Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **95**, 1–22; https://doi.org/10.14232/ejqtde.2019.1.95 www.math.u-szeged.hu/ejqtde/

On the composition conjecture for a class of rigid systems

Zhengxin Zhou ✉ [1] and **Yuexin Yan** [1,2]

[1]School of Mathematical Sciences, Yangzhou University, China
[2]Guangling College of Yangzhou University, China

Received 10 April 2019, appeared 21 December 2019
Communicated by Armengol Gasull

Abstract. In this paper, we prove that for a class of rigid systems the Composition Conjecture is correct. We show that the Moments Condition is the sufficient and necessary conditions for these rigid systems to have a center at origin point. By the obtained conclusions we can derive all the focal values of these higher order polynomial differential systems and their expressions are more succinct and beautiful.

Keywords: Composition Conjecture, rigid system, Center Condition, Moments Condition.

2010 Mathematics Subject Classification: 34C07, 34C05, 34C25, 37G15.

1 Introduction

Consider the planar differential system

$$\begin{cases} x' = -y + p(x,y), \\ y' = x + q(x,y) \end{cases} \tag{1.1}$$

where p, q are analytic functions starting with second order terms.

If p and q are polynomials of degree n and $yp - xq = 0$, the system (1.1) in polar coordinates becomes

$$\frac{dr}{d\theta} = r^2 \sum_{i=0}^{n-2} A_i(\theta) r^i, \tag{1.2}$$

where $A_i(\theta)$ $(i = 0, 1, 2, \dots, n-2)$ are 2π-periodic functions. Therefore, the system (1.1) has a center at $(0,0)$ if and only if all solutions $r(\theta)$ of equation (1.2) near the solution $r = 0$ are periodic. In such case it is said that equation (1.2) has a center at $r = 0$ [15,20].

If p and q are homogeneous polynomials of degree n, via the Cherkas [9] transformation equation (1.2) becomes the Abel equation

$$\frac{d\rho}{d\theta} = \rho^2(\tilde{A}_1(\theta) + \tilde{A}_2(\theta)\rho), \tag{1.3}$$

✉Corresponding author. Email: zxzhou@yzu.edu.cn

where $\tilde{A}_i(\theta)$ $(i = 1, 2)$ are 2π-periodic functions. Thus, finding the center conditions for (1.1) is equivalent to studying when the Abel equation (1.3) has a center at $\rho = 0$. This problem has been investigated in [5, 8, 10, 14, 20] among other works.

The problem of determining necessary and sufficient conditions on p and q for system (1.1) to have a center at the origin is known as the center-focus problem. Due to the Hilbert' basis theorem we know that when p and q are polynomials of a given degree this set of conditions is finite. To get the necessary and sufficient conditions is very complexity, up to now only for a very few families of polynomial system (1.1) the center conditions are known. The problem is solved for quadratic system and some families of cubic systems and systems in the form of the linear center perturbed by homogeneous quartic and quintic nonlinearities, see e.g. [1, 4, 9–16, 22] and references given there.

Alwash and Lloyd [6, 7] give the following simple sufficient condition for the Abel equation to have a center.

Theorem 1.1 ([6, 7])**.** *If there exists a differentiable function u of period 2π such that*

$$\tilde{A}_1(\theta) = u'(\theta)\check{A}_1(u(\theta)), \qquad \tilde{A}_2(\theta) = u'(\theta)\check{A}_2(u(\theta))$$

for some continuous functions $\check{A}_1$ and $\check{A}_2$, then the Abel differential equation (1.3) *has a center at the origin.*

The following statement presents a generalization of Theorem 1.1.

Theorem 1.2 ([2, 22])**.** *If there exists a differentiable function u of period 2π such that*

$$A_i(\theta) = u'\hat{A}_i(u), \qquad (i = 1, 2, \dots, n-2)$$

for some continuous functions $\hat{A}_i$ $(i = 1, 2, \dots, n-2)$, then the differential equation (1.2) *has a center at $r = 0$.*

The condition in Theorem 1.1 (or Theorem 1.2) is called the **Composition Condition**. When an Abel equation (or (1.2)) has a center because its coefficients satisfy the composition condition we will say that this equation has a **CC-center**. In [7, 18] it was shown that this condition is not necessary to have a center.

