Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **60**, 1–14; https://doi.org/10.14232/ejqtde.2019.1.60 www.math.u-szeged.hu/ejqtde/

Variable exponent perturbation of a parabolic equation with $p(x)$-Laplacian

Aldo T. Lourêdo[1], **Manuel Milla Miranda**[1] and **Marcondes R. Clark**[✉2]

[1]Universidade Estadual da Paraíba, R. Baraúnas, 351 - C. Grande - PB, CEP 58429-500, Brasil
[2]Universidade Federal do Piauí, B. Ininga - Teresina - PI, CEP 64049-550, Brasil

Received 26 March 2019, appeared 14 August 2019
Communicated by Vilmos Komornik

Abstract. This paper is concerned with the study of the global existence and the decay of solutions of an evolution problem driven by an anisotropic operator and a nonlinear perturbation, both of them having a variable exponent. Because the nonlinear perturbation leads to difficulties in obtaining a priori estimates in the energy method, we had to significantly modify the Tartar method. As a result, we could prove the existence of global solutions at least for small initial data. The decay of the energy is derived by using a differential inequality and applying a non-standard approach.

Keywords: variable exponent, energy not defined, existence of solutions.

2010 Mathematics Subject Classification: 35K92, 35K58, 35K55, 35L60, 35L70.

1 Introduction

Let Ω be an open bounded set of $\mathbb{R}^n$ with boundary Γ of class C^2. Consider $p, \sigma \in L^\infty(\Omega)$. The objective of this paper is to analyze the global existence and the decay of solutions of the following parabolic problem:

$$u' - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p(x)-2} \frac{\partial u}{\partial x_i} \right) + |u|^{\sigma(x)} = 0 \qquad \text{in } \Omega \times (0,\infty), \tag{1.1a}$$

$$u = 0 \qquad \text{on } \Gamma \times (0,\infty) \tag{1.1b}$$

$$u(x,0) = u^0(x) \qquad \text{in } \Omega. \tag{1.1c}$$

The $p(x)$-Laplacian operator $\mathcal{A}$ given by $\mathcal{A}u = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(\big|\frac{\partial u}{\partial x_i}\big|^{p(x)-2} \frac{\partial u}{\partial x_i}\big)$, arises in some physical problems. For example, in the theory of elasticity and in mechanics of fluids, more precisely, in fluids of electrorheological type (see [8,19,20]), whose equation of motion is given by

$$u' + \operatorname{div} S(u) + (u \cdot \nabla u) + \nabla \pi = f,$$

✉Corresponding author. Email: aldolouredo@gmail.com

where $u : \mathbb{R}^{3+1} \to \mathbb{R}^3$ is the velocity of the fluid at a point in space-time , $\nabla = (\partial_1, \partial_2, \partial_3)$ the gradient operator, $\pi : \mathbb{R}^{3+1} \to \mathbb{R}$ the pressure, $f : \mathbb{R}^{3+1} \to \mathbb{R}^3$ represents external forces and S is the stress tensor $S : W^{1,1}_{\text{loc}} \to \mathbb{R}^{3\times 3}$. This operator has the form

$$S(u)(x) = \mu(x)(1 + |D(u(x))|^{\frac{p(x)-2}{2}})D(u(x))$$

where $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the symmetric part of the gradient of u. Note that if $p(x) \equiv 2$, then this equation reduces to the usual Navier–Stokes equation.

To obtain the existence of global solutions of (1.1a)–(1.1c) we cannot apply the energy method because the term $\int_\Omega |u(x)|^{\sigma(x)} u(x) dx$ does not have a definite sign. To overcome this difficulty we apply a new method which has its motivation in the work of Tartar [24] (see also [17]). With this approach and results on monotone operators (see [6,7,25]) we succeed in obtaining a global solution of (1.1a) with small initial data. This is the main contribution of the paper. The decay of the energy is derived by using differential inequalities and applying a new approach.

Problem (1.1a) is an example of an evolution problem driven by an anisotropic operator with variable exponents and a nonlinear perturbation, which has also a variable exponent. Recent contributions to the study of anisotropic problems can be found, for instance, in [14,20] and the references contained therein. Parabolic problems with variable exponents can be seen in [3,4,10–12,18]. In [2], Antontsev analized the wave equation with $p(x,t)$-Laplacian. In [5], those authors considered the energy decay for a class of plate equations with memory and a lower order perturbation of p-Laplacian type. We can find elliptic problems with operators having variable exponents in [1,22] and the references contained therein. Because the energy method works very well, the proof of the existence of a solution in those papers is based on the Galerkin method.

The paper is organized as following. In Section 2, we introduce notation and state the results in form of theorems, whose proofs are given in Section 3.

