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On monotone solutions and a self-adjoint spectral problem for a functional-differential equation of even order

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Abstract. For a self-adjoint boundary value problem for a functional-differential equation of even order, the basis property of the system of eigenfunctions and the equivalence of such statements as the positivity of the corresponding quadratic functional, the Jacobi condition and the positivity of the Green function are established.

Keywords: quadratic functional, monotone solutions, spectrum, Jacobi condition.

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1 The problem, notation, results

1.1 The problem

If the length of the interval $[a, b]$ is less than the distance between the zeros of solutions of an ordinary second-order differential equation, then the Green function and the corresponding quadratic functional are positive. Due to their importance, these properties have been generalized many times. The case of a self-adjoint operator is interesting because of its applications in physics. A second order self-adjoint functional differential operator with Sturm–Liouville boundary conditions was considered in [12, 13]. In this paper, we establish the equivalence of the analogue of the Jacobi condition and the analogs of the statements about the differential and integral inequality and positive definiteness of the quadratic functional for a two-term functional differential equation.

Let the operator $\mathcal{L}$ be defined by (:= means *'is equal to' by definition*)

$$\mathcal{L}u(x) := \frac{1}{\rho(x)}\left((-1)^m u^{(2m)} - \int_0^l u(s)q(x, ds)\right), \quad x \in [0, l] \quad (m \geq 1). \tag{1.1}$$

($\rho(x)$ is a fixed positive weight function). Under boundary conditions

$$u^{(k)}(0) = 0,\ k = 0, \dots, m-1, \tag{1.2}$$

$$u^{(k)}(l) = 0,\ k = m, \dots, 2m-1, \tag{1.3}$$

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operator $\mathcal{L}$ will be self-adjoint. These conditions are a special case of the boundary conditions considered in [6] and [11]. The study of such boundary conditions is related to the oscillatory property of solutions (see, for example, [7]). Let

$$\mathcal{L}_0 u(x) := \frac{1}{\rho(x)}(-1)^m u^{(2m)}, \tag{1.4}$$

$$Qu(x) := \frac{1}{\rho(x)}\int_0^l u(s)q(x,ds). \tag{1.5}$$

Then $\mathcal{L} = \mathcal{L}_0 - Q$. Let's call $\mathcal{L}_0$ the main part of the operator $\mathcal{L}$. Below $\mathcal{L}_0$ and Q will be defined in a special space.

1.2 Notation and assumptions

1.2.1 Basic notation and assumptions

In (1.1) $q(x,\cdot)$ is a measure depending on the parameter x. Instead of $q(x,ds)$ it can be written $d_s q(x,s)$, considering $q(x,\cdot)$ as usual non-decreasing function. If $\int_0^l u(s)q(x,ds) = \sum_{i=1}^\infty q_i(x)u(h_i(x))$, we have an equation with deviating argument. Let us introduce the following notation, definitions and assumptions.

- BVP is 'boundary value problem', $:=$ means *equal by definition*, $\not\equiv$ means *not equivalent* for measurable functions.
- $\Delta := [0,l]$.
- $[u,v]$, $\langle u,v\rangle$, (f,g) and $Q(u,v)$ are bilinear forms defined by the equalities

$$[u,v] := \int_0^l u^{(m)}v^{(m)}\,dx, \tag{1.6}$$

$$\langle u,v\rangle := [u,v] - Q(u,v), \tag{1.7}$$

$$(f,g) := \int_0^l f(x)g(x)\rho(x)\,dx, \tag{1.8}$$

$$Q(u,v) := \int_{\Delta\times\Delta} u(s)v(x)\,d\xi. \tag{1.9}$$

  In (1.9) the measure ξ is defined on $\Delta\times\Delta$ and it is symmetric (see below).

- $L_2(\Delta,\rho)$ is the space of Lebesgue quadratic integrable on Δ with positive weight $\rho(x)$ and scalar product (1.8). $L_2(\Delta) := L_2(\Delta,1)$. Assume that $\int_0^l \rho(x)\,dx < \infty$.
- $q(x,\cdot)$ is non-decreasing on Δ for almost all $x\in\Delta$, for any $s\in\Delta$ the function $q(\cdot,s)$ is measurable on Δ, $q(x) := q(x,l) - q(x,0) = q(x,\Delta)$. Assume that

$$\frac{q}{\rho} \in L_2(\Delta,\rho). \tag{1.10}$$

  $\xi(x,y) := \int_0^x q(t,y)\,dt$ is assumed to be symmetric: $\xi(y,x) = \xi(x,y)$. It defines a symmetric measure ($\xi(e\times g) = \xi(g\times e)$) on $\Delta\times\Delta$ denoted by the same letter.

- AC^k ($k\ge 0$) is the set of functions u that have absolutely continuous on $[0,l]$ derivative $u^{(k)}$, $u^{(0)} := u$.