The **Composition Conjecture** is that the composition condition in Theorem 1.1 (or Theorem 1.2) is not only the sufficient but also necessary condition for a center. This conjecture first appeared in [6] with classes of coefficients which are polynomial functions in t, or trigonometric polynomials. A counterexample was presented in [7, 13] to demonstrate that the conjecture is not true. To find the restrictive conditions under which the composition conjecture is true, this is an open problem which has attracted during the last years a wide interest [2–8, 10, 12, 14, 17, 18, 21, 22]. The authors in paper [11] give the sufficient and necessary conditions for the $r = 0$ of the Abel equation (1.3) to be a CC-center.

The condition

$$\int_0^{2\pi} \left(\int_0^{\theta} \tilde{A}_1(\tau)d\tau \right)^k \tilde{A}_2(\theta)d\theta = 0 \qquad (k \geq 0)$$

is called **Moments Condition** [10]. In [18] an example of a polynomial Abel equation satisfying the moments condition and not satisfying the composition condition is given. Later on, in [17] a full algebraic characterization of the moments condition in the polynomial case is done. In [10] prove that a natural trigonometric analogous to it does not hold.

In [21], it was proved that for the rigid system

$$\begin{cases} x' = -y + xP, \\ y' = x + yP \end{cases} \tag{1.4}$$

with $P = P_1 + P_m$, P_m is a homogeneous polynomial of degree m and which is an arbitrary natural number greater than 1, the composition conjecture is true, i.e., its origin point is a center and a CC-center, and shown that for its corresponding 2π-periodic equation

$$\frac{dr}{d\theta} = r(P_1(\cos\theta, \sin\theta)r + P_m(\cos\theta, \sin\theta)r^m),$$

$r = 0$ is a center if and only if it satisfies the moments conditions:

$$\int_0^{2\pi} \left(\int_0^{\theta} P_1(\cos\tau, \sin\tau)d\tau \right)^k P_m(\cos\theta, \sin\theta)d\theta = 0, \qquad (k = 0, 1, 2, \ldots, m).$$

In [1] the authors used the methods of the normal form theory to prove that the rigid system (1.4) has a center if and only if it is reversible. In [19], the author has calculated by computer and obtained the center condition for system (1.4) with $P = P_m + P_{2m}$, m is a finite number that does not exceed 5.

In this paper, we will study the rigid system

$$\begin{cases} x' = -y + x(P_1(x, y) + P_m(x, y) + P_{2m+1}(x, y)), \\ y' = x + y(P_1(x, y) + P_m(x, y) + P_{2m+1}(x, y)), \end{cases} \tag{1.5}$$

where $P_k(x, y)$ is a homogeneous polynomial in x, y of degree k and $P_1 \neq 0$, m is an arbitrary positive integer greater than 1, and give the necessary and sufficient conditions for the origin of (1.5) to be a center. We prove that under some restrictions conditions the composition conjecture is true for its corresponding periodic differential equation

$$\frac{dr}{d\theta} = r(P_1(\cos\theta, \sin\theta)r + P_m(\cos\theta, \sin\theta)r^m + P_{2m+1}(\cos\theta, \sin\theta)r^{2m+1}). \tag{1.6}$$

By this, we can derive all the focal values of system (1.5) and they contain exactly $\frac{[m]}{2} + m + 2$ relations. As m is an arbitrary number, in general, even with the help of computers, it is difficult to get the center conditions. However, in this paper, we have obtained these results only by using the mathematical analysis technique. We firmly believe that the method of this paper can be used to solve the center-focus problem of more high-order polynomial differential systems.

