.15}$$

where

$$\begin{aligned} H_2 &= c_2 E|z(t_0)|^2 + E \int_{t_0}^{\frac{t_0}{q^2}} \Big(\alpha_0 + \alpha_2 |x(qt)|^2 + \alpha_4 |x(qt)|^\gamma\Big) dt \\ &= H_1 + E \int_{\frac{t_0}{q}}^{\frac{t_0}{q^2}} \Big(\alpha_0 + \alpha_2 |x(qt)|^2 + \alpha_4 |x(qt)|^\gamma\Big) dt \\ &= H_1 + \frac{1}{q} E \int_{t_0}^{\frac{t_0}{q}} \Big(\alpha_0 + \alpha_2 |x(t)|^2) + \alpha_4 |x(t)|^\gamma\Big) dt < \infty. \end{aligned}$$

Consequently

$$E|z(\tau_k \wedge t_1)|^2 \le \frac{H_2}{c_1}, \quad t_0 \le t_1 \le \frac{t_0}{q^2}.$$

Similar to (2.12), we can obtain that

$$E|x(\tau_k \wedge t_1)|^2 \le \frac{H_2}{c_1(1-\sqrt{k_0})} + \frac{\sqrt{k_0}}{1-\sqrt{k_0}} E\|\xi\|^2, \quad t_0 \le t_1 \le \frac{t_0}{q^2}. \tag{2.16}$$

In particular, $E|x(\tau_k \wedge \frac{t_0}{q^2})|^2 \le \frac{H_2}{c_1(1-\sqrt{k_0})} + \frac{\sqrt{k_0}}{1-\sqrt{k_0}} E\|\xi\|^2$, $\forall k \ge \bar{k}_0$. This implies $k^2 P(\tau_k \le \frac{t_0}{q^2}) \le \frac{H_2}{c_1(1-\sqrt{k_0})} + \frac{\sqrt{k_0}}{1-\sqrt{k_0}} E\|\xi\|^2$. Letting $k \to \infty$, we then obtain that $P(\tau_\infty \le \frac{t_0}{q^2}) = 0$, namely $P(\tau_\infty > \frac{t_0}{q^2}) = 1$. Letting $k \to \infty$ in (2.16) yields

$$E|x(t_1)|^2 \le \frac{H_2}{c_1(1-\sqrt{k_0})} + \frac{\sqrt{k_0}}{1-\sqrt{k_0}} E\|\xi\|^2, \quad t_0 \le t_1 \le \frac{t_0}{q^2}.$$

Repeating this procedure, we can show that, for any integer $i \ge 1$, $\tau_\infty > \frac{t_0}{q^i}$ a.s. and

$$E|x(t)|^2 \le \frac{H_i}{c_1(1-\sqrt{k_0})} + \frac{\sqrt{k_0}}{1-\sqrt{k_0}} E\|\xi\|^2, \quad t_0 \le t_1 \le \frac{t_0}{q^i},$$

where $H_i = 2c_2(1+k_0)E\|\xi\|^2 + E\int_{t_0}^{\frac{t_0}{q^i}} (\alpha_0 + \alpha_2|x(qt)|^2 + \alpha_4|x(qt)|^\gamma)dt < \infty$. We must therefore have $\tau_\infty = \infty$ a.s. and the required assertion (2.7) holds as well. The proof is therefore complete. □

Remark 2.7. In [12,20,31,36,38], the authors proved that stochastic pantograph differential systems has a unique solution $x(t)$ under the local Lipschitz condition and the generalized Khasminskii-type condition. In fact, the key of their proof is that the coefficients $\alpha_i, i = 1,2,3,4$ of (2.6) are required to satisfy $\alpha_1 \ge \alpha_2$ and $\alpha_3 \ge \alpha_4$. However, in our theorem, we remove this condition and prove that Eq. (2.1) has a unique global solution $x(t)$. Hence, we improve and generalize the corresponding existence results of [12,20,31,36,38].

3 Stability of neutral stochastic pantograph systems

In this section, we shall study the almost sure stability with general decay rate of NSPEwMSs (2.1). Let us first introduce the following ψ-type function, which will be used as the decay function.