- W is the Hilbert space (Lemma 3.5) of functions in AC^{m-1}, satisfying the conditions (1.2) and $[u,u] < \infty$, with scalar product $[u,v]$.
- $R(A)$ is the range of an operator A.
- $r(A)$ is the spectral radius of an operator A.
- $T\colon W \to L_2(\Delta,\rho)$ is the operator defined by $Tu(x) := u(x)$, $x \in \Delta$. The definition is correct and T is continuous (Lemma 3.6). T^* is the adjoint operator to T.
- $D_{\mathcal{L}_0} := \{u \in AC^{2m-1}\colon \rho^{-1}u^{(2m)} \in L_2(\Delta,\rho)\}$ is domain of $\mathcal{L}_0$. However, note that from $\rho^{-1}u^{(2m)} \in L_2(\Delta,\rho)$ it follows $u \in AC^{2m-1}$, since $\int_\Delta \rho(x)\,dx < \infty$.
- λ_0 is minimal eigenvalue of the operator $\mathcal{L}$ (λ is an eigenvalue, if $\mathcal{L}u = \lambda Tu$ for some $u \neq 0$).
- $\lambda_0(\mathcal{L}_0)$ is minimal eigenvalue of the operator $\mathcal{L}_0$.
- B is the boundary conditions operator defined on the set AC^{2m-1} by
$$B(u) := \left(u(0),\dots,u^{(m-1)}(0),u^{(m)}(l),-u^{(m+1)}(l),\dots,(-1)^{m-1}u^{(2m-1)}(l)\right).$$
- $U\alpha$ is the solution to the problem $\mathcal{L}_0 u = 0$, $B(u) = \alpha$ (it is a polynomial of the degree not higher than $2m-1$).
- $C_m \subset D_{\mathcal{L}_0}$ is the set of functions satisfying
$$u^{(k)} \geq 0 \ (k = 0,\dots,m-1), \quad (-1)^{k-m}u^{(k)} \geq 0 \ (k = m,\dots,2m-1). \tag{1.11}$$
It is easy to see that C_m is a cone*.
- G_0 is the Green operator of the problem
$$\mathcal{L}_0 u = z, \quad B(u) = \alpha \tag{1.12}$$
It means (Lemma 3.1) that the solution of this problem for any $z \in L_2(\Delta,\rho)$ has the form
$$u = G_0 z + U\alpha. \tag{1.13}$$
- G is the Green operator of the problem $\mathcal{L}u = f$, (1.2), (1.3), that is, $u = Gf$, if the problem is uniquely solvable.

1.2.2 The Green functions

The operator G_0 is integral operator (this can be verified directly)
$$G_0 z(x) = \int_0^l G_0(x,s)z(s)\rho(s)\,ds,$$
where
$$G_0(x,s) = \int_0^{\min\{x,s\}} \frac{(x-t)^{m-1}(s-t)^{m-1}}{((m-1)!)^2}\,dt, \tag{1.14}$$
moreover, the Green function is symmetric: $G_0(x,s) = G_0(s,x)$. It is easy to verify the following lemma.

*a closed convex set C of a Banach space is called *cone* if from $x \in C$, $x \neq 0$ it follows $\alpha x \in C$ for $\alpha \geq 0$ and $-x \notin C$ (see, for example [8])

Lemma 1.1. *If $z \geq 0$, $\alpha \geq 0$, the solution to the problem* (1.12) *belongs to the cone C_m.*

The Green operator G has integral representation (see for example [2])

$$u(x) = \int_0^l G(x,s)f(s)\rho(s)\,ds.$$

The Green function $G(x,s)$ is symmetric, that is $G(x,s) = G(s,x)$. For each s the section $G(\cdot,s)$ is the solution to the problem $\mathcal{L}u = 0$, (1.2), (1.3) (considering the jump of the derivative of order $2m-1$).

1.3 Results

Theorem 1.2. *The spectral problem $\mathcal{L}u = \lambda Tu$ under boundary conditions* (1.2), (1.3) *has complete and orthogonal in $L_2(\Delta,\rho)$ system of eigenfunctions: $\mathcal{L}u_k = \lambda_k Tu_k$, $k = 0,1,2,\dots$ The eigenvalues are bounded from below and have the unique density point $+\infty$, that is, $\lambda_0 \leq \lambda_1 \leq \cdots$, and $\lambda_k \to \infty$. If λ is not an eigenvalue, the BVP $\mathcal{L}u - \lambda Tu = f$ has unique solution in W for any $f \in L_2(\Delta,\rho)$.*

Define *truncated* operator $\mathcal{L}^{(\nu)}$ by

$$\mathcal{L}^{(\nu)}u := \frac{1}{\rho}\left((-1)^m u^{(2m)} - \int_\nu^l u(s)q(x,ds)\right), \quad x \in [\nu,l],$$

and boundary condition

$$u^{(k)}(\nu) = 0, \quad k = 0,\dots,m-1. \tag{1.15}$$

The following theorem is presented in the form of equivalence of several assertions. This naturally arises in similar boundary-value problems (see for example [2, 10, 14]), which are sometimes called focal [1].