In the following we denote $\bar{P} = \int_0^{\theta} P(\cos\theta, \sin\theta)d\theta$; $C_n^k = \frac{n!}{k!(n-k)!}$ $(0 \leq k \leq n)$; $C_n^k = 0$, if $k < 0$ or $n < 0$; $\sum_{i+j=k} = 0$, if $k < 0$.

2 Several lemmas

In order to prove the main result, first of all, we give the following lemmas.

Lemma 2.1 ([21]). *If $P_1 \neq 0$ and for an arbitrary positive integer m,*

$$\int_0^{2\pi} \bar{P}_1^i(\cos\theta, \sin\theta)P_m(\cos\theta, \sin\theta)d\theta = 0 \qquad (i = 0, 1, 2, \ldots, m),$$

then

$$P_m = P_1 \sum_{j=1}^{m} j\lambda_j \bar{P}_1^{j-1}, \tag{2.1}$$

where $\lambda_j\,(j = 1, 2, \ldots, m)$ *are real numbers.*

Lemma 2.2. *Suppose that* P_m *satisfies* (2.1) *and*

$$h_0' = P_m,\ h_k' = 2P_1 \sum_{i+j=k-1} \bar{P}_1^i h_j + P_m C_{m+k}^m \bar{P}_1^k, \qquad (k = 1, 2, 3, \ldots) \tag{2.2}$$

Then

$$h_k = \sum_{j=0}^{k} h_k^j \bar{P}_1^{k-j} \overline{\bar{P}_1^j P_m} = \hat{h}_k \bar{P}_1^{m+k} + \sum_{j=0}^{k-1} \hat{h}_{kj} \bar{P}_1^{m+j}, \tag{2.3}$$

where $\hat{h}_{kj}\,(j = 0, 1, 2, \ldots, k-1)$ *are real numbers,*

$$h_k^j = \frac{2}{k-j} \sum_{i=j}^{k-1} h_i^j = (k-j+1) C_{m+j-2}^j, \qquad (j = 0, 1, 2, \ldots, k-1), \tag{2.4}$$

$$h_k^k = C_{m+k}^k - \sum_{j=0}^{k-1} h_k^j = C_{m+k-2}^k, \tag{2.5}$$

$$\hat{h}_k = \lambda_m \sum_{j=0}^{k} \frac{m}{m+j} h_k^j = \lambda_m \sum_{j=0}^{k} \frac{m}{m+j} (k-j+1) C_{m+j-2}^j. \tag{2.6}$$

Proof. By (2.1) we get

$$\bar{P}_m = \int_0^\theta P_m d\theta = \lambda_m \bar{P}_1^m + \sum_{j=1}^{m-1} \lambda_j \bar{P}_1^j, \tag{2.7}$$

$$\overline{\bar{P}_1^k P_m} = \int_0^\theta P_1 \sum_{j=1}^{m} j\lambda_j \bar{P}_1^{j+k-1} = \frac{m}{k+m} \lambda_m \bar{P}_1^{m+k} + \sum_{j=1}^{m-1} \frac{j}{j+k} \lambda_j \bar{P}_1^{j+k}, \tag{2.8}$$

where $\lambda_j\,(j = 1, 2, \ldots, m)$ are real numbers.

By (2.2) and (2.7) and (2.8) we get $h_0' = P_m$,

$$h_0 = \bar{P}_m = h_0^0 \bar{P}_m = \hat{h}_0 \bar{P}_1^m + \sum_{j=1}^{m-1} \lambda_j \bar{P}_1^j,\ h_0^0 = 1, \hat{h}_0 = \lambda_m h_0^0 = \lambda_m.$$

$$h_1' = 2P_1 h_0 + P_m C_{m+1}^m \bar{P}_1,$$

$$h_1 = 2h_0^0 \bar{P}_1 \bar{P}_m + (C_{m+1}^1 - 2h_0^0) \overline{\bar{P}_1 P_m} = h_1^0 \bar{P}_1 \bar{P}_m + h_1^1 \overline{\bar{P}_1 P_m} = \hat{h}_1 \bar{P}_1^{m+1} + \sum_{j=1}^{m} \hat{h}_{1j} \bar{P}_1^j,$$