2 Notations and main results

The scalar product and norm of $L^2(\Omega)$ are denoted by (u, v) and $|u|$, respectively. Consider a function $q \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} q(x) = q^- \geq 1$. The space

$$L^{q(x)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{q(x)} dx < \infty \right\},$$

equipped with the Luxemburg's norm

$$\|u\|_{L^{q(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\}$$

is a Banach space. With the notation $q^+ = \operatorname{ess\,inf}_{x\in\Omega} q(x)$ and the fact $1 \leq q^- \leq q(x) \leq q^+$ *a.e.* in $x \in \Omega$, we have

$$\|u\|^{q^+}_{L^{q(\cdot)}(\Omega)} \leq \int_\Omega |u(x)|^{q(x)} dx \leq \|u\|^{q^-}_{L^{q(\cdot)}(\Omega)} \quad \text{if } \|u\|_{L^{q(\cdot)}(\Omega)} \leq 1; \tag{2.1a}$$

$$\|u\|^{q^-}_{L^{q(\cdot)}(\Omega)} \leq \int_\Omega |u(x)|^{q(x)} dx \leq \|u\|^{q^+}_{L^{q(\cdot)}(\Omega)} \quad \text{if } \|u\|_{L^{q(\cdot)}(\Omega)} > 1. \tag{2.1b}$$

Assume that

$$p \in C(\overline{\Omega}), \quad p \text{ is Lipschitzian and } \quad p(x) \geq 2, \quad \forall x \in \overline{\Omega}; \tag{2.2a}$$
$$\sigma \in C(\overline{\Omega}), \quad \sigma(x) > 1, \quad \forall x \in \overline{\Omega}. \tag{2.2b}$$

Introduce the notations

$$p^- = \min_{x\in\overline{\Omega}} p(x), \qquad p^+ = \max_{x\in\overline{\Omega}} p(x), \qquad \sigma^- = \min_{x\in\overline{\Omega}} \sigma(x), \qquad \sigma^+ = \max_{x\in\overline{\Omega}} \sigma(x).$$

Thus

$$2 \leq p^- \leq p(x) \leq p^+ \quad \text{and} \quad 1 < \sigma^- \leq \sigma(x) \leq \sigma^+. \tag{2.3}$$

The space

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p(x)}(\Omega), i = 1, 2, \ldots, n \right\},$$

provided with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}, \qquad u \in W^{1,p(\cdot)}(\Omega).$$

is a reflexive Banach space. The closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ is denoted by $W_0^{1,p(x)}(\Omega)$. This reflexive Banach space is equipped with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p(\cdot)}(\Omega)}.$$

The dual space of $W^{1,p(x)}(\Omega)$ is denoted by $W^{-1,p'(x)}(\Omega)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \qquad \forall x \in \overline{\Omega}.$$

Let denote by $\mathcal{A}$ the operator

$$\mathcal{A} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$$

defined by

$$\langle \mathcal{A}u, v\rangle = \sum_{i=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p(x)-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx.$$

It is known that $\mathcal{A}$ is monotone and hemicontinuous (see Diening [9]).

Proposition 2.1. *The operator $\mathcal{A}$ takes bounded subsets of $W_0^{1,p(x)}(\Omega)$ into bounded subsets of $W_0^{-1,p'(x)}(\Omega)$.*

Proof. In fact, it holds that

$$|\langle \mathcal{A}u, v\rangle| \leq \sum_{i=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p(x)-1} \left| \frac{\partial v}{\partial x_i} \right| dx.$$

In order to facilitate the notations, we denote the space $W_0^{1,p(x)}(\Omega)$ by X.

Note that $\left|\frac{\partial u}{\partial x_i}\right| \in L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)$ since $\frac{\partial u}{\partial x_i} \in L^{p(\cdot)}(\Omega)$. So by the Hölder inequality for the spaces $L^{p(\cdot)}(\Omega)$ (cf. [13, p. 341]), we obtain the estimate

$$\begin{aligned}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p(x)-1} \left|\frac{\partial v}{\partial x_i}\right| dx &\leq 2 \left\| \left|\frac{\partial u}{\partial x_i}\right|^{p(x)-1} \right\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)} \left\|\frac{\partial v}{\partial x_i}\right\|_{L^{p(\cdot)}(\Omega)} \\ &\leq 2 \left\| \left|\frac{\partial u}{\partial x_i}\right|^{p(x)-1} \right\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)} \|v\|_X.\end{aligned}$$

Therefore, from the above two inequalities we have

$$|\langle \mathcal{A}u, v\rangle| \leq 2 \left(\sum_{i=1}^n \left\| \left|\frac{\partial u}{\partial x_i}\right|^{p(x)-1} \right\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)} \right) \|v\|_X. \tag{2.4}$$

Let us define α and β as follows:

$$\left[\frac{p(x)}{p(x)-1}\right]^- = \alpha, \qquad \left[\frac{p(x)}{p(x)-1}\right]^+ = \beta.$$