Theorem 1.3. *The following affirmations are equivalent.*

1. *The quadratic functional $\langle u,u\rangle$ is positive definite in W ($\langle u,u\rangle \geq \varepsilon[u,u]$ for some $\varepsilon > 0$).*
2. *The minimal eigenvalue λ_0 of the spectral problem $\{\mathcal{L}u = \lambda Tu,\ B(u) = 0\}$ is positive.*
3. *The BVP $\{\mathcal{L}u = f,\ B(u) = 0\}$ is uniquely solvable and its solution is positive with respect to the cone C_m for any $f \geq 0$.*
4. *The Green function of the BVP $\{\mathcal{L}u = f,\ B(u) = 0\}$ is positive in the square $(0,l] \times (0,l]$, and for any $s > 0$ the section $g(x) = G(x,s)$ satisfies* (1.11).
5. *$r(QG_0) < 1$.*
6. *There exists $v \in C_m$ such that $\mathcal{L}v = \psi \geq 0$ and either $\psi \not\equiv 0$ or $B(v) \neq 0$.*
7. *For any $\nu \in [0,l)$ the truncated BVP $\mathcal{L}^{(\nu)}u = 0$,* (1.15), (1.3) *has only trivial solution.*

Consider some corollaries from this theorem. The statement about the existence of a function $v(x)$ with nonnegative $\mathcal{L}v$ is de la Vallée-Poussin like theorem [4] about differential inequality. Using concrete functions v lets to obtain effective positivity conditions of the quadratic functional. For example, letting $v(x) = (x+\varepsilon)^{m-0.5}$, we obtain the following.

Corollary 1.4. *If for some $\varepsilon > 0$*

$$\int_0^l (s+\varepsilon)^{m-0.5} d_s q(x,s) \leq \frac{((2m-1)!!)^2}{2^{2m}} (x+\varepsilon)^{-m-0.5} \tag{1.16}$$

then $\lambda_0 > 0$.

In particular case $\int_0^l u(s) d_s q(x,s) = q(x)u(x)$ we have $q(x) \leq \frac{((2m-1)!!)^2}{2^{2m}(x+\varepsilon)^{2m}}$, that for $m = 1$ coincides with well known estimate $q(x) \leq 1/(4(x+\varepsilon)^2)$.

Note that from (2.2) and (3.6) for the minimal eigenvalue $\lambda_0(\mathcal{L}_0)$ of the operator $\mathcal{L}_0$ it follows the estimate

$$\lambda_0(\mathcal{L}_0) \geq ((m-1)!)^2(2m-1) \left(\int_0^l x^{2m-1} \rho(x)\, dx \right)^{-1}.$$

If $\rho(x) \equiv 1$ then

$$\lambda_0(\mathcal{L}_0) \geq \frac{((m-1)!)^2(2m-1)2m}{l^{2m}}.$$

For $m = 1$ obtain $\lambda_0(\mathcal{L}_0) \geq 2/l^2$. The exact value is $\lambda_0(\mathcal{L}_0) = \pi^2/4l^2 \approx 2.47/l^2$.

An estimate in integral form can be obtained by Corollary 3.18.

Corollary 1.5. *Consider the case $Qu(x) = q(x)u(x)$. If*

$$\int_x^l s^{2m-3/2} \rho(s) q(s)\, ds \leq \frac{m((m-1)!)^2}{2\sqrt{x}}, \tag{1.17}$$

then $\lambda_0 > 0$.

Proof. It is easy to obtain the estimate $G_0(x,s) \leq \frac{x^{m-1}}{((m-1)!)^2} \frac{s^m}{m}$ for $x \geq s$. Let $v(x) = x^{m-1/2}$. From (1.17) it follows the integral inequality $\int_0^l G_0(x,s)\rho(s)q(s)v(s)\, ds \leq v(x)$. □

Note another important statement.

Theorem 1.6. *The first eigenfunction $u_0(x)$ is positive on $(0,l]$ and satisfies the equalities (1.11) that is it positive with respect to the cone C_m if $\lambda_0 > 0$ or $\lambda_0 \leq 0$ but has* small *absolute value.*

2 Boundary value and spectral problems. Variational method

The study we realize in abstract form referring to the properties of operators and spaces, confirmed in the relevant lemmas in the section 3. First we consider the operator $\mathcal{L}_0$ defined in (1.4), then the operator $\mathcal{L} = \mathcal{L}_0 - Q$ defined by the equality (1.1). The representation $u = G_0 f + U\alpha$ of the solution of the problem 1.12, as well as the properties of G_0 are verified directly (Lemma 3.1).

However, the application of the variational method gives more information. We use the scheme from [13]. According to the variational method, equation $\mathcal{L}_0 u = f$ will be obtained from the equation in the variational form.

We will use a separate space W for solutions that is different from $L_2(\Delta,\rho)$. This small simplification avoids the consideration of unbounded operators in the spectral theory.

2.1 The main part of the differential operator

The problem of the minimum of a quadratic functional $(1/2)[u,u]-(f,Tu)$ leads to the equation in the variational form

$$[u,v]=(f,Tv),\quad \forall v\in W. \tag{2.1}$$

The equation is considered in $u\in W$ for a given $f\in L_2(\Delta,\rho)$. The following statement is the short form of Lemma 3.4.

Lemma 2.1. *The equation* (2.1) *is equivalent to the BVP* $\{\mathcal{L}_0u=f,\ B(u)=0\}$.

Corollary 2.2. *If* $u\in D_{\mathcal{L}_0}$ *and* $B(u)=0$*, then* $[u,v]=(\mathcal{L}_0u,Tv)$.

The solution to the problem $\{\mathcal{L}_0u=f,\ B(u)=0\}$ is G_0f. On the other hand, T is bounded (Lemma 3.6), therefore (2.1) has the unique solution $u=T^*f$. So, we obtain the following corollary.

Corollary 2.3. $G_0=T^*$.

Remark 2.4. $T^*\colon L_2(\Delta,\rho)\to W$, $G_0\colon L_2(\Delta,\rho)\to D_{\mathcal{L}_0}$, but $R(T^*)=R(G_0)$.