where $\hat{h}_{1j}\,(j = 1, 2, \ldots, m)$ are real numbers,

$$h_1^0 = 2h_0^0, \qquad h_1^1 = C_{m+1}^1 - h_1^0 = C_{m-1}^1, \qquad \hat{h}_1 = \left(h_1^0 + h_1^1 \frac{m}{1+m}\right) \lambda_m = \frac{m^2+m+2}{m+1} \lambda_m,$$

this means that the relations (2.3)–(2.6) are valid for $k = 0, 1$. Now suppose that these identifies are valid for integer k, next we will prove they are correct for integer $k+1$.

Indeed, by (2.2)–(2.5) and (2.7) and (2.8) we have

$$h'_{k+1} = 2P_1 \sum_{i+j=k} \bar{P}_1^i h_j + P_m C_{m+k+1}^m \bar{P}_1^{k+1}$$
$$= 2P_1 \sum_{j=0}^{k} \bar{P}_1^{k-j} \overline{\bar{P}_1^j P_m} \sum_{i=j}^{k} h_i^j + P_m C_{m+k+1}^m \bar{P}_1^{k+1},$$

solving this equation we get

$$h_{k+1} = \sum_{j=0}^{k+1} h_{k+1}^j \bar{P}_1^{k+1-j} \overline{\bar{P}_1^j P_m} = \hat{h}_{k+1} \bar{P}_1^{m+k+1} + \sum_{j=1}^{m+k} \hat{h}_{k+1\,j} \bar{P}_1^j,$$

where $\hat{h}_{k+1\,j}$ $(j = 1, 2, \dots, m+k)$ are real numbers,

$$\hat{h}_{k+1} = \lambda_m \sum_{j=0}^{k+1} \frac{m}{m+j} h_{k+1}^j,$$

$$h_{k+1}^j = \frac{2}{k+1-j} \sum_{i=j}^{k} h_i^j,\ (j = 0, 1, 2, ..., k),\ h_{k+1}^{k+1} = C_{m+k+1}^{k+1} - \sum_{j=0}^{k} h_{k+1}^j,$$

By Lemma 3.2 of [21], we get

$$h_{k+1}^j = (k-j+2) C_{m+j-2}^j,\ h_{k+1}^{k+1} = C_{m+k-1}^{k+1}.$$

By mathematical induction, the present lemma holds. □

Similarly, we can prove the following lemma.

Lemma 2.3. *If*

$$\delta'_k = 2P_1 \sum_{i+j=k-1} \bar{P}_1^i \delta_j + P_{2m+1} C_{2m+k+1}^{2m+1} \bar{P}_1^k, \tag{2.9}$$

then

$$\delta_k = \sum_{j=0}^{k} \delta_k^j \bar{P}_1^{k-j} \overline{\bar{P}_1^j P_{2m+1}}, \tag{2.10}$$

where

$$\delta_k^j = \frac{2}{k-j} \sum_{i=j}^{k-1} \delta_i^j = (k-j+1) C_{2m+j-1}^j, \qquad (j = 0, 1, 2, \dots, k-1),$$

$$\delta_k^k = C_{2m+k+1}^{2m+1} - \sum_{j=0}^{k-1} \delta_k^j = C_{2m+k-1}^k,\ \delta_0^0 = 1.$$