If $l_i = \left\| \left|\frac{\partial u}{\partial x_i}\right|^{p(x)-1} \right\|_{L^{\frac{p(\cdot)}{p(\cdot)-1}}(\Omega)} \leq 1$, by (2.1a), we obtain

$$\begin{aligned} l_i^\beta &\leq \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p(x)} dx \leq \left\|\frac{\partial u}{\partial x_i}\right\|^{p^-}_{L^{p(\cdot)}(\Omega)} + \left\|\frac{\partial u}{\partial x_i}\right\|^{p^+}_{L^{p(\cdot)}(\Omega)} \\ &\leq \|u\|_X^{p^-} + \|u\|_X^{p^+}.\end{aligned}$$

Thus,

$$l_i \leq \left(\|u\|_X^{p^-} + \|u\|_X^{p^+}\right)^{\frac{1}{\beta}}.$$

In similar way, if $l_i > 1$ we find

$$l_i \leq \left(\|u\|_X^{p^-} + \|u\|_X^{p^+}\right)^{\frac{1}{\alpha}}.$$

These last two inequalities imply

$$\sum_{i=1}^n l_i \leq n \left(\|u\|_X^{p^-} + \|u\|_X^{p^+}\right)^{\frac{1}{\beta}} + n \left(\|u\|_X^{p^-} + \|u\|_X^{p^+}\right)^{\frac{1}{\alpha}}.$$

Now, this inequality and (2.4) provide

$$\|\mathcal{A}u\|_{W^{-1,p'(x)}(\Omega)} \leq 2n \left(\|u\|_X^{p^-} + \|u\|_X^{p^+}\right)^{\frac{1}{\beta}} + 2n \left(\|u\|_X^{p^-} + \|u\|_X^{p^+}\right)^{\frac{1}{\alpha}}$$

which proves the proposition. □

We also assume that

$$(p^+ - p^-)n < p^+ p^-, \tag{2.5a}$$

$$p^+ < \sigma^- + 1 \leq \sigma(x) + 1 \leq \sigma^+ + 1 < \frac{np(x)}{n - p(x)}, \qquad \forall x \in \overline{\Omega} \tag{2.5b}$$

if $p(x) < n$, for all $x \in \overline{\Omega}$; and that

$$\sigma \text{ satisfies hypothesis (2.2b)} \tag{2.6}$$

if $p(x) \geq n$ for all $x \in \overline{\Omega}$.

Note that by (2.5b) we have

$$p^+ < \frac{np(x)}{n - p(x)}, \qquad \forall x \in \overline{\Omega}.$$

Under the hypotheses (2.2a), (2.5a) and (2.5b), we obtain

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\sigma^+ +1}(\Omega) \hookrightarrow L^{\sigma(x)+1}(\Omega) \hookrightarrow L^{\sigma^- +1}(\Omega) \hookrightarrow L^2(\Omega) \tag{2.7}$$

where $\hookrightarrow$ denotes continuous embedding. Note that

$$\text{the embeddding of } W_0^{1,p(x)}(\Omega) \text{ in } L^{\sigma^+ +1}(\Omega) \text{ is compact.} \tag{2.8}$$

See Diening et al. [9], Fan and Zhao [13], Rădulescu et al. [20] and Kováčik and Rákosník [23] for detailed proofs of all these results on spaces with variable exponents that we have used in the present paper.

By (2.7) there exists a positive constant K such that

$$\|u\|_{L^{\sigma(\cdot)+1}(\Omega)} \leq K\|u\|_{W_0^{1,p(\cdot)}(\Omega)}, \qquad \forall u \in W_0^{1,p(\cdot)}(\Omega). \tag{2.9}$$

Consider positive constants a_0, a_1, b_0 and b_1 satisfying

$$a_0 \left(\sum_{i=1}^n |\xi_i|^{p^+} \right)^{\frac{1}{p^+}} \leq \sum_{i=1}^n |\xi_i| \leq a_1 \left(\sum_{i=1}^n |\xi_i|^{p^+} \right)^{\frac{1}{p^+}}, \qquad \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n; \tag{2.10a}$$

$$b_0 \left(\sum_{i=1}^n |\xi_i|^{p^-} \right)^{\frac{1}{p^-}} \leq \sum_{i=1}^n |\xi_i| \leq b_1 \left(\sum_{i=1}^n |\xi_i|^{p^-} \right)^{\frac{1}{p^-}}, \qquad \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \tag{2.10b}$$

Further, set the notations

$$d = \frac{1}{2p^+ a_1^{p^+}}, \qquad M = K^{\sigma^- +1} + K^{\sigma^+ +1} \tag{2.11a}$$

$$\lambda_0 = \min \left\{ 1, \left[\frac{dp^+}{M(\sigma^- + 1)} \right]^{\frac{1}{\sigma^- +1-p^+}} \right\}. \tag{2.11b}$$

Under the above considerations we have the following result.