We will keep in mind that in order to consider the spectrum it would be necessary to deal with complex spaces. The spectrum of the operator $\mathcal{L}_0$ is determined by the spectral problem $\mathcal{L}_0u=\lambda Tu$, that is, by the resolvent $(\mathcal{L}_0-\lambda T)^{-1}$. This problem is equivalent to $u=\lambda T^*Tu$. The operators T and T^* are compact (Lemma 3.8), therefore the spectrum of the operator $\mathcal{L}_0$ is discrete and real. The minimal eigenvalue of the operator $\mathcal{L}_0$ is determined by

$$\lambda_0(\mathcal{L}_0)=\inf_{u\neq 0}\frac{(\mathcal{L}_0u,Tu)}{(Tu,Tu)}=\inf_{u\neq 0}\frac{[u,u]}{(Tu,Tu)}=\|T\|^{-2}. \tag{2.2}$$

2.2 General case

2.2.1 Boundary value problem

The substitution $u=T^*z+U\alpha$ converts BVP $\{\mathcal{L}u=f,B(u)=\alpha\}$ to equation

$$z-QT^*z=QU\alpha+f \tag{2.3}$$

with compact operator QT^* (Lemmas 3.8, 3.9). If the unit is not an eigenvalue of QT^*, then $z=(I-QT^*)^{-1}f$ (I is the identity operator). The operator QT^* is positive. Therefore, if its spectral radius $r(QT^*)<1$, then $(I-QT^*)^{-1}$ is positive. The Green operator

$$G=T^*(I-QT^*)^{-1}, \tag{2.4}$$

is integral operator with symmetric kernel, has ordinary properties. By Lemma 1.1 and Corollary 2.3, G is positive in the sense that it maps the cone of non-negative functions from $L_2(\Delta,\rho)$ to the cone C_m:

$$r(QT^*)<1\Rightarrow G\geq 0\quad\text{in the sense of } C_m. \tag{2.5}$$

2.2.2 The spectral problem

The spectral problem (under condition $B(u) = 0$) is written in the form

$$\mathcal{L}u = \mathcal{L}_0 u - Qu = \lambda Tu. \tag{2.6}$$

Theorem 2.5. *The spectrum of the $\mathcal{L}$ is real and discrete.*

Proof. The substitution $u = T^*z$ leads the spectral problem (2.6) to the equation $z - QT^*z = \lambda TT^*z$. If the unit is not a point of the spectrum of QT^*, the last equation is converted to

$$z = \lambda(I - QT^*)^{-1}TT^*z.$$

If the unit is an eigenvalue of QT^*, then for a small ε

$$z = (\lambda + \varepsilon)(I - QT^* - \varepsilon TT^*)^{-1}TT^*z.$$

In both cases we can conclude that the spectrum of the problem (2.6) is discrete, since TT^* is compact.

Since $Q(u, v) = (Qu, Tv)$ (equation (3.11)),

$$\langle u, v\rangle = [u, v] - (Qu, Tv) = (\mathcal{L}u, Tv). \tag{2.7}$$

If $\mathcal{L}u = \lambda Tu$, then $\langle u, u\rangle = \lambda(Tu, Tu)$. So, λ is real. □

If $\lambda_1 \neq \lambda_2$ are two eigenvalue, the corresponding eigenvectors Tu_1 and Tu_2 are orthogonal: $(Tu_1, Tu_2) = 0$.

Since $\langle u, u\rangle = (\mathcal{L}u, Tu)$, from [3, Chapter 6]

$$\lambda_0 = \inf_{u \neq 0} \frac{\langle u, u\rangle}{(Tu, Tu)} \tag{2.8}$$

is exact lower bound of the spectrum of the operator $\mathcal{L}$.

Remark 2.6. The minimal eigenvalue λ_0 exists, because the form $\langle u, u\rangle$ is semibounded from below (Lemma 3.10).

Lemma 2.7. *Positive definiteness of the form $\langle u, u\rangle$ is equivalent to $r(QT^*) < 1$.*

Proof. The exact upper bound of the operator T^*Q is equal to exact upper bound of the spectrum

$$\sup_{u \neq 0} \frac{[T^*Qu, u]}{[u, u]} = r(T^*Q) = r(QT^*) = r.$$

So, $\langle u, u\rangle = [u, u] - (Qu, Tu) = [u, u] - [T^*Qu, u] \geq (1 - r)[u, u]$. If $r < 1$, then $\langle u, u\rangle$ is positive definite.

Conversely, if $\langle u, u\rangle \geq \varepsilon[u, u]$ for some $\varepsilon > 0$, that is, $[u, u] - [T^*Qu, u] \geq \varepsilon[u, u]$, then $r \leq 1 - \varepsilon < 1$. □

2.3 Proofs of theorems

Proof of Theorem 1.2. See Section 2.2.2. □

Proof of Theorem 1.3. Proof consists of a series of consecutive implications. First, consider the chain $6 \Rightarrow 5 \Rightarrow 3 \Rightarrow 6$.

