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Oscillatory behavior of the second order noncanonical differential equations

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Abstract. Establishing monotonical properties of nonoscillatory solutions we introduce new oscillatory criteria for the second order noncanonical differential equation with delay/advanced argument

$$(r(t)y'(t))' + p(t)y(\tau(t)) = 0.$$

Our oscillatory results essentially extend the earlier ones. The progress is illustrated via Euler differential equation.

Keywords: half-linear neutral differential equation, delay, second-order, oscillation.

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1 Introduction

We consider the second order noncanonical differential equation

$$(r(t)y'(t))' + p(t)y(\tau(t)) = 0, \tag{E}$$

where

(H_1) $p(t) \in C([t_0,\infty))$ is positive;

(H_2) $r(t) \in C([t_0,\infty))$ is positive;

(H_3) $\tau(t) \in C^1([t_0,\infty))$ and $\tau'(t) \geq 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

By a solution of (E) we mean a function $y(t)$ with $(r(t)y'(t))$ in $C^1([t_0,\infty))$, which satisfies Eq. (E) on $[t_0,\infty)$. We consider only those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq t_0$. A solution of (E) is said to be oscillatory if it has arbitrarily large zeros and otherwise, it is called nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

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We say that (E) is in noncanonical form if

$$\pi(t) = \int_t^\infty \frac{1}{r(s)} \mathrm{d}s < \infty.$$

In this paper we establish new differential inequalities that lead to new monotonicity properties of solutions which are applied to obtain new oscillatory criteria for delay and advanced differential equations.

In the theory of differential equations, comparison theorems assert particular properties of solutions of a differential equation provided that an auxiliary equation/inequality possesses a certain property. See enclosed references [1–18]. In the paper we use the comparison technique to establish the main results.

There is a significant difference in the structure of nonoscillatory (say positive) solutions between canonical and non-canonical equations. It is well known that the first derivative of any positive solution y of canonical equation is of one sign eventually, while for noncanonical one both sign possibilities of the first derivative of any positive solution y have to be treated. A common approach in the literature (see [2,7,8,15,16,18]) for investigation of such equations consists in extending known results for canonical ones.

Very recently, Džurina and Jadlovská [4] established, contrary to most existing results, one-condition oscillation criterion for (E) Particularly, they showed that (E) is oscillatory if

$$\limsup_{t\to\infty} \pi(t) \int_{t_0}^t p(s)\mathrm{d}s > 1. \tag{1.1}$$

Recently, Baculíková in [3] extended the technique of Koplatadze et al. [12] to noncanonical equations. The objective of this paper is to study further the oscillatory and asymptotic properties of (E) in non-canonical form and provide new results, which would improve those obtained for linear equations discussed above.

We assume that all functional inequalities hold eventually, i.e., they are satisfied for all t large enough.

2 Preliminary results

It follows from a generalization of lemma of Kiguradze [10] that the set of positive solutions of (E) has the following structure.

Lemma 2.1. *Assume that $y(t)$ is an eventually positive solution of (E). Then $y(t)$ satisfies one of the following conditions*

$$\begin{aligned} (N_1):\ & r(t)y'(t) > 0, \quad \big(r(t)y'(t)\big)' < 0, \\ (N_*):\ & r(t)y'(t) < 0, \quad \big(r(t)y'(t)\big)' < 0 \end{aligned}$$

for $t \ge t_1 \ge t_0$.

The following considerations are intended to show that the class (N_*) is the essential one.

Lemma 2.2. *If*

$$\int_{t_0}^\infty \pi(s)p(s)\mathrm{d}s = \infty, \tag{2.1}$$

then positive solution $y(t)$ of (E) satisfies (N_*) and, moreover, *(i)* $\lim_{t\to\infty} y(t) = 0$;

(ii) $y(t) + r(t)y'(t)\pi(t) \geq 0$;