Lemma 2.4. *If the conditions of Lemma 2.2 are satisfied and*

$$\begin{aligned}
\alpha'_0 &= C_{m+1}^1 P_m h_0, \\
\alpha'_k &= P_1 \sum_{i+j=k-1} (h_i h_j + 2\bar{P}_1^i \alpha_j) + 2P_1 \sum_{i+j=k-1-m} h_i \alpha_j + P_m C_{m+1}^1 \sum_{i+j=k} C_{m+i-1}^{m-1} \bar{P}_1^i h_j \\
&\quad + P_m \sum_{i+j+l=k-m} C_{m+1}^2 C_{m+i-2}^{m-2} \bar{P}_1^i h_j h_l + P_m \sum_{i+j=k-m} C_{m+1}^1 C_{m+i-1}^{m-1} \bar{P}_1^i \alpha_j,
\end{aligned} \tag{2.11}$$

then

$$\alpha_k = \hat{\alpha}_k \bar{P}_1^{2m+k} + \sum_{j=2}^{2m+k-1} \hat{\alpha}_{kj} \bar{P}_1^j, \qquad (k = 0, 1, 2, \dots), \tag{2.12}$$

where $\hat{\alpha}_{kj}$, $(j = 2, 3, \dots, 2m+k-1)$ *are real numbers and*

$$\begin{aligned}
\hat{\alpha}_0 &= \frac{m+1}{2} \lambda_m^2, \\
\hat{\alpha}_k &= \frac{1}{2m+k} \Bigg(\sum_{i+j=k-1} \hat{h}_i \hat{h}_j + 2 \sum_{j=0}^{k-1} \hat{\alpha}_j + 2 \sum_{i+j=k-1-m} \hat{h}_i \hat{\alpha}_j + m \lambda_m C_{m+1}^1 \sum_{i+j=k} C_{m+i-1}^{m-1} \hat{h}_j \\
&\quad + \lambda_m m \sum_{i+j+l=k-m} C_{m+1}^2 C_{m+i-2}^{m-2} \hat{h}_j \hat{h}_l + \lambda_m m \sum_{i+j=k-m} C_{m+1}^1 C_{m+i-1}^{m-1} \hat{\alpha}_j \Bigg), \quad (k = 1, 2, \dots).
\end{aligned} \tag{2.13}$$

Proof. Solving equation $\alpha_0' = C_{m+1}^1 P_m h_0$, by Lemma 2.2 we get

$$\alpha_0 = \frac{1}{2} C_{m+1}^1 h_0^0 \bar{P}_m^2 = \hat{\alpha}_0 \bar{P}_1^{2m} + \sum_{j=2}^{2m-1} \hat{\alpha}_{0j} \bar{P}_1^j.$$

where $\hat{\alpha}_{0j}$ are real numbers, $\hat{\alpha}_0 = \frac{1}{2} C_{m+1}^1 \lambda_m^2$. Thus, the relation (2.12) is valid for $k = 0$.

Suppose that the conclusion of the present lemma is correct for integer $k-1$, next we will show that they are held for integer k.

Indeed, by Lemma 2.2 and (2.11) we get

$$\begin{aligned}
\alpha_k' = P_1 \bar{P}_1^{2m+k-1} \Bigg(&\sum_{i+j=k-1} \hat{h}_i \hat{h}_j + 2 \sum_{j=0}^{k-1} \hat{\alpha}_j + 2 \sum_{i+j=k-1-m} \hat{h}_i \hat{\alpha}_j + \lambda_m C_m^1 C_{m+1}^1 \sum_{i+j=k} C_{m+i-1}^{m-1} \hat{h}_j \\
&+ \lambda_m C_m^1 \sum_{i+j+l=k-m} C_{m+1}^2 C_{m+i-2}^{m-2} \hat{h}_j \hat{h}_l + \lambda_m C_m^1 \sum_{i+j=k-m} C_{m+1}^1 C_{m+i-1}^{m-1} \hat{\alpha}_j \Bigg) + \cdots
\end{aligned}$$

which implies that the identities (2.12) and (2.13) are valid. By mathematical induction, the present lemma holds. □