Definition 3.1. The function $\psi : R_+ \to (0,\infty)$ is said to be ψ-type function if this function satisfies the following conditions:

(i) It is continuous and nondecreasing in R and continuously differentiable in R_+.

(ii) $\psi(0) = 1$, $\psi(\infty) = \infty$ and $\phi = \sup_{t>0} |\frac{\psi'(t)}{\psi(t)}| < \infty$.

(iii) For any $s, t \ge 0$, $\psi(t) \le \psi(s)\psi(t-s)$.

Definition 3.2. The solution of Eq. 2.1 is said to be almost surely ψ-type stable if there exists a constant $\bar{\gamma}$ such that

$$\limsup_{t\to\infty} \frac{\log|x(t)|}{\log\psi(t)} < -\bar{\gamma} \quad a.s.$$

Obviously, when $\psi(t) = e^t$ and $\psi(t) = 1 + t$, this ψ-type stability implies the exponential stability and polynomial stability, respectively.

In order to obtain the almost sure ψ-type stability of Eq. (2.1), we shall impose the following conditions on the neutral term D.

Assumption 3.3. For all $u, v \in R^n$ and $(t, i) \in [t_0, \infty) \times S$, there exists a constant $k_0 \in (0, 1)$ and $\varepsilon \geq 0$ such that

$$|D(u,t,i) - D(v,t,i)|^2 \leq k_0 \psi^{-\varepsilon}((1-q)t)|u - v|^2 \tag{3.1}$$

and $D(0, t, i) = 0$.

Theorem 3.4. *Let Assumptions 2.1, 3.3 and 2.3 hold except (2.6) which is replaced by*

$$LV(x,y,t,i) \leq -\alpha_1|x|^2 + \alpha_2 q\psi^{-\varepsilon}((1-q)t)|y|^2 - \alpha_3|x|^\gamma + \alpha_4 q\psi^{-\varepsilon}((1-q)t)|y|^\gamma \tag{3.2}$$

for all $(x, y, t, i) \in R^n \times R^n \times [t_0, \infty) \times S$, where $\alpha_1 > \alpha_2 \geq 0$ and $\alpha_3 > \alpha_4 \geq 0$. Then for any given initial data ξ, the solution $x(t)$ of Eq. (2.1) has the property that

$$\limsup_{t\to\infty} \frac{\log|x(t)|}{\log\psi(t)} < -\frac{\eta}{2}, \quad a.s. \tag{3.3}$$

where $\eta \in (0, \varepsilon \wedge \bar{\eta})$ while $\bar{\eta}$ is the unique root to the following equation

$$\alpha_1 = \alpha_2 + 2\left(1 + \frac{k_0}{q}\right)c_2 C_\psi \bar{\eta}.$$

Proof. We first observe that (3.2) is stronger than (2.6). So, by Theorem 2.6, for any given initial data ξ, Eq.(2.1) has a unique global solution $x(t)$ on $t \geq t_0$. Let $\eta \in (0, \varepsilon)$. For any $t \geq t_0$, by the generalized Itô formula to $\psi^\eta(t)V(z(t), t, r(t))$, we obtain that

$$\begin{aligned}\psi^\eta(t)V(z(t),t,r(t)) = \psi^\eta(t_0)V(z(t_0),t_0,r(t_0)) + \int_{t_0}^t \eta\frac{\psi'(s)}{\psi(s)}\psi^\eta(s)V(z(s),s,r(s))ds \\ + \int_{t_0}^t \psi^\eta(s)LV(x(s),x(qs),s,r(s))ds + M_t,\end{aligned}$$

where $M_t = \int_{t_0}^t \psi^\eta(s)V_x(z(s),s,r(s))g(x(s),x(qs),s,r(s))dw(s)$. By conditions (2.5) and (3.2), we then compute