(iii) $\frac{y(t)}{\pi(t)}$ *is increasing.*

Proof. Assume on the contrary that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_1) for $t \geq t_1 \geq t_0$. Integrating (E) from t_1 to ∞, we get

$$r(t_1)y'(t_1) \geq \int_{t_1}^{\infty} p(s)y(\tau(s))\mathrm{d}s.$$

Since $y(t)$ is positive and increasing, there exists positive constant k that $y(t) \geq k$ and $y(\tau(t)) \geq k$ eventually. Therefore, we obtain

$$r(t_1)y'(t_1) \geq k\int_{t_1}^{\infty} p(s)\mathrm{d}s \geq k\int_{t_1}^{\infty} \pi(s)p(s)\mathrm{d}s$$

which contradicts to (2.1) and we conclude that $y(t)$ satisfies (N_*). Consequently, there exists a finite $\lim_{t\to\infty} y(t) = \ell$. We claim that $\ell = 0$. If not, then $y(t) \geq \ell > 0$. An integration of (E) from t_1 to t yields

$$-r(t)y'(t) \geq \ell\int_{t_1}^{t} p(s)\mathrm{d}s.$$

Integrating once more from t_1 to ∞, one gets

$$y(t_1) \geq \ell\int_{t_1}^{\infty} \frac{1}{r(u)}\int_{t_1}^{u} p(s)\mathrm{d}s\mathrm{d}u = \ell\int_{t_1}^{\infty} \pi(s)p(s)\mathrm{d}s = \infty.$$

A contradiction and we conclude that $\ell = 0$.

To verify part (ii) we proceed as follows. The monotonicity of $r(t)y'(t)$ implies that

$$y(t) \geq \int_{t}^{\infty} \frac{-r(s)y'(s)}{r(s)}\mathrm{d}s \geq -r(t)y'(t)\int_{t}^{\infty} \frac{1}{r(s)}\mathrm{d}s = -r(t)y'(t)\pi(t),$$

which implies that part (iii) holds true. The proof is complete now. □

In the previous results we do not distinguish whether (E) is delay or advanced differential equation. But it what follows we separately establish oscillatory criteria for delay and advanced differential equations.

3 Delay equation

Throughout this section we assume that (E) is delay equation, that is

$$\tau(t) \leq t. \tag{3.1}$$

We are about to establish new monotonic properties for solutions of (E) from the class (N_*).

Lemma 3.1. *Let* (2.1) *and* (3.1) *hold. Assume that there exists a* $\beta_0 > 0$ *such that*

$$p(t)\pi^2(t)r(t) \geq \beta_0 \tag{3.2}$$

eventually. If $y(t)$ *is a positive solution of* (E)*, then* *(i)* $\frac{y(t)}{\pi^{\beta_0}(t)}$ *is decreasing;*

(ii) $\lim_{t\to\infty} \frac{y(t)}{\pi^{\beta_0}(t)} = 0$;

(iii) $\frac{y(t)}{\pi^{1-\beta_0}(t)}$ *is increasing.*

Proof. Assume that $y(t)$ is an eventually positive solution of (E). Then (2.1) ensures that $y(t)$ and $y(\tau(t))$ satisfies condition (N_*) for $t \ge t_1 \ge t_0$. An integration of (E) from t_1 to t yields

$$-r(t)y'(t) = -r(t_1)y'(t_1) + \int_{t_1}^{t} p(s)y(\tau(s))\mathrm{d}s \ge -r(t_1)y'(t_1) + y(t)\int_{t_1}^{t} p(s)\mathrm{d}s,$$

which in view of (3.2) leads to

$$\begin{aligned} -r(t)y'(t) &\ge -r(t_1)y'(t_1) + \beta_0 y(t)\int_{t_1}^{t} \frac{1}{\pi^2(s)r(s)}\mathrm{d}s \\ &= -r(t_1)y'(t_1) - \beta_0\frac{y(t)}{\pi(t_1)} + \beta_0\frac{y(t)}{\pi(t)} \ge \beta_0\frac{y(t)}{\pi(t)}, \end{aligned} \tag{3.3}$$

where we have used that $y(t) \to 0$ as $t \to \infty$. Consequently,

$$\left(\frac{y(t)}{\pi^{\beta_0}(t)}\right)' = \frac{\pi^{\beta_0-1}(t)\left[r(t)y'(t)\pi(t) + \beta_0 y(t)\right]}{r(t)\pi^{2\beta_0}(t)} \le 0.$$