Lemma 2.5. *Suppose that h_k and δ_k are the solutions of equations (2.2) and (2.9) respectively, and*

$$\begin{aligned}
\beta_k' = 2P_1 &\sum_{i+j=k-1} (\bar{P}_1^i \beta_j + h_i \delta_j) + P_m C_{m+1}^1 \sum_{i+j=k} C_{m+i-1}^{m-1} \bar{P}_1^i \delta_j \\
&+ P_{2m+1} C_{2m+2}^1 \sum_{i+j=k} C_{2m+i}^{2m} \bar{P}_1^i h_j, \qquad (k = 0, 1, 2, \dots, m+1)
\end{aligned} \tag{2.14}$$

Then

$$\beta_k = \sum_{j=0}^{k} (\beta_k^j \bar{P}_1^{m+k-j} \overline{\bar{P}_1^j P_{2m+1}} + \beta_k^{m+j} \bar{P}_1^{k-j} \overline{\bar{P}_1^{m+j} P_{2m+1}}) + \sum_{j=0}^{m+k-1} \sum_{i=0}^{j} (\beta_{kji} \bar{P}_1^{j-i} \overline{\bar{P}_1^i P_{2m+1}}), \tag{2.15}$$

where β_{kji}, $(i = 0, 1, 2, \dots, j)$ *are real numbers,*

$$\beta_k^j = C_{2m+j-1}^j C_{m+k-j+1}^1 \hat{h}_{k-j}, \qquad (j = 0, 1, 2, \dots, k), \tag{2.16}$$

$$\beta_k^{m+j} = (k-j+1) \beta_j^{m+j}, \ \beta_0^m = C_{m+1}^1 \hat{h}_0, \qquad (j = 0, 1, 2, \dots, k-1), \tag{2.17}$$

$$\beta_k^{m+k} = \sum_{j=0}^{k} (C_{2m+2}^1 C_{2m+j}^j - C_{2m+j-1}^j C_{m+k-j+1}^1) \hat{h}_{k-j} - \sum_{j=0}^{k-1} (k-j+1) \beta_j^{m+j}. \tag{2.18}$$

Proof. By (2.14) we get

$$\beta_0' = C_{m+1}^1 P_m \delta_0 + P_{2m+1} C_{2m+2}^1 h_0,$$

using Lemma 2.2 and Lemma 2.3 we get

$$h_0 = h_0^0 \bar{P}_m,\ \delta_0 = \delta_0^0 \bar{P}_{2m+1}, h_0^0 = \delta_0^0 = 1,$$

thus

$$\begin{aligned}
\beta_0' &= C_{m+1}^1 P_m \delta_0^0 \bar{P}_{2m+1} + P_{2m+1} C_{2m+2}^1 h_0^0 \bar{P}_m,\\
\beta_0 &= C_{m+1}^1 \delta_0^0 \bar{P}_m \bar{P}_{2m+1} + (C_{2m+2}^1 h_0^0 - C_{m+1}^1 h_0^0)\overline{\bar{P}_m P_{2m+1}}\\
&= \beta_0^0 \bar{P}_1^m \bar{P}_{2m+1} + \beta_0^m \overline{\bar{P}_1^m P_{2m+1}} + \sum_{j=0}^{m-1} (\beta_{0j0} \bar{P}_1^j \bar{P}_{2m+1} + \beta_{00j} \overline{\bar{P}_1^j P_{2m+1}}),
\end{aligned}$$

where β_{0j0}, β_{00j} $(j = 0, 1, 2, \dots, m-1)$ are real numbers,

$$\beta_0^0 = \lambda_m C_{m+1}^1 \delta_0^0 = \lambda_m C_{m+1}^1, \qquad \beta_0^m = \lambda_m (C_{2m+2}^1 h_0^0 - C_{m+1}^1 \delta_0^0) = \lambda_m C_{m+1}^1.$$

Therefore, the conclusion of the present lemma is correct for $k = 0$. Now we suppose that

$$\beta_n = \sum_{j=0}^{n} (\beta_n^j \bar{P}_1^{m+n-j} \overline{\bar{P}_1^j P_{2m+1}} + \beta_n^{m+j} \bar{P}_1^{n-j} \overline{\bar{P}_1^{m+j} P_{2m+1}}) + \cdots \qquad (n