$$\begin{aligned}c_1\psi^\eta(t)|z(t)|^2 \leq c_2\psi^\eta(t_0)|z(t_0)|^2 + c_2\eta C_\psi \int_{t_0}^t \psi^\eta(s)|z(s)|^2 ds - \alpha_1\int_{t_0}^t \psi^\eta(s)|x(s)|^2 ds \\ + \alpha_2 q\int_{t_0}^t \psi^\eta(s)\psi^{-\varepsilon}((1-q)s)|x(qs)|^2 ds - \alpha_3\int_{t_0}^t \psi^\eta(s)|x(s)|^\gamma ds \\ + \alpha_4 q\int_{t_0}^t \psi^\eta(s)\psi^{-\varepsilon}((1-q)s)|x(qs)|^\gamma ds + M_t,\end{aligned} \tag{3.4}$$

where $C_\psi = \sup_{t_0\leq t<\infty}|\frac{\psi'(t)}{\psi(t)}| < \phi < \infty$. By the basic inequality $|a+b|^2 \leq 2(|a|^2+|b|^2)$ and the definition of ψ function, we have

$$\begin{aligned}\int_{t_0}^{t}\psi^{\eta}(s)|z(s)|^2ds &\le 2\int_{t_0}^{t}\psi^{\eta}(s)\left(|x(s)|^2+k_0\psi^{-\varepsilon}((1-q)s)|x(qs)|^2\right)ds\\ &\le 2\int_{t_0}^{t}\psi^{\eta}(s)|x(s)|^2ds+2k_0\int_{t_0}^{t}\psi^{\eta}(s)\psi^{-\varepsilon}((1-q)s)|x(qs)|^2ds\\ &\le 2\int_{t_0}^{t}\psi^{\eta}(s)|x(s)|^2ds+2k_0\int_{t_0}^{t}\psi^{\eta}(s)\psi^{-\eta}((1-q)s)|x(qs)|^2ds\\ &\le 2\int_{t_0}^{t}\psi^{\eta}(s)|x(s)|^2ds+2k_0\int_{t_0}^{t}\psi^{\eta}(qs)|x(qs)|^2ds\\ &\le 2\frac{k_0}{q}\int_{qt_0}^{t_0}\psi^{\eta}(s)|x(s)|^2ds+2\left(1+\frac{k_0}{q}\right)\int_{t_0}^{t}\psi^{\eta}(s)|x(s)|^2ds. \end{aligned}\tag{3.5}$$

Similarly, we get

$$\alpha_2 q\int_{t_0}^{t}\psi^{\eta}(s)\psi^{-\varepsilon}((1-q)s)|x(qs)|^2ds \le \alpha_2\int_{qt_0}^{t_0}\psi^{\eta}(s)|x(s)|^2ds+\alpha_2\int_{t_0}^{t}\psi^{\eta}(s)|x(s)|^2ds \tag{3.6}$$

and

$$\alpha_4 q\int_{t_0}^{t}\psi^{\eta}(s)\psi^{-\varepsilon}((1-q)s)|x(qs)|^{\gamma}ds \le \alpha_4\int_{qt_0}^{t_0}\psi^{\eta}(s)|x(s)|^{\gamma}ds+\alpha_4\int_{t_0}^{t}\psi^{\eta}(s)|x(s)|^{\gamma}ds. \tag{3.7}$$

Hence,

$$\begin{aligned}c_1\psi^{\eta}(t)|z(t)|^2 &\le C-\left(\alpha_1-\alpha_2-2\left(1+\frac{k_0}{q}\right)c_2C_{\psi}\eta\right)\int_{t_0}^{t}\psi^{\eta}(s)|x(s)|^2ds\\ &\quad-(\alpha_3-\alpha_4)\int_{t_0}^{t}\psi^{\eta}(s)|x(s)|^{\gamma}ds+M_t,\end{aligned}\tag{3.8}$$

where

$$\begin{aligned}C &= c_2\psi^{\eta}(t_0)|z(t_0)|^2+\left(2c_2\eta C_{\psi}\frac{k_0}{q}+\alpha_2\right)\int_{qt_0}^{t_0}\psi^{\eta}(s)|x(s)|^2ds+\alpha_4\int_{qt_0}^{t_0}\psi^{\eta}(s)|x(s)|^{\gamma}ds\\ &\le c_2\psi^{\eta}(t_0)(|x(t_0)|^2+k_0|x(qt_0)|^2)\\ &\quad+\left(2c_2\eta C_{\psi}\frac{k_0}{q}+\alpha_2\right)\int_{qt_0}^{t_0}\psi^{\eta}(s)|x(s)|^2ds+\alpha_4\int_{qt_0}^{t_0}\psi^{\eta}(s)|x(s)|^{\gamma}ds<\infty.\end{aligned}$$