- $6 \Rightarrow 5$. Let $v \in C_m$ satisfy the inequality $\mathcal{L}v = \psi \geq 0$. Then $v = T^*z + U\alpha$, where $\alpha = B(v)$, and either $\psi \not\equiv 0$ or $\alpha \neq 0$. From (2.3) $z - QT^*z = QU\alpha + \psi$. If $QU\alpha + \psi \equiv 0$, then $\psi \equiv 0$, and $\int_0^l U\alpha(s) d_s q(x,s) \equiv 0$. Since the polynomial $U\alpha(x) > 0$ for $x \in (0, l]$, the last identity can be valid only if $Qu(x) = q(x)u(0)$. In this case the operator QG_0 is equal zero.

  If $QU\alpha + \psi \not\equiv 0$, then $z = Qv + \psi \geq\not\equiv 0$. The inequality $r(QT^*) < 1$ it follows from Corollary 3.17.
- $5 \Rightarrow 3$. The affirmation is proved in Section 2.2.1, implication (2.5).
- $3 \Rightarrow 6$ is obvious because v can be any solution to the problem $\mathcal{L}u = f$, (1.2), (1.3) with nonzero $f \geq 0$.
- $3 \Leftrightarrow 4$, obviously.
- $1 \Leftrightarrow 2$ follows from (2.8).
- $1 \Leftrightarrow 5$ follows from Lemma 2.7.
- $1 \Leftrightarrow 7$. See Theorem 2.8. □

Theorem 2.8 (Analogue of Jacobi's theorem)**.** *The statements* 1 *and* 7 *of Theorem* 1.3 *are equivalent.*

Proof. Consider the bilinear form

$$\langle u, v \rangle_\nu := \int_\nu^l u^{(m)} v^{(m)}\, dx - \int_{[\nu,l]\times[\nu,l]} u(s)v(x)\, d\xi \tag{2.9}$$

in the space $W_\nu = \{u \in W\colon u(x) = 0 \text{ if } x \in [0,\nu]\}$. It is clear that W_ν has the same properties as W. Let $\lambda_0^{(\nu)}$ be the minimal eigenvalue of $\mathcal{L}^{(\nu)}$. Note, $\lambda_0^{(\nu)} = \inf_{u\neq 0, u\in W_\nu} \frac{\langle u,u\rangle_\nu}{(Tu,Tu)}$.

It can be shown that the function $F(\nu) := \min\{\langle u,u\rangle_\nu : u \in W_\nu, \|u\| = 1\}$ is continuous. The proof of continuity is based on estimation of the function $u(x)$ and its derivatives with relation to $[u,u]$. Note, that $F(\nu)$ does not decrease. Not also that $F(\nu) = 0$ iff $\lambda_0^{(\nu)} = 0$.

If 1 holds, then $\langle u,u\rangle_\nu > 0$ for any $\nu \in [0,l)$ and $u \in W_\nu$. If, for some $\nu > 0$, the BVP $\mathcal{L}^{(\nu)}u = 0$, (1.15), (1.3) has a nonzero solution, then $\langle u,u\rangle_\nu = 0$ (see (2.7)). This is contradiction.

Conversely, suppose 7 holds, but $\lambda_0 \leq 0$. By virtue of continuity, $F(\nu) = 0$ for some $\nu \geq 0$, therefore $\lambda_0^{(\nu)} = 0$. Then BVP $\mathcal{L}^{(\nu)}u = 0$, (1.15), (1.3) has a nonzero solution. This is contradiction. □

The proof of Theorem 1.6 relies on the statement of positivity with respect to the cone of the first eigenvector of the compact operator [9]. Let K be almost almost reproducing cone** in a Banach space E, and $A\colon E \to E$ is linear compact operator. Let A be positive with respect to K, that is $AK \subset K$. Let $r = r(A)$ be the spectral radius of A (see[8]).

**K is almost reproducing cone, if closure of its linear span is all the space E

Theorem 2.9 (M. Krein, M. Rutman [9])**.** *If the spectrum of A contains points different from zero, then its spectral radius r is eigenvalue of both the A and its adjoint A^*, this eigenvalue is simple, and it is associated with an eigenvector $v_0 \in K$: $Av_0 = rv_0$.*

Proof of Theorem 1.6. Let $\lambda_0 > 0$. From $\mathcal{L}u_0 = \lambda_0 T u_0$ it follows $z_0 = \lambda_0 TGz_0$, where $u_0 = Gz_0$. The operator TG is compact and positive with respect to the cone of nonnegative functions in $L_2(\Delta, \rho)$. Therefore its spectral radius $r(TG)$ is eigenvalue, associated with a positive eigenvector. This eigenvalue is simple and it is greater than modulo of others eigenvalues. From Theorem 1.2 it is clear, that $r(TG) = 1/\lambda_0$, and z_0 is mentioned eigenvector. The vector $u_0 = Gz_0$ is positive with respect to C_m.

In the case of $\lambda_0 \leq 0$, the equation $\mathcal{L}u = \lambda_0 u$ can be written as $\mathcal{L}_0 u + \mu T u - Qu = (\mu + \lambda_0) T u$. It is easy to show that for small positive *mu* the Green's function of the operator $\mathcal{L}_0 + \mu T$ remains positive (in the sense of the same cone (1.11)). Other statements of Theorem 1.3 remain valid for this operator. Therefore, in the case of $\mu + \lambda_0 > 0$, the eigenfunction u_0 is positive with respect to C_m. □

3 Lemmas. Properties of the space and of operators

Lemma 3.1. *Under condition $\int_0^l \rho(x)\,dx < \infty$ the problem $\mathcal{L}_0 u = f$, $B(u) = 0$ is uniquely solvable in $D_{\mathcal{L}_0}$ for any $f \in L_2(\Delta, \rho)$.*

Proof. Product $f\rho$ is integrable on Δ, because $\left(\int_0^l f\rho\,dx\right)^2 \leq \int_0^l f^2\rho\,dx \int_0^l \rho\,dx$. By sequential integration, we see that the equation $(-1)^m u^{(2m)} = \rho f$ under condition $B(u) = 0$ has unique solution in AC^{2m-1} (see definition of $D_{\mathcal{L}_0}$). □

3.1 Euler equation

The following two statements are obtained by integration by parts.