So $\frac{y(t)}{\pi^{\beta_0}(t)}$ is decreasing, and there exists $\lim_{t\to\infty}\frac{y(t)}{\pi^{\beta_0}(t)} = \ell \ge 0$. We claim that $\ell = 0$. If not, then $\frac{y(t)}{\pi^{\beta_0}(t)} \ge l > 0$ eventually. On the other hand, we introduce the auxiliary function

$$z(t) = (r(t)y'(t)\pi(t) + y(t))\pi^{-\beta_0}(t).$$

Lemma 2.2 (ii) implies that $z(t) > 0$ and

$$\begin{aligned} z'(t) &= (r(t)y'(t))'\pi^{1-\beta_0}(t) + \beta_0 y'(t)\pi^{-\beta_0}(t) + \beta_0\frac{\pi^{-\beta_0-1}(t)y(t)}{r(t)} \\ &= -p(t)y(\tau(t))\pi^{1-\beta_0}(t) + \beta_0 y'(t)\pi^{-\beta_0}(t) + \beta_0\frac{\pi^{-\beta_0-1}(t)y(t)}{r(t)} \\ &\le -\beta_0\frac{y(\tau(t))\pi^{-\beta_0-1}(t)}{r(t)} + \beta_0 y'(t)\pi^{-\beta_0}(t) + \beta_0\frac{\pi^{-\beta_0-1}(t)y(t)}{r(t)} \\ &\le \beta_0 y'(t)\pi^{-\beta_0}(t). \end{aligned}$$

Employing (3.3) and the fact that $y(t) \ge \ell\pi^{\beta_0}(t)$, we get that

$$z'(t) \le \frac{-\beta_0^2\ell}{\pi(t)r(t)} < 0.$$

Integrating the last inequality from t_1 to t, we obtain

$$z(t_1) \ge \beta_0^2\ell\ln\frac{\pi(t_1)}{\pi(t)} \to \infty \quad \text{as } t \to \infty.$$

which is a contradiction and we conclude that $\lim_{t\to\infty}\frac{y(t)}{\pi^{\beta_0}(t)} = 0$.

Finally, we shall show that $\frac{y(t)}{\pi^{1-\beta_0}(t)}$ is increasing. Equation (E), we can rewrite in equivalent form

$$(r(t)y'(t)\pi(t) + y(t))' + \pi(t)p(t)y(\tau(t)) = 0. \tag{3.4}$$

It follows from Lemma 2.2 (iii) that $\frac{y(t)}{\pi(t)}$ is increasing. An integration of (3.4) from t to ∞ yields

$$\begin{aligned} r(t)y'(t)\pi(t) + y(t) &\geq \int_t^\infty \pi(s)p(s)y(\tau(s))\mathrm{d}s \geq \int_t^\infty \pi(s)p(s)y(s)\mathrm{d}s \\ &\geq \frac{y(t)}{\pi(t)} \int_t^\infty \pi^2(s)p(s)\mathrm{d}s \geq \beta_0 y(t). \end{aligned} \tag{3.5}$$

The last inequality implies that

$$\left(\frac{y(t)}{\pi^{1-\beta_0}(t)}\right)' = \frac{\pi^{-\beta_0}(t)\,[r(t)y'(t)\pi(t) + y(t)(1-\beta_0)]}{r(t)\pi^{2-2\beta_0}(t)} \geq 0.$$

The proof is complete. □

Lemma 3.1 provides

$$\frac{y(t)}{\pi^{\beta_0}(t)} \downarrow \quad \text{and} \quad \frac{y(t)}{\pi^{1-\beta_0}(t)} \uparrow,$$

which immediately guarantees the following oscillatory criterion.

Theorem 3.2. *Let (2.1), (3.1), and (3.2) hold. If*

$$\beta_0 > \frac{1}{2},$$

then (E) is oscillatory.

If $\beta_0 \leq 1/2$, then the we are able to improve the results presented in Lemma 3.1. Since $\pi(t)$ is decreasing, there exists a constant $\alpha \geq 1$ such that

$$\frac{\pi(\tau(t))}{\pi(t)} \geq \alpha. \tag{3.6}$$

We introduce the constant $\beta_1 > \beta_0$ as follows

$$\beta_1 = \frac{\alpha^{\beta_0}\beta_0}{1-\beta_0}. \tag{3.7}$$

Lemma 3.3. *Let (2.1), (3.1), and (3.2) hold. If $y(t)$ is a positive solution of (E), then*