Theorem 2.2. *Assume that hypotheses (2.2a), (2.5a) and (2.6) hold. If $u^0 \in W_0^{1,p(\cdot)}(\Omega)$ satisfies*

$$\|u^0\|_{W_0^{1,p(\cdot)}(\Omega)} < \lambda_0, \tag{2.12}$$

and

$$\left(\frac{1}{b_0^{p^-}} + \frac{1}{a_0^{p^+}} \right) \|u^0\|^{p^-}_{W_0^{1,p(\cdot)}(\Omega)} + M\|u^0\|^{\sigma^- +1}_{W_0^{1,p(\cdot)}(\Omega)} < d\lambda_0^{p^+}. \tag{2.13}$$

Then there exists a function $u \in L^\infty(0, \infty; W_0^{1,p(\cdot)}(\Omega))$, with $u' \in L^2(0, \infty; L^2(\Omega))$ that satisfies

$$u' - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p(\cdot)-2} \frac{\partial u}{\partial x_i} \right) + |u|^{\sigma(\cdot)} = 0 \quad \textit{in } L^2_{\text{loc}}(0, \infty; W^{-1,p'(\cdot)}(\Omega)), \tag{2.14}$$

$$u(0) = u^0 \quad \textit{in } \Omega. \tag{2.15}$$

Remark 2.3. We note that in the particular case $p(x) = c$, $c \geq 2$, hypothesis (2.5a) is always holds. Thus by applying the same method used to prove Theorem 2.2, we obtain global solutions to Problem (1.1a)–(1.1c) under only the hypotheses (2.5a) and

$$\begin{aligned} &\sigma^+ + 1 < \frac{np}{n-p} && \text{if } p < n \\ &\text{no restriction on } \sigma && \text{if } p \geq n. \end{aligned}$$

In order to state the result of the decay of solutions, we introduce some notations.
By (2.7), there exists a constant $L > 0$ such that

$$|v| \leq L\|v\|_{W_0^{1,p(\cdot)}(\Omega)}, \qquad \forall v \in W_0^{1,p(\cdot)}(\Omega), \tag{2.16}$$

where $|\cdot| = |\cdot|_{L^2(\Omega)}$. Set the notation

$$\eta = \frac{1}{a_1^{p^+} L^{p^+}}.$$

Let u be the solution given by Theorem 2.2. Define the energy $E(t)$ by

$$E(t) = |u(t)|^2, \qquad \forall t \geq 0. \tag{2.17}$$

By $u \in L^\infty(0,\infty; W_0^{1,p(\cdot)}(\Omega))$ and $u' \in L^2(0,\infty; L^2(\Omega))$, we have that $E \in C([0,\infty); L^2(\Omega))$.

Theorem 2.4. *Let u be the solution given by Theorem 2.2. Then*

(i) *if $p^+ = 2$, that is, $p(x) = 2$, $\forall x \in \overline{\Omega}$, we have*

$$E(t) \leq E(0)e^{-\eta t}, \qquad \forall t \geq 0; \tag{2.18}$$

(ii) *if $p^+ > 2$, we set $\frac{p^+}{2} = 1 + \gamma$, $\gamma > 0$. In this case we have*

$$E(t) \leq E(0)(1 + E(0)^\gamma \eta \gamma t)^{-\frac{1}{\gamma}}, \qquad \forall t \geq 0. \tag{2.19}$$

3 Proof of the results

Proof of Theorem 2.2. Consider a Schauder basis $\{w_1, w_2, w_3, \ldots\}$ of $W_0^{1,p(x)}(\Omega)$. Let u_m be an approximate solution of Problem (1.1a)–(1.1c), more precisely,

$$u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x),$$

$$\begin{aligned} &(u_m'(t), v) + \sum_{i=1}^{n} \int_\Omega \left| \frac{\partial u_m(t)}{\partial x_i} \right|^{p(\cdot)-2} \frac{\partial u_m(t)}{\partial x_i} \frac{\partial v}{\partial x_i} dx \\ &\qquad + \int_\Omega |u_m(t)|^{\sigma(\cdot)} v dx = 0, \quad \text{for all } v \in V_m = [w_1, \ldots, w_m]; \\ &u_m(0) = u_m^0,\ u_m^0 \in V_m,\ u_m^0 \to u^0 \text{ in } W_0^{1,p(x)}(\Omega). \end{aligned} \tag{3.1}$$

We denote by $[0, t_m)$ the maximal interval of existence of the solution u_m.