Since $\eta\in(0,\varepsilon\wedge\bar{\eta})$ and $\alpha_3>\alpha_4$, then

$$c_1\psi^{\eta}(t)|z(t)|^2\le C+M_t. \tag{3.9}$$

By Lemma 2.5, we have that $\limsup_{t\to\infty}\psi^{\eta}(t)|z(t)|^2\le\infty$ a.s. Hence, there is a finite positive random variable ζ such that

$$\sup_{t_0<t<\infty}\psi^{\eta}(t)|z(t)|^2\le\zeta \quad \text{a.s.} \tag{3.10}$$

Similar to (2.12), it follows that for any $t_1>t_0$

$$\begin{aligned}\sup_{t_0\le t\le t_1}\psi^{\eta}(t)|x(t)|^2 &\le \frac{\sup_{t_0\le t\le t_1}\psi^{\eta}(t)|z(t)|^2}{(1-\sqrt{k_0})^2}+\frac{\sqrt{k_0}}{1-\sqrt{k_0}}\psi^{\eta}(t_0)\|\xi\|^2.\\ &\le\frac{\zeta}{(1-\sqrt{k_0})^2}+\frac{\sqrt{k_0}}{1-\sqrt{k_0}}\psi^{\eta}(t_0)\|\xi\|^2.\end{aligned}$$

This implies

$$\limsup_{t\to\infty} \frac{\log|x(t)|}{\log\psi(t)} < -\frac{\eta}{2} \quad \text{a.s.}$$

as required. The proof is therefore complete. □

Remark 3.5. In Theorem 3.4, if $\psi(t) = e^t$ and $\psi(t) = 1+t$, then (3.3) implies that Eq. (2.1) is almost surely exponentially stable and polynomially stable. Hence, we obtain the general stability result as it contains both exponential and polynomial stability as special cases. In other words, we extend these two classes of stability into the general decay stability in this paper. And this will be fully illustrated by Examples 3.11 and 3.12.

Remark 3.6. From Theorem 3.4, the almost sure stability with general decay rate of Eq. (2.1) has been examined and the upper bound of the convergence rate has been estimated.

Obviously, it is not convenient to check condition (3.2) of Theorem 3.4, since it is not related to coefficients f and g explicitly. Now, we shall impose some conditions on f and g to guarantee Theorem 3.4 and establish a sufficient criteria on almost sure ψ-type stability in terms of M-matrix.

Let us now state our hypothesis in terms of an M-matrix, which will replace condition (3.2).

Assumption 3.7. Let $\gamma > 2$ and assume that for each $i \in S$, there are nonnegative numbers $\alpha_{2i}, \alpha_{3i}, \alpha_{4i}, \beta_{1i}, \beta_{2i}, \beta_{3i}, \beta_{4i}$ and a real number α_{1i} as well as bounded functions $h_i(\cdot)$ such that

$$\begin{aligned}&(x - D(y,t,i))^\top f(x,y,t,i)\\ &\quad \le \alpha_{1i}|x|^2 + \alpha_{2i}q\psi^{-\varepsilon}((1-q)t)|y|^2 - \alpha_{3i}|x|^\gamma + \alpha_{4i}q\psi^{-\varepsilon}((1-q)t)|y|^\gamma\end{aligned} \tag{3.11}$$

and

$$|g(x,y,t,i)|^2 \le \beta_{1i}|x|^2 + \beta_{2i}q\psi^{-\varepsilon}((1-q)t)|y|^2 + \beta_{3i}|x|^\gamma + \beta_{4i}q\psi^{-\varepsilon}((1-q)t)|y|^\gamma \tag{3.12}$$

for any $(x,y,t) \in R^n \times R^n \times [t_0,\infty)$.