(i) $\frac{y(t)}{\pi^{\beta_1}(t)}$ *is decreasing;*

(ii) $\lim\limits_{t\to\infty} \frac{y(t)}{\pi^{\beta_1}(t)} = 0$;

(iii) $\frac{y(t)}{\pi^{1-\beta_1}(t)}$ *is increasing.*

Proof. Assume that $y(t)$ is an eventually positive solution of (E) satisfying condition (N_*) for $t \geq t_1 \geq t_0$. Integrating (E) from t_1 to t and using the fact that $\frac{y(t)}{\pi^{\beta_0}(t)}$ is decreasing, we get

$$\begin{aligned} -r(t)y'(t) &\geq -r(t_1)y'(t_1) + \int_{t_1}^{t} \frac{p(s)y(s)\pi^{\beta_0}(\tau(s))}{\pi^{\beta_0}(s)} \mathrm{d}s \\ &\geq -r(t_1)y'(t_1) + \frac{y(t)}{\pi^{\beta_0}(t)} \int_{t_1}^{t} p(s)\pi^{\beta_0}(\tau(s))\mathrm{d}s, \end{aligned} \tag{3.8}$$

which in view of (3.6) implies

$$-r(t)y'(t) \geq -r(t_1)y'(t_1) + \frac{\alpha^{\beta_0}\beta_0 y(t)}{\pi^{\beta_0}(t)} \int_{t_1}^{t} \frac{\pi^{\beta_0-2}(s)}{r(s)} \mathrm{d}s.$$

Evaluating the integral, we see that

$$-r(t)y'(t) \geq -r(t_1)y'(t_1) - \beta_1 \pi^{\beta_0-1}(t_1)\frac{y(t)}{\pi^{\beta_0}(t)} + \beta_1 \frac{y(t)}{\pi(t)}.$$

Since $\frac{y(t)}{\pi^{\beta_0}(t)} \to 0$ as $t \to \infty$, we obtain

$$-r(t)y'(t) \geq \beta_1 \frac{y(t)}{\pi(t)}, \tag{3.9}$$

from which exactly as in the proof of Lemma 3.1 follows that $\frac{y(t)}{\pi^{\beta_1}(t)}$ is decreasing.

Proceeding exactly as in the proof of Lemma 3.1 we can verify the rest of the assertions. □

If $\beta_1 < 1$, we can repeat the above procedure and introduce $\beta_2 > \beta_1$ as follows

$$\beta_2 = \beta_0 \frac{\alpha^{\beta_1}}{1 - \beta_1}.$$

In generally, as follows as $\beta_j < 1$ for $j = 1, 2, \ldots, n-1$ we can define

$$\beta_n = \beta_0 \frac{\alpha^{\beta_{n-1}}}{1 - \beta_{n-1}}, \tag{3.10}$$

provided that $\beta_n < 1$. And what is more, proceeding exactly as in proof of Lemma 4, we can verify that

$$\frac{y(t)}{\pi^{\beta_n}(t)} \downarrow \quad \text{and} \quad \frac{y(t)}{\pi^{1-\beta_n}(t)} \uparrow.$$

Consequently, the following result is obvious.

Theorem 3.4. *Let (2.1), (3.1), (3.2) and (3.10) hold. If there exists $n \in N$ such that*

$$\beta_n > \frac{1}{2}, \tag{3.11}$$

then (E) is oscillatory.

Now we are prepared to present the main result of this section.

Theorem 3.5. *Let* (2.1), (3.1), (3.2) *and* (3.10) *hold. If there exists* $n \in N$ *such that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\pi(s)\mathrm{d}s > \frac{1-\beta_n}{\mathrm{e}}, \tag{3.12}$$

then (*E*) *is oscillatory.*

Proof. Assume on the contrary that (E) possesses an eventually positive solution $y(t)$. Condition (2.1) guarantees that $y(t)$ satisfies condition (N_*). We construct sequence $\{\beta_n\}$ by (3.10). We consider the auxiliary function

$$w(t) = r(t)y'(t)\pi(t) + y(t).$$

It follows from Lemma 2.2 (ii) that $w(t) > 0$ and, moreover,

$$w'(t) = (r(t)y'(t))'\pi(t) = -p(t)\pi(t)y(\tau(t)). \tag{3.13}$$

On the other hand, since $\frac{y(t)}{\pi^{\beta_n}(t)}$ is decreasing, then $r(t)y'(t)\pi(t) + \beta_n y(t) \le 0$ and so

$$w(t) \le (1-\beta_n)y(t).$$

Setting the last inequality into (3.13) we see that $w(t)$ is a positive solution of

$$w'(t) + \frac{p(t)\pi(t)}{1-\beta_n}w(\tau(t)) \le 0. \tag{3.14}$$

This is a contradiction since by Theorem 2.1.1 in [14], condition (3.12) guarantees that (3.14) has no positive solution. The proof is complete. □

We illustrate the importance of the obtained results via illustrative examples.