By (2.12) and (2.13), we obtain

$$\|u_m^0\|_X < \lambda_0, \ \forall m \geq m_0; \tag{3.2}$$

$$\left(\frac{1}{b_0^{p^-}} + \frac{1}{a_0^{p^+}}\right) \|u_m^0\|_X^{p^-} + M\|u_m^0\|_X^{\sigma^-+1} < d\lambda_0^{p^+}, \ \forall m \geq m_0. \tag{3.3}$$

Fixing m such that $m \geq m_0$, we have the following estimate:

Lemma 3.1. *We have $\|u_m(t)\|_X < \lambda_0$, $\forall t \in [0, \infty)$.*

Proof. We argue by contradiction. Assume that there exists $t_1 \in (0, t_m)$ such that

$$\|u_m(t_1)\|_X \geq \lambda_0.$$

Consider the set

$$\mathcal{O} = \{\tau \in (0, t_m) : \|u_m(\tau)\|_X \geq \lambda_0\}$$

and

$$\inf_{\tau \in \mathcal{O}} \tau = t^*.$$

We have

$$\|u_m(t^*)\|_X = \lambda_0 \quad \text{and} \quad t^* > 0.$$

In fact, the function $\beta(t) = \|u_m(t)\|_X$ is continuous on $[0, t_m)$ then $\|u_m(t^*)\|_X \geq \lambda_0$. If $\|u_m(t^*)\|_X > \lambda_0$, the Intermediate Value Theorem and noting that $\|u_m(0)\|_X < \lambda_0$, imply that t^* is not the infimum on $\mathcal{O}$, which is a contradiction. Thus $\|u_m(t^*)\|_X = \lambda_0$. Also $t^* > 0$ because $\|u_m(0)\|_X < \lambda_0$. Note that

$$\|u_m(t)\|_X < \lambda_0, \quad \forall t \in [0, t^*).$$

Consider $t \in [0, t^*)$ and $v = u_m'$ in (3.1), we obtain

$$|u_m'(t)|^2 + \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)-2} \frac{\partial u_m(t)}{\partial x_i} \frac{\partial u_m'(t)}{\partial x_i} dx + \int_\Omega |u_m(t)|^{\sigma(x)} u_m'(t) dx = 0.$$

It follows

$$|u_m'(t)|^2 + \sum_{i=1}^{n} \frac{d}{dt} \int_\Omega \frac{1}{p(x)} \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx + \frac{d}{dt} \int_\Omega \frac{1}{\sigma(x)+1} |u_m(t)|^{\sigma(x)} u_m(t) \ dx = 0.$$

Integrating on $[0, t]$, we find

$$\begin{aligned}\int_0^t |u_m'(s)|^2 ds &+ \sum_{i=1}^{n} \int_\Omega \frac{1}{p(x)} \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx + \int_\Omega \frac{1}{\sigma(x)+1} |u_m(t)|^{\sigma(x)} u_m(t) dx \\ &= \sum_{i=1}^{n} \int_\Omega \frac{1}{p(x)} \left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)} dx + \int_\Omega \frac{1}{\sigma(x)+1} |u_m^0|^{\sigma(x)} u_m^0 dx.\end{aligned}$$

By (2.3), we get

$$\begin{aligned}\int_0^t |u_m'(s)|^2 ds &+ \sum_{i=1}^{n} \int_\Omega \frac{1}{p^+} \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx + \int_\Omega \frac{1}{\sigma^+ + 1} |u_m(t)|^{\sigma(x)} u_m(t) dx \\ &\leq \sum_{i=1}^{n} \int_\Omega \frac{1}{p^-} \left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)} dx + \int_\Omega \frac{1}{\sigma^- + 1} |u_m^0|^{\sigma(x)+1} dx.\end{aligned} \tag{3.4}$$

As $\lambda_0 \leq 1$ and $t \in [0, t^*)$, we have $\|\frac{\partial u_m(t)}{\partial x_i}\|_{L^{p(\cdot)}(\Omega)} < \lambda_0 \leq 1$, $i = 1, 2, \ldots, n$. Therefore it follows from (2.1a) that

$$\frac{1}{p^+}\sum_{i=1}^{n}\left\|\frac{\partial u_m(t)}{\partial x_i}\right\|_{L^{p(\cdot)}(\Omega)}^{p^+} \leq \frac{1}{p^+}\sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx.$$

By (2.10a) we obtain

$$\left(\sum_{i=1}^{n}\left\|\frac{\partial u_m(t)}{\partial x_i}\right\|_{L^{p(\cdot)}(\Omega)}\right)^{p^+} \leq a_1^{p^+}\sum_{i=1}^{n}\left\|\frac{\partial u_m(t)}{\partial x_i}\right\|_{L^{p(\cdot)}(\Omega)}^{p^+}.$$

These last two inequalities furnish

$$\frac{1}{p^+ a_1^{p^+}}\|u_m(t)\|_{W_0^{1,p(\cdot)}(\Omega)}^{p^+} \leq \frac{1}{p^+}\sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u_m(t)}{\partial x_i}\right|^{p(x)} dx. \tag{3.5}$$

We modify the third term of (3.4). From the inequalities (2.9) and (2.1a) and noting that $\|u_m(t)\|_X < 1$, because $t \in [0, t^*)$ it follows that

$$\begin{aligned}\left|\int_\Omega |u_m(t)|^{\sigma(x)} u_m(t) dx\right| &\leq \int_\Omega |u_m(t)|^{\sigma(x)+1} dx \leq \|u_m(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^-+1} + \|u_m(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^++1} \\ &\leq K^{\sigma^-+1}\|u_m(t)\|_X^{\sigma^-+1} + K^{\sigma^++1}\|u_m(t)\|_X^{\sigma^-+1}.\end{aligned}$$

Thus, noting that $\frac{1}{\sigma^++1} < 1$,

$$\left|\int_\Omega \frac{1}{\sigma^++1}|u_m(t)|^{\sigma(x)} u_m(t) dx\right| \leq M\|u_m(t)\|_X^{\sigma^-+1} \tag{3.6}$$

where M was defined in (2.11a).