Assumption 3.8. Assume that

$$\mathcal{A} := -\text{diag}(2\alpha_{11}+\beta_{11},\dots,2\alpha_{1N}+\beta_{1N}) - (1+\sqrt{k_0})\Gamma \tag{3.13}$$

is a nonsingular M-matrix.

Lemma 3.9 (see [25]). *If $A \in Z^{N\times N} = \{A = (a_{ij})_{N\times N} : a_{ij} \le 0,\ i \ne j\}$, then the following statements are equivalent:*

(1) A is a nonsingular M-matrix.

(2) A is semi-positive; that is, there exists $x \gg 0$ in R^N such that $Ax \gg 0$.

(3) A^{-1} exists and its elements are all nonnegative.

(4) All the leading principal minors of A are positive; that is

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \textit{for every } k = 1,2,\dots,N.$$

In fact, by Assumption 3.8 and Lemma 3.9, it follows that

$$\theta = (\theta_1, \ldots, \theta_N)^\top := \mathcal{A}^{-1}\overrightarrow{1} > 0 \tag{3.14}$$

for all $i \in S$, where $\overrightarrow{1} = (1, \ldots, 1)^\top$.

Theorem 3.10. *Let Assumptions 2.1, 3.7 and 3.8 hold. Assume that*

$$\max_{i\in S}\left((2\alpha_{2i} + \beta_{2i})\theta_i + \sqrt{k_0}(1+\sqrt{k_0})\sum_{j=1}^{N}\gamma_{ij}\theta_j\right) < 1 \tag{3.15}$$

and

$$\min_{i\in S}(2\alpha_{3i} - \beta_{3i})\theta_i > \max_{i\in S}(2\alpha_{4i} + \beta_{4i})\theta_i. \tag{3.16}$$

Then for any given initial data ξ, there is a unique global solution $x(t)$ of Eq. (2.1) and the solution is almost surely ψ-type stable.

Proof. Let us define the function $V(x - D(y,t,i), t, i) = \theta_i|x - D(y,t,i)|^2$. Clearly, V obeys conditions (2.5) with $c_1 = \min_{i\in S}\theta_i$ and $c_2 = \max_{i\in S}\theta_i$. To verify condition (3.2), we compute the operator LV as follows

$$LV(x,y,t,i) = 2\theta_i(x - D(y,t,i))^\top f(x,y,t,i) + \theta_i|g(x,y,t,i)|^2 + \sum_{j=1}^{N}\gamma_{ij}\theta_j|x - D(y,t,i)|^2. \tag{3.17}$$

By the basic inequality

$$|a+b|^2 \le (1+\varepsilon)|a|^2 + \left(1+\frac{1}{\varepsilon}\right)|b|^2, \quad \text{for any } a, b \ge 0 \text{ and } r \in [0,1]$$

and Assumption 3.3, we have

$$\begin{aligned} |x - D(y,t,i)|^2 &\le (1+\sqrt{k_0})|x|^2 + \left(1+\frac{1}{\sqrt{k_0}}\right)|D(y,t,i)|^2 \\ &\le (1+\sqrt{k_0})|x|^2 + \sqrt{k_0}(1+\sqrt{k_0})\psi^{-\varepsilon}((1-q)t)|y|^2. \end{aligned} \tag{3.18}$$

By Assumption 3.7, it follows from (3.17) that

$$\begin{aligned} LV(x,y,t,i) \le &\left((2\alpha_{1i}+\beta_{1i})\theta_i + (1+\sqrt{k_0})\sum_{j=1}^{N}\gamma_{ij}\theta_j\right)|x|^2 \\ &+ \left((2\alpha_{2i}+\beta_{2i})\theta_i + \sqrt{k_0}(1+\sqrt{k_0})\sum_{j=1}^{N}\gamma_{ij}\theta_j\right)q\psi^{-\varepsilon}((1-q)t)|y|^2 \\ &- (2\alpha_{3i}-\beta_{3i})\theta_i|x|^\gamma + (2\alpha_{4i}+\beta_{4i})\theta_i q\psi^{-\varepsilon}((1-q)t)|y|^\gamma. \end{aligned} \tag{3.19}$$