Lemma 3.2. *Let $u^{(2m-1)}$ be absolutely continuous on $[0, l]$. Then*

$$\int_0^l u^{(m)} v^{(m)}\,dx = \sum_{i=1}^{m} (-1)^{i-1} u^{(m+i-1)} v^{(m-i)} \Big|_0^l + (-1)^m \int_0^l u^{(2m)} v\,dx. \tag{3.1}$$

Lemma 3.3. *Let φ be Lebesgue integrable on $[0, l]$, and the function v has absolutely continuous derivative $v^{(m-1)}$. Then*

$$\int_0^l \varphi v\,dx = \sum_{i=0}^{m-1} (-1)^i F^{(m-1-i)} v^{(i)} \Big|_0^l + (-1)^{(m)} \int_0^l F(x) v^{(m)}\,dx, \tag{3.2}$$

where $F^{(m)} = \varphi$.

Let $f \in L_2(\Delta, \rho)$, $u \in W$ be the solution of the equation in variational form

$$\int_0^l u^{(m)} v^{(m)}\,dx = \int_0^l f v \rho\,dx \quad (\forall v \in W) \tag{3.3}$$

and $F^{(m)} = \varphi = f\rho$. From (3.3), (3.2) it follows (since $v \in W$ it satisfies (1.2))

$$\int_0^l \left(u^{(m)} - (-1)^m F\right) v^{(m)}\,dx = \sum_{i=0}^{m-1} (-1)^i F^{(m-1-i)} v^{(i)} \Big|_0^l. \tag{3.4}$$

Lemma 3.4 (Euler equation)**.** *Let $f \in L_2(\Delta, \rho)$ and $u \in W$ be solution to (3.3). Then $u \in AC^{2m-1}$ and is solution to the BVP $(-1)^m u^{(2m)} = \rho f$, (1.2), (1.3).*

Proof. The product $f\rho$ is integrable on Δ, since $\left(\int_0^l f\rho\, dx\right)^2 \le \int_0^l f^2\rho\, dx \int_0^l \rho\, dx$. In equality (3.4) we can assume that $F^{(m-1-i)}(l) = 0$, $i = 0, \dots, m-1$. Then $\int_0^l \left(u^{(m)} - (-1)^m F\right) z\, dx = 0$ for all $z = v^{(m)} \in L_2(\Delta)$. Thus, $u^{(m)} - (-1)^m F = 0$. This implies existence $u^{(2m)}$ and equality $(-1)^m u^{(2m)} = f\rho$. From (3.1) and (3.3) it follows

$$\sum_{i=1}^{m} (-1)^{i-1} u^{(m+i-1)} v^{(m-i)} \Big|_0^l = 0$$

for any $v \in W$. From here it follows (1.3). □

3.2 Space W. Boundedness and compactness of T

Lemma 3.5. *The space W with inner product $[u, v]$ is Hilbert one.*

Proof. W and $L_2(\Delta)$ are related by $y = u^{(m)}$ and

$$u(x) = \int_0^x \frac{(x-s)^{m-1}}{(m-1)!} y(s)\, ds \tag{3.5}$$

($u \in W$, $z \in L_2(\Delta)$). Moreover, these relations preserve scalar products. Therefore (3.5) is isomorphism. □

Lemma 3.6. *The operator T acts from W to $L_2(\Delta, \rho)$ and is bounded.*

Proof. Let $y = u^{(m)}$. The affirmation follows from the estimate

$$\begin{aligned}(Tu, Tu) &= \int_0^l \left(\int_0^x \frac{(x-s)^{m-1}}{(m-1)!} y(s)\, ds\right)^2 \rho(x)\, dx \\ &\le \int_0^l \rho(x) dx \int_0^x \left(\frac{(x-s)^{m-1}}{(m-1)!}\right)^2 ds \int_0^x y(s)^2\, ds \\ &\le [u, u] \int_0^l \rho(x)\, dx \int_0^x \left(\frac{(x-s)^{m-1}}{(m-1)!}\right)^2 ds. \end{aligned} \tag{3.6}$$

□

Lemma 3.7. *The range $T(W)$ is dense in $L_2(\Delta, \rho)$.*

Proof. Suppose the closure $\overline{T(W)}$ does not coincide with $L_2(\Delta, \rho)$. Then there exists $h \in L_2(\Delta, \rho)$, orthogonal to $T(W)$, that is

$$(\forall u \in W) \int_0^l h(x) u(x) \rho(x)\, dx = 0.$$

Integrating by parts obtain

$$\int_0^l hu\rho\, dx = \sum_{k=1}^{m} (-1)^{k-1} H^{(m-k)} u^{(k-1)} \Big|_0^l + (-1)^{(m)} \int_0^l H(x) u^{(m)}(x)\, dx, \tag{3.7}$$

where $H^{(m)} = h\rho$. Letting $H^{(m-k)}(l) = 0$, $k = 1, \dots, m$, obtain

$$0 = \int_0^l H(x) u^{(m)}(x)\, dx.$$

Since $u^{(m)}$ *runs through* all the space $L_2(\Delta)$, $H \equiv 0$. So $h \equiv 0$. □

Lemma 3.8. *The operator T is compact one.*

Proof. Even in the non-singular case, it is worth to use the general Gelfand's criterium of compactness scheme. Namely, in the Banach space E the set A is relatively compact if and only if for any sequence f_n of continuous linear functionals, converging to zero for any $z \in E$, convergence on the set A will be uniform.