Example 3.6. Consider the second order delay differential equation

$$\left(t^2y'(t)\right)' + a\,y(0.2\,t) = 0, \tag{E_{x1}}$$

with $a > 0$. For considered equation $\tau(t) = 0.2\,t$, $\pi(t) = 1/t$, $\beta_0 = a$, and $\alpha = 5$. So condition (3.12) reduces to

$$a\ln 5 = \frac{1-\beta_n}{\mathrm{e}} \tag{3.15}$$

with β_n iterative defined by (3.10).

A simple computation reveals that for $a = 0.155$ desired sequence

$$\beta_1 = 0.2354048140,$$
$$\beta_2 = 0.2961017968,$$
$$\beta_3 = 0.3546403245,$$

and (3.15) holds for $n = 3$, that is for $a = 0.155$ (E_{x1}) is oscillatory. We mention that condition (3.11) fails in this case.

What is more, we can establish oscillation of (E_{x1}) even for smaller value of a, but some mathematical software is needed because e.g. for $a = 0.13009$ condition (3.15) is satisfied for $\beta_{171} = 0.4316960062$.

We note that criterion (1.1) reduces to $a > 1$ for considered equation.

Example 3.7. Consider the second order delay differential equation

$$\left(t^3y'(t)\right)' + \frac{at^3+t}{t^2}\,y(\lambda\, t) = 0, \qquad a>0, \quad \lambda\in(0,1) \quad t\ge 1. \tag{E_{x2}}$$

Now $\pi(t) = 1/(2t^2)$ and consequently $\beta_0 = a/4$ and $\alpha = (1/\lambda)^2$.

If we set $a = 0.66$ and $\lambda = 0.5$, we can verify that $\beta_4 = 0.4333899$ and condition (3.12) holds for $n = 4$, which implies oscillation of (E_{x2}). We mention that condition (3.11) fails for $n = 4$.

On the other hand, for $a = 0.9$ and $\lambda = 0.8$, we have $\beta_5 = 0.53666$ and condition (3.11) holds for $n = 5$, which guarantees oscillation of (E_{x2}), but condition (3.12) fails for $n = 5$.

4 Advanced equation

The above mentioned method can be modified to serve also for advanced differential equations, namely when

$$\tau(t) \ge t. \tag{4.1}$$

We slightly modify the key constant β_0 to γ_0 as follows.

Lemma 4.1. *Let* (2.1) *and* (4.1) *hold. Assume that there exists a* $\gamma_0 > 0$ *such that*

$$p(t)\pi(t)\pi(\tau(t))r(t) \ge \gamma_0, \tag{4.2}$$

eventually. If $y(t)$ *is a positive solution of* (E), *then*

- *(i)* $\frac{y(t)}{\pi^{\gamma_0}(t)}$ *is decreasing;*
- *(ii)* $\lim_{t\to\infty} \frac{y(t)}{\pi^{\gamma_0}(t)} = 0$;
- *(iii)* $\frac{y(t)}{\pi^{1-\gamma_0}(t)}$ *is increasing.*

Proof. Assume that $y(t)$ is an eventually positive solution of (E). Then (2.1) ensures that $y(t)$ satisfies condition (N_*) for $t \ge t_1 \ge t_0$. By Lemma 2.2 (iii)

$$y(\tau(t)) \ge \frac{\pi(\tau(t))}{\pi(t)} y(t).$$

An integration of (E) from t_1 to t yields

$$-r(t)y'(t) = -r(t_1)y'(t_1) + \int_{t_1}^{t} p(s)y(\tau(s))\mathrm{d}s \ge -r(t_1)y'(t_1) + y(t)\int_{t_1}^{t} p(s)\frac{\pi(\tau(s))}{\pi(s)}\mathrm{d}s,$$