By the definition of θ_i, we have

$$(2\alpha_{1i}+\beta_{1i})\theta_i + (1+\sqrt{k_0})\sum_{j=1}^{N}\gamma_{ij}\theta_j = -1.$$

Hence,

$$LV(x,y,t,i) \le -\alpha_1|x|^2 + \alpha_2 q\psi^{-\varepsilon}((1-q)t)|y|^2 - \alpha_3|x|^\gamma + \alpha_4 q\psi^{-\varepsilon}((1-q)t)|y|^\gamma, \tag{3.20}$$

where

$$\alpha_1 = 1, \quad \alpha_2 = \max_{i\in S}\left((2\alpha_{2i}+\beta_{2i})\theta_i + \sqrt{k_0}(1+\sqrt{k_0})\sum_{j=1}^{N}\gamma_{ij}\theta_j\right),$$
$$\alpha_3 = \min_{i\in S}(2\alpha_{3i}-\beta_{3i})\theta_i, \quad \alpha_4 = \max_{i\in S}(2\alpha_{4i}+\beta_{4i})\theta_i. \tag{3.21}$$

Recalling (3.15) and (3.16), condition (3.2) is fulfilled. By Theorem 3.4, we can conclude that for any given initial data ξ, there is a unique global solution $x(t)$ and the solution of Eq. (2.1) is almost surely ψ-type stable. The proof is therefore complete. □

Finally, we shall give two examples to illustrate the applications of our results.

Example 3.11. Let $w(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1,2\}$ with the generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}.$$

Of course, $w(t)$ and $r(t)$ are assumed to be independent. Consider the following scalar NSPEwMSs

$$d[x(t) - D(x(0.75t),t,r(t))] = f(x(t),t,r(t))dt + g(x(0.75t),t,r(t))dw(t), \quad t \geq 1, \tag{3.22}$$

with initial data $\xi(t) = x_0$ $(0.75 \leq t \leq 1)$ and $r(1) = 1$. Moreover, for $(x,y,t,i) \in R \times R \times [0.75,\infty) \times S$,

$$D(y,t,i) = \begin{cases} 0.1(1+0.25t)^{-0.5}y, & \text{if } i = 1, \\ 0.2(1+0.25t)^{-0.5}y, & \text{if } i = 2, \end{cases}$$
$$f(x,t,i) = \begin{cases} -x-2x^3, & \text{if } i = 1, \\ x - x^3, & \text{if } i = 2, \end{cases}$$

and

$$g(y,t,i) = \begin{cases} 0.1(1+0.25t)^{-0.5}y^2, & \text{if } i = 1, \\ 0.5(1+0.25t)^{-0.5}y^2, & \text{if } i = 2. \end{cases}$$

We note that Eq. (3.22) can be regarded as the result of the two equations

$$\begin{aligned} &d[x(t) - 0.1(1+0.25t)^{-0.5}x(0.25t)] \\ &\quad = (-x(t) - 2x^3(t))dt + 0.1(1+0.25t)^{-0.5}x^2(0.25t)dw(t) \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &d[x(t) - 0.2(1+0.25t)^{-0.5}x(0.25t)] \\ &\quad = (x(t) - x^3(t))dt + 0.05(1+0.25t)^{-0.5}x^2(0.25t)dw(t) \end{aligned} \tag{3.24}$$

switching among each other according to the movement of the Markov chain $r(t)$. It is easy to see that Eq. (3.23) is polynomially stable but Eq. (3.24) is unstable. However, we shall see that due to the Markovian switching, the overall system (3.22) will be polynomially stable. Note that the coefficients f and g satisfy the local Lipschitz condition but they do not satisfy the linear growth condition. Through a straight computation, we have

$$\begin{aligned}(x-D(y,t,1))^\top f(x,t,1) \leq & -0.8|x|^2+0.05(1+0.25t)^{-1}|y|^2 \\ & -1.85|x|^4+0.05(1+0.25t)^{-1}|y|^4, \quad (3.25)\\ (x-D(y,t,2))^\top f(x,t,2) \leq & \, 0.6|x|^2+0.1(1+0.25t)^{-1}|y|^2 \\ & -0.85|x|^4+0.05(1+0.25t)^{-1}|y|^4, \quad (3.26)\\ |g(y,t,1)|^2 \leq & \, 0.01(1+0.25t)^{-1}|y|^4, \quad (3.27)\\ |g(y,t,2)|^2 \leq & \, 0.25(1+0.25t)^{-1}|y|^4 \quad (3.28)\end{aligned}$$