We modify the last two terms of (3.4). Note that $\frac{1}{p^-} \leq 1$, $\frac{1}{\sigma^-+1} \leq 1$. By (2.1a), (2.10a) and observing that $\|u_m^0\|_X < 1$, we obtain

$$\begin{aligned}\sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)} dx &\leq \sum_{i=1}^{n}\left\|\frac{\partial u_m^0}{\partial x_i}\right\|_{L^{p(\cdot)}(\Omega)}^{p^-} + \sum_{i=1}^{n}\left\|\frac{\partial u_m^0}{\partial x_i}\right\|_{L^{p(\cdot)}(\Omega)}^{p^+} \\ &\leq \frac{1}{b_0^{p^-}}\|u_m^0\|_X^{p^-} + \frac{1}{a_0^{p^+}}\|u_m^0\|_X^{p^+}.\end{aligned}$$

Thus

$$\sum_{i=1}^{n}\int_\Omega \frac{1}{p^-}\left|\frac{\partial u_m^0}{\partial x_i}\right|^{p(x)} dx \leq \left(\frac{1}{b_0^{p^-}} + \frac{1}{a_0^{p^+}}\right)\|u_m^0\|_X^{p^-}. \tag{3.7}$$

In a similar way, we find

$$\begin{aligned}\int_\Omega |u_m^0|^{\sigma(x)+1} dx &\leq \|u_m^0\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^-+1} + \|u_m^0\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^++1} \\ &\leq K^{\sigma^-+1}\|u_m^0\|_X^{\sigma^-+1} + K^{\sigma^++1}\|u_m^0\|_X^{\sigma^++1}.\end{aligned}$$

That is

$$\frac{1}{\sigma^-+1}\int_\Omega |u_m^0|^{\sigma(x)+1} dx \leq M\|u_m^0\|_X^{\sigma^-+1}. \tag{3.8}$$

Plugging (3.5)–(3.8) into (3.4), we obtain

$$\int_0^t |u_m'(s)|^2 ds + d\|u_m(t)\|_X^{p^+} + d\|u_m(t)\|_X^{p^+} - M\|u_m(t)\|_X^{\sigma^-+1} \leq \left(\frac{1}{b_0^{p^-}} + \frac{1}{a_0^{p^+}}\right)\|u_m^0\|_X^{p^-} + M\|u_m^0\|_X^{\sigma^-+1} = I(u_m^0) \tag{3.9}$$

where d, M, a_0 and b_0 were defined, respectively, in (2.11a), (2.10a) and (2.10b).

We now compare the third and four term of the last expression. Consider the function

$$\theta(\lambda) = d\lambda^{p^+} - M\lambda^{\sigma^-+1}, \qquad \lambda \geq 0.$$

By hypothesis (2.5a) we have that $\sigma^- + 1 - p^+ > 0$. We find that if

$$0 \leq \lambda \leq \left(\frac{dp^+}{M(\sigma^-+1)}\right)^{\frac{1}{\sigma^-+1-p^+}} = P$$

then

$$\theta(\lambda) \geq 0.$$

In particular if

$$0 \leq \lambda \leq \min\{1, P\} = \lambda_0$$

(λ_0 defined in (2.11a)), we have

$$\theta(\lambda) \geq 0.$$

As $\|u_m(t)\|_X < \lambda_0$, for all $t \in [0, t^*)$, we deduce that

$$\theta(\|u_m(t)\|_X) = d\|u_m(t)\|_X^{p^+} - M\|u_m(t)\|_X^{\sigma^-+1} \geq 0, \qquad t \in [0, t^*). \tag{3.10}$$

Thus by (3.9) we get

$$\int_0^t |u_m'(s)|^2 ds + d\|u_m(t)\|_X^{p^+} \leq I(u_m^0), \qquad t \in [0, t^*).$$

By (3.3) and (3.4), we obtain

$$I(u_m^0) < d\lambda_0^{p^+}.$$

Therefore,

$$d\|u_m(t)\|_X^{p^+} < I(u_m^0) < r < d\lambda_0^{p^+}, \quad \text{for some } r \in \mathbb{R}.$$

Taking the limit $t \to t^*$, $t < t^*$, in the above inequality, we obtain

$$d\|u_m(t^*)\|_X^{p^+} \leq r < d\lambda_0^{p^+}$$

which is a contradiction because $\|u_m(t^*)\|_X = \lambda_0$. Thus the lemma is proved. □