We are interested in the set $\Omega = \{Tu\colon \|u\|_W \le 1\}$. Here $\|u\|_W = \sqrt{[u,u]}$.

Let $f_n(z) \to 0$, $\forall z \in L_2(\Delta, \rho)$. Using the substitute (3.5) obtain

$$f_n(Tu)^2 = \left(\int_0^l dx\, \rho(x) f_n(x) \int_0^x \frac{(x-s)^{m-1}}{(m-1)!} y(s)\, ds\right)^2$$
$$= \left(\int_0^l y(s)\, ds \int_s^l \frac{(x-s)^{m-1}}{(m-1)!} f_n(x)\rho(x)\, dx\right)^2 \le \int_0^l y(s)^2\, ds \int_0^l \varphi_n(s)^2\, ds,$$

where

$$\varphi_n(s) = \int_s^l \frac{(x-s)^{m-1}}{(m-1)!} f_n(x)\rho(x)\, dx.$$

Since $\int_0^l y(s)^2\, dx = [u,u] \le 1$ it is sufficient to show $\int_0^l \varphi_n(s)^2\, ds \to 0$. This ensures uniform convergence. Since $\varphi_n(s) = f_n(g_s)$, where

$$g_s(x) = \begin{cases} 0, & \text{if } x < s \\ \dfrac{(x-s)^{m-1}}{(m-1)!}, & \text{if } x \ge s, \end{cases}$$

and the sequence f_n converges on the element g_s, pointwise convergence $\varphi_n(s) \to 0$ for each $s \in \Delta$ is valid. To apply the Lebesgue theorem, we note that

$$\varphi_n(s)^2 \le \int_0^l g_s(x)^2 \rho(x)\, dx \int_0^l f_n(x)^2 \rho(x)\, dx,$$

and the first factor on the right side is a bounded function of s, and the second is a bounded sequence. □

3.3 The second part of the operator

Lemma 3.9. *The operator $Q\colon W \to L_2(\Delta, \rho)$ is bounded.*

Proof. Let $u \in W$, $y = u^{(m)}$. Since $q/\rho \in L_2(\Delta, \rho)$, the assertion follows from the inequalities

$$u(x)^2 = \left(\int_0^x \frac{(x-s)^{m-1}}{(m-1)!} y(s)\, ds\right)^2 \le \int_0^x \frac{(x-s)^{2m-2}}{(m-1)!^2}\, ds \int_0^x y(s)^2\, ds \le C^2[u,u],$$

$$|Qu(x)| \le \frac{1}{\rho(x)} \int_0^l |u(s)|\, q(x, ds) \le C\sqrt{[u,u]}\, \frac{q(x)}{\rho(x)}$$

and

$$(Qu, Qu) \le C^2[u,u] \int_\Delta \left(\frac{q(x)}{\rho(x)}\right)^2 \rho(x)\, dx.$$

□

3.4 Semi-boundedness from below and representation of the form

Consider first the special case when $\langle u,u\rangle = [u,u] - \int_0^l q(x)u(x)^2\,dx$, and $q/\rho \in L_2(\Delta,\rho)$. Let $M > 0$ and $E := \{x\colon q(x)/\rho(x) > M\}$. From relation (3.5), which can be written as $u(x) = \int_0^l H(x,s)y(s)\,ds$, it follows

$$\int_E qu^2\,dx \le (\max H)^2 \int_E q(x) \left(\int_0^l |y(s)|\,ds\right)^2 dx$$
$$\le (\max H)^2 \int_E q(x) \int_0^l |y(s)|^2\,ds \int_0^l 1\,ds\,dx = (\max H)^2 \cdot [u,u] \cdot l \cdot \int_E q(x)\,dx.$$

Choose M so that

$$(\max H)^2 l \int_E q(x)\,dx \le 1. \tag{3.8}$$

Then $\int_E qu^2\,dx \le [u,u]$ and

$$[u,u] - \int_0^l qu^2\,dx \ge -\int_{\Delta\setminus E} qu^2\,dx \ge -M\int_\Delta u^2\rho(x)\,dx = -M(Tu,Tu).$$

This confirms the semi-boundedness in the case of

$$\int_{\Delta\times\Delta} u(s)v(x)\,d\xi = \int_\Delta quv\,dx.$$

The general case is reduced to that considered with the help of

$$\int_{\Delta\times\Delta} u(s)u(x)\,d\xi \le \frac{1}{2}\int_{\Delta\times\Delta} \left(u(s)^2 + u(x)^2\right) d\xi = \int_{\Delta\times\Delta} u(x)^2\,d\xi$$
$$= \int_0^l dx \int_0^l u(x)^2 q(x,ds) = \int_0^l u(x)^2 q(x)\,dx,$$

where $q(x) = q(x,\Delta)$. So, the following lemma is proved.