where $\psi^{-\varepsilon}(0.25t)=(1+0.25t)^{-1},(\varepsilon=1)$ and $\alpha_{11}=-0.8,\ \alpha_{21}=0.2,\ \alpha_{31}=1.85,\ \alpha_{41}=0.2,\ \alpha_{12}=0.6,\ \alpha_{22}=0.4,\ \alpha_{32}=0.85,\ \alpha_{42}=0.2,\ \beta_{11}=0,\ \beta_{21}=0,\ \beta_{31}=0,\ \beta_{41}=0.04,\ \beta_{12}=0,\ \beta_{22}=0,\ \beta_{32}=0,\ \beta_{42}=1,\ \gamma=4$. So the inequalities (3.25)–(3.28) show that the Assumption 3.7 holds. By (3.13), we get the matrix $\mathcal{A}$

$$\begin{aligned}\mathcal{A} &= -\mathrm{diag}(2\alpha_{11}+\beta_{11}, 2\alpha_{12}+\beta_{12})-(1+\sqrt{k_0})\Gamma \\ &= \begin{pmatrix} 2.8 & -1.2 \\ -4.8 & 3.6 \end{pmatrix}.\end{aligned}$$

It is easy to compute

$$\mathcal{A}^{-1}=\begin{pmatrix} 0.833 & 0.278 \\ 1.111 & 0.648 \end{pmatrix}.$$

By Lemma 3.9, we see that $\mathcal{A}$ is a non-singular M-matrix. Compute

$$(\theta_1,\theta_2)^T=\mathcal{A}^{-1}\vec{1}=(1.111,1.759)^T,$$

and by (3.21), we have

$$\begin{aligned}&\alpha_2=\max_{i=1,2}\left((2\alpha_{2i}+\beta_{2i})\theta_i+\sqrt{k_0}(1+\sqrt{k_0})\sum_{j=1}^{2}\gamma_{ij}\theta_j\right)=0.7851,\\ &\alpha_3=\min_{i=1,2}(2\alpha_{3i}\theta_i-\beta_{3i}\theta_i)=2.9903, \quad \alpha_4=\max_{i=1,2}(2\alpha_{4i}\theta_i+\beta_{4i}\theta_i)=2.4626.\end{aligned}$$

Hence, we conclude that the conditions (3.15), (3.16) hold. By Theorem 3.4, we can obtain that

$$\limsup_{t\to\infty}\frac{\log|x(t)|}{\log t}\leq-\frac{\eta}{2} \quad a.s.$$

where $\eta\in(0,0.1159)$. That is to say, the solution of Eq. (3.22) decays at the polynomial rate of at least 0.05795.

Example 3.12. Let $w(t)$ is a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S=\{1,2\}$ with the generator

$$\Gamma=\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Of course, $w(t)$ and $r(t)$ are assumed to be independent. Consider the following scalar NSPEwMSs

$$\begin{aligned} &d[x(t) - D(x(0.5t), t, r(t))] \\ &\quad = f(x(t), x(0.5t), t, r(t))dt + g(x(t), x(0.5t), t, r(t))dw(t), \quad t \geq 1, \end{aligned} \tag{3.29}$$

with initial data $\xi(t) = x_0$ $(0.5 \leq t \leq 1)$ and $r(1) = 1$. Moreover, for $(x, y, t, i) \in R \times R \times [0.5, \infty) \times S$, $D(y, t, i) = 0.25e^{-0.5t}y, i = 1, 2$.