Returning to the Proof of Theorem 2.2

By Lemma 3.1, (3.9) and properties of operator $\mathcal{A}$, we obtain that there exists a subsequence of (u_m), still denoted by (u_m), and a function u such that

$$u_m \to u \text{ weak star in } L^\infty(0, \infty; W_0^{1,p(\cdot)}(\Omega)); \tag{3.11a}$$

$$u_m' \to u' \text{ weak in } L^2(0, \infty; L^2(\Omega)); \tag{3.11b}$$

$$\mathcal{A}(u_m) \to \chi \text{ weak star in } L^\infty(0, \infty; W^{-1,p'(\cdot)}(\Omega)). \tag{3.11c}$$

The next step is to prove that $\chi = \mathcal{A}u$ and for that we need to show that

$$\int_0^T \int_\Omega |u_m|^{\sigma(x)} u_m dxdt \to \int_0^T \int_\Omega |u|^{\sigma(x)} u dxdt \tag{3.12}$$

for any $T > 0$. Introduce the notations:

$$\begin{aligned} Q_T &= \Omega \times (0,T); \\ F_m &= \{(x,t) \in Q_T; |u_m(x,t)| \leq 1\}; \\ G_m &= \{(x,t) \in Q_T; |u_m(x,t)| > 1\}. \end{aligned}$$

By compactness (2.8) (see [16] or Corollary 6 in [21]) and convergences (3.11a) and (3.11b), we find

$$u_m \to u \text{ in } C([0,T]; L^{\sigma^+ +1}(\Omega))$$

therefore,

$$u_m \to u \text{ in } L^{\sigma^+ +1}(Q_T) \tag{3.13}$$

and

$$u_m(x,t) \to u(x,t) \text{ a.e. in } Q_T.$$

Hence

$$|u_m(x,t)|^{\sigma(x)} \to |u(x,t)|^{\sigma(x)} \text{ a.e. in } Q_T. \tag{3.14}$$

By (3.13), we have

$$\begin{aligned} \int_{Q_T} [|u_m(x,t)|^{\sigma(x)}]^{\frac{\sigma^+ +1}{\sigma^+}} dxdt &= \int_{F_m} [|u_m(x,t)|^{\sigma(x)}]^{\frac{\sigma^+ +1}{\sigma^+}} dxdt + \int_{G_m} [|u_m(x,t)|^{\sigma(x)}]^{\frac{\sigma^+ +1}{\sigma^+}} dxdt \\ &\leq T(\operatorname{meas}\Omega) + \int_{Q_T} |u_m(x,t)|^{\sigma^+ +1} dxdt \leq C, \qquad \forall m \in \mathbb{N}, \end{aligned}$$

that is,

$$\int_{Q_T} [|u_m(x,t)|^{\sigma(x)}]^{\frac{\sigma^+ +1}{\sigma^+}} dxdt \leq C, \qquad \forall m \in \mathbb{N}. \tag{3.15}$$

From (3.14), (3.15) and Lions' Lemma (see [15] or [16]) it follows that

$$|u_m|^{\sigma(x)} \to |u|^{\sigma(x)} \quad \text{weak in } L^{\frac{\sigma^+ +1}{\sigma^+}}(Q_T). \tag{3.16}$$

This result and convergence (3.13) imply convergence (3.12).

By the theory of monotone operators and the convergences (3.12) and (3.16), we deduce (see Lions [15])

$$\chi = \mathcal{A}u. \tag{3.17}$$

Also by applying the diagonalization process to the sequence of (u_m), we find from (3.16)

$$|u_m|^{\sigma(x)} \to |u|^{\sigma(x)} \quad \text{weak in } L^{\frac{\sigma^+ +1}{\sigma^+}}(Q_T), \qquad \forall T > 0. \tag{3.18}$$

Convergences (3.11a), (3.18) and equality (3.17) allows us to pass to the limit in the approximate equation (3.1) and so it holds that

$$\int_0^\infty (u', \varphi) dt + \int_0^\infty \langle \mathcal{A}u, \varphi \rangle dt + \int_0^\infty \int_\Omega |u|^\sigma \varphi dxdt = 0$$
$$\forall \varphi \in L^2_{\text{loc}}(0,\infty; W_0^{1,p(x)}(\cdot)), \text{ supp } \varphi \text{ compact in } (0,\infty).$$

Taking $\varphi \in C_0^\infty(\Omega \times (0,T))$ in the last equality, we find equation (2.14). The initial condition (2.15) follows by convergences (3.11a) and (3.11b). This concludes the proof of Theorem 2.2. $\square$

Proof of Theorem 2.4. Multiply both sides of (2.14) by u and integrate on Ω. We obtain

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p(x)} dx + \int_\Omega |u|^{\sigma(x)} u dx = 0. \tag{3.19}$$

By Lemma 3.1 we have $\|u(t)\|_X \le \lambda_0 < 1$ then $\|\frac{\partial u}{\partial x_i}\|_X < 1$, $i = 1, 2, \ldots, n$. Therefore, from (2.1a) it follows that

$$\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(\cdot)}(\Omega)}^{p^+} \le \sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p(x)} dx.$$