which in view of (4.2) yields

$$\begin{aligned} -r(t)y'(t) &\ge -r(t_1)y'(t_1) + \gamma_0 y(t)\int_{t_1}^{t} \frac{1}{\pi^2(s)r(s)}\mathrm{d}s \\ &= -r(t_1)y'(t_1) - \gamma_0\frac{y(t)}{\pi(t_1)} + \gamma_0\frac{y(t)}{\pi(t)} \ge \gamma_0\frac{y(t)}{\pi(t)}, \end{aligned} \tag{4.3}$$

where we have used that $y(t) \to 0$ as $t \to \infty$. Therefore,

$$\left(\frac{y(t)}{\pi^{\gamma_0}(t)}\right)' \le 0.$$

To prove parts (ii) and (iii) we proceed exactly as in the proof of Lemma 3.1. The proof is complete. □

Assuming that $\gamma_0 < 1$ we can introduce the constant $\gamma_1 > \gamma_0$ as follows. Since $\pi(t)$ is decreasing, there exist a constant $\omega \geq 1$ such that

$$\frac{\pi(t)}{\pi(\tau(t))} \geq \omega$$

and define

$$\gamma_1 = \gamma_0 \frac{\omega^{\gamma_0}}{1-\gamma_0}.$$

In generally as far as $\gamma_{n-1} < 1$ we can define

$$\gamma_n = \gamma_0 \frac{\omega^{\gamma_{n-1}}}{1-\gamma_{n-1}} \tag{4.4}$$

and verify that

$$\frac{y(t)}{\pi^{\gamma_n}(t)} \downarrow \quad \text{and} \quad \frac{y(t)}{\pi^{1-\gamma_n}(t)} \uparrow .$$

Similarly as in the "delay" section we can establish the following oscillatory criteria for advanced differential equations.

Theorem 4.2. *Let (2.1), (4.1), (4.2) and (4.4) hold. If there exists $n \in N$ such that*

$$\gamma_n > \frac{1}{2},$$

then (E) is oscillatory.

Theorem 4.3. *Let (2.1), (4.1), (4.2) and (4.4) hold. If there exists $n \in N$ such that*

$$\liminf_{t\to\infty} \int_t^{\tau(t)} p(s)\pi(\tau(s))\,\mathrm{d}s > \frac{1-\gamma_n}{\mathrm{e}}, \tag{4.5}$$

then (E) is oscillatory.

Example 4.4. Consider the second order advanced differential equation

$$\left(t^2 y'(t)\right)' + \frac{at+\ln t}{t}\, y(5t) = 0, \qquad a > 0, \quad t \geq 1. \tag{E_{x3}}$$

Now $\gamma_0 = a/5$, $\omega = 5$. It is easy to see that for $a = 0.8$, $\gamma_2 = 0.3156681513$, thus (4.5) is satisfied for $n = 2$ and Theorem 4.3 ensures oscillation of (E_{x3}).

5 Ordinary equation

The above mentioned results can be applied also for ordinary differential equation ($\tau(t) \equiv t$)

$$(r(t)y'(t))' + p(t)y(t) = 0. \tag{E_0}$$

Now the sequences β_n and γ_n are identical and defined by

$$\beta_n = \frac{\beta_0}{1-\beta_{n-1}} \tag{5.1}$$

with β_0 adjusted in (3.2). Both Theorems 3.4, 4.2 reduces to the following.

Theorem 5.1. *Let* (2.1) *and* (5.1) *hold. If there exists* $n \in N$ *such that*

$$\beta_n > \frac{1}{2},$$

then (E_0) *is oscillatory.*

Example 5.2. Consider the second order advanced differential equation

$$\left(t^{5/2}y'(t)\right)' + \frac{at + \arctan t}{\sqrt{t}}\, y(t) = 0, \qquad a > 0, \quad t \geq 1. \tag{E_{x4}}$$

On can see that $\beta_0 = 4a/9$, $\pi(t) = 2/(3t^{3/2})$ moreover, for $a = 0.57$, $\gamma_{12} = 0.5022329499$, thus Theorem 5.1 guarantees oscillation of (E_{x4}) for considered case.

6 Summary

In this paper we provided complete oscillation analyses for ordinary, delay and advanced differential equations.

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