$$f(x, y, t, i) = \begin{cases} -3x - 2.5x^3 + e^{-0.5t}y, & \text{if } i = 1 \\ 2x - 1.5x^3 + 0.8e^{-0.5t}y, & \text{if } i = 2, \end{cases}$$

and

$$g(x, y, t, i) = \begin{cases} \rho_1 e^{-0.5t}y^2, & \text{if } i = 1 \\ \rho_2 e^{-0.5t}y^2, & \text{if } i = 2, \end{cases}$$

but ρ_1 and ρ_2 are unknown parameters. Eq. (3.29) can be regarded as a stochastically perturbed system of the follwing neutral pantograph equations with Markovian switching

$$\frac{d[x(t) - D(x(0.5t), t, r(t))]}{dt} = f(x(t), x(0.5t), t, r(t)).$$

Our aim here is to get the bounds on the unknown parameters ρ_1 and ρ_2 so that Eq. (3.29) remain stable. To apply Theorem 3.10, we let $\gamma = 4$. Noting

$$(x - D(y, t, 1))^\top f(x, y, t, 1) \leq -1.469|x|^2 + 0.25e^{-t}|y|^2 - 2.125|x|^4 + 0.125e^{-t}|y|^4, \tag{3.30}$$

$$(x - D(y, t, 2))^\top f(x, y, t, 2) \leq 2.045|x|^2 + 0.3e^{-t}|y|^2 - 1.219|x|^4 + 0.094e^{-t}|y|^4, \tag{3.31}$$

$$|g(x, y, t, 1)|^2 \leq \rho_1^2 e^{-t}|y|^4, \quad |g(x, y, t, 2)|^2 \leq \rho_2^2 e^{-t}|y|^4 \tag{3.32}$$

where $\psi^{-\varepsilon}(0.5t) = e^{-t}, (\varepsilon = 2)$ and $\alpha_{11} = -1.469$, $\alpha_{21} = 0.5$, $\alpha_{31} = 2.125$, $\alpha_{41} = 0.25$, $\alpha_{12} = 2.045$, $\alpha_{22} = 0.6$, $\alpha_{32} = 1.219$, $\alpha_{42} = 0.188$, $\beta_{11} = 0$, $\beta_{21} = 0$, $\beta_{31} = 0$, $\beta_{41} = 2\rho_1^2$, $\beta_{12} = 0$, $\beta_{22} = 0$, $\beta_{32} = 0$, $\beta_{42} = 2\rho_2^2$. Then, the inequalities (3.30)–(3.32) show that the Assumption 3.7 holds. By (3.13), we see that the matrix $\mathcal{A}$ is

$$\begin{aligned} \mathcal{A} &= -\text{diag}(2\alpha_{11} + \beta_{11}, 2\alpha_{12} + \beta_{12}) - (1 + \sqrt{k_0})\Gamma \\ &= \begin{pmatrix} 4.188 & -1.25 \\ -2.5 & 6.59 \end{pmatrix}. \end{aligned}$$

It is easy to compute

$$\mathcal{A}^{-1} = \begin{pmatrix} 0.269 & 0.051 \\ 0.102 & 0.171 \end{pmatrix}.$$

By Lemma 3.9, we see that $\mathcal{A}$ is a non-singular M-matrix. By (3.14), we then have $\theta_1 = 0.32$ and $\theta_2 = 0.273$. Clearly, $\alpha_2 = \max_{i=1,2}\left((2\alpha_{2i} + \beta_{2i})\theta_i + \sqrt{k_0}(1 + \sqrt{k_0})\sum_{j=1}^{2}\gamma_{ij}\theta_j\right) = 0.3569$, while condition (3.16) becomes

$$\min\{1.36, 0.665\} > \max\{0.16 + 0.64\rho_1^2, 0.103 + 0.546\rho_2^2\}.$$

i.e.,

$$\rho_1^2 < 0.789, \; \rho_2^2 < 1.0293. \tag{3.33}$$

By Theorem 3.10, we can conclude that if the parameters $\rho_i, i = 1, 2$ satisfy (3.33), then for any initial data x_0, there is a unique global solution $x(t)$ to Eq. (2.1) on $t \in [1, \infty)$. Moreover, the solution has the property that

$$\limsup_{t \to \infty} \frac{\log |x(t)|}{t} \leq -\frac{\eta}{2} \quad a.s.$$

where $\eta \in (0, 0.8932)$. That is to say, the solution of Eq. (3.29) decays at the exponential rate of at least 0.4466.

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