On the other side, by (2.10a) we obtain

$$\|u(t)\|_X^{p^+} = \left(\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(\cdot)}(\Omega)}\right)^{p^+} \le a_1^{p^+}\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p(\cdot)}(\Omega)}^{p^+}.$$

These two preceding inequalities furnish

$$\frac{1}{a_1^{p^+}}\|u(t)\|_X^{p^+} \le \sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p(x)} dx. \tag{3.20}$$

Also by (2.1a) and (2.1b), we obtain

$$\left|\int_\Omega |u(t)|^{\sigma(x)} u(t) dx\right| \le \|u(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^-+1} + \|u(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^++1}$$

and by (2.9),

$$\|u(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^-+1} + \|u(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^++1} \le K^{\sigma^-+1}\|u(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^-+1} + K^{\sigma^++1}\|u(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^++1}.$$

As $\|u(t\|_X \le 1$, we find

$$\|u(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^-+1} + \|u(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^++1} \le K^{\sigma^-+1}\|u(t)\|_X^{\sigma^-+1} + K^{\sigma^++1}\|u(t)\|_X^{\sigma^++1}.$$

As $\|u(t)\|_X \le 1$, we find

$$\|u(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^-+1} + \|u(t)\|_{L^{\sigma(\cdot)+1}(\Omega)}^{\sigma^++1} \le M\|u(t)\|_X^{\sigma^{-1}+1}$$

where M was defined in (2.11a). Then three preceding inequalities provide

$$\left|\int_\Omega |u(t)|^{\sigma(x)} u(t) dx\right| \le M\|u(t)\|_X^{\sigma^-+1}. \tag{3.21}$$

Plugging inequalities (3.20) and (3.21) in (3.19), we obtain

$$\frac{d}{dt}|u(t)|^2 + \frac{2}{a_1^{p^+}}\|u(t)\|_X^{p^+} - 2M\|u(t)\|_X^{\sigma^-+1} \le 0.$$

Noting that $\frac{1}{a_1^{p^+}} \ge 2d$ because $p^+ \ge 1$, we derive of the last inequality

$$\frac{d}{dt}|u(t)|^2 + \frac{1}{a_1^{p^+}}\|u(t)\|_X^{p^+} + 2\left(d\|u(t)\|_X^{p^+} - M\|u(t)\|_X^{\sigma^-+1}\right) \le 0.$$

By (2.16) and (3.10), we get

$$\frac{d}{dt}|u(t)|^2 + \frac{1}{a_1^{p^+} L^{p^+}}|u(t)|^{p^+} \leq 0$$

that is

$$\frac{d}{dt}E(t) + \eta E(t)^{\frac{p^+}{2}} \leq 0. \tag{3.22}$$

We prove (i). For $p(x) = 2$, for all $x \in \overline{\Omega}$, that is, $p^+ = 2$, we have

$$\frac{d}{dt}E(t) + \eta E(t) \leq 0$$

that implies (2.18).

Before proving (ii), we make the following considerations. If $u^0 = 0$, we take $u \equiv 0$ as the solution of Problem (1.1a)–(1.1c). Assume $u^0 \neq 0$. If there exists $t_1 \in (0, \infty)$ such that $E(t_1) = 0$, we consider the set

$$\mathcal{P} = \{\tau \in (0, \infty); E(\tau) = 0\}$$

and

$$t^* = \inf_{\tau \in \mathcal{P}} \tau.$$

Then $t^* > 0$ because $E(0) > 0$. Also $E(t^*) = 0$. As $E'(t) \leq 0$ a.e. in $(0, \infty)$, then $E(t)$ is decreasing, therefore $E(t) = 0$ for all $t \geq t^*$. Thus

$$\text{either } E(t) = 0, \quad \text{for all } t \geq t^* \quad \text{or} \quad E(t) > 0, \quad \text{for all } t > 0.$$

We prove inequality (2.19) for the second case, that is, $E(t) > 0$, for all $t \in [0, \infty)$. The inequality (2.19) for $t \in [0, t^*)$ is derived in a similar way. Recalling that $\frac{p^+}{2} = 1 + \gamma$, $\gamma > 0$. By (3.22), we obtain

$$\frac{(-\gamma)E'(t)}{E(t)^{1+\gamma}} - \eta\gamma \geq 0,$$

which implies

$$\left([E(t)]^{-\gamma}\right)' \geq \eta\gamma.$$

Thus

$$E^{-\gamma}(t) \geq E^{-\eta}(0) + \eta\gamma t$$

that is,

$$E^{-\gamma}(t) \geq \frac{(1 + E^{\gamma}(0)\eta\gamma t)}{E^{\gamma}(0)}.$$

This implies inequality (2.19). □

Acknowledgement

We thank to the Referee for his/her very important remarks.

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