Lemma 3.10. *The form $\langle u,u\rangle$ is semi-bounded from below*

$$\inf_{u\neq 0} \frac{\langle u,u\rangle}{(Tu,Tu)} \ge -M, \tag{3.9}$$

where M is defined by (3.8).

Lemma 3.11 ([5]). *Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be measurable spaces, μ be a measure on $(X,\mathcal{A})$, $K\colon X\times\mathcal{B}\to[0,\infty]$ be kernel (i.e. for μ-almost all $x\in X$ $K(x,\cdot)$ is a measure on $(Y,\mathcal{B})$, $\forall B\in\mathcal{B}$ $K(\cdot,B)$ is μ-measurable on X). Then*

1. *The function ν defined on $\mathcal{A}\times\mathcal{B}$ by the equality*

$$\nu(E) = \int_X K(x,E_x)\mu(dx), \quad E_x = \{y\colon (x,y)\in E\},$$

   is measure.

2. *if $f\colon X\times Y\to[-\infty,\infty]$ is ν-measurable on $X\times Y$, then*

$$\int_{X\times Y} f(x,y)d\nu = \int_X \left(\int_Y f(x,y)K(x,dy)\right)\mu(dx).$$

From Lemma 3.11 we obtain the following lemma.

Lemma 3.12. *Let $f(x,y)$ be ξ-measurable function, where ξ is defined in Section 1.2. Then*

$$\int_{\Delta\times\Delta} f(x,s)\,d\xi = \int_\Delta dx \int_\Delta f(x,s)q(x,ds). \tag{3.10}$$

Corollary 3.13. *From* (3.10)

$$Q(u,v) = \int_\Delta \left(\rho(x)^{-1}\int_\Delta u(s)q(x,ds)\right) v(x)\rho(x)\,dx = (Qu,Tv). \tag{3.11}$$

3.5 Lemmas for a de la Vallée-Poussin type theorem

To establish the statement about the differential inequality (as in [4]) we need the following lemma, which is close to a similar statement in [13]. Let

$$E := \{x \colon q(x,l) = q(x,0+)\}. \tag{3.12}$$

Lemma 3.14. *If $z \geq 0$, $z \not\equiv 0$, and $y = QG_0 z$, then $y(x) = 0$, if $x \in E$, and $y(x) > 0$, if $x \in \Delta \setminus E$.*

Proof. The function $u = G_0 z > 0$ on $(0,l]$ (see the kernel of the Green operator (1.14)), $u(0) = 0$. Therefore $y(x) = (\rho(x))^{-1}\int_0^l u(s)d_s q(x,s)$ satisfies the required property. □

It is known (Theorem 2.9), that the spectral radius $r = r(QG_0) = r(QT^*)$ is an eigenvalue of both the QT^* operator and the adjoint TQ^*. The eigenvectors of both operators corresponding to this value are non-negative.

Lemma 3.15. *The eigenfunction of the operator TQ^*, corresponding to the eigenvalue $r = r(TQ^*)$, is positive almost everywhere on $[0,l]$.*

Proof. Let $TQ^*\varphi = r\varphi$, $\varphi \neq 0$. Suppose $\varphi(s) = 0$ on the set $\Delta \setminus E$ (E is defined in (3.12)). By Lemma 3.14 for any z

$$(TQ^*\varphi, z) = (\varphi, QT^*z) = \int_{\Delta\setminus E} \varphi(s)QT^*z(s)\rho(s)\,ds = 0,$$

and $TQ^*\varphi \equiv 0$. This contradicts $TQ^*\varphi = r\varphi \neq 0$. Therefore $\varphi(s) \neq 0$ on the set $E_1 \subset \Delta \setminus E$ of positive measure. In this case for any $z \geq\not\equiv 0$

$$(TQ^*\varphi, z) = (\varphi, QT^*z) > 0.$$

It means that $r\varphi(s) = TQ^*\varphi(s) > 0$ almost everywhere on Δ. □

Remark 3.16. The function $\varphi(s) \in AC^m$, since T is an embedding from W to $L_2(\Delta,\rho)$.

Corollary 3.17. *Suppose there exists $z \in L_2(\Delta,\rho)$, $z \geq 0$, satisfying the inequality $z - QT^*z = \psi \geq\not\equiv 0$. Then $r(QT^*) < 1$.*

Proof. Let $r = r(QT^*)$ and $r\varphi = TQ^*\varphi$. Then $(\varphi,z) - (\varphi,QT^*z) = (\varphi,\psi) > 0$. Since $(\varphi,QT^*z) = (TQ^*\varphi,z) = r(\varphi,z)$, $0 < (\varphi,\psi) = (1-r)(\varphi,z)$. So $1-r > 0$. □

Corollary 3.18 (Theorem about integral inequality)**.** *Suppose there exists a function $v \in W$, $v(x) > 0$ on $(0,l]$, such that $v - G_0 Qv = g$, $Qg \geq\not\equiv 0$. Then $r(QG_0) < 1$.*

Proof. Let $z = Qv$. Then $z - QG_0 z = Qg$. The assertion follows from Corollary 3.17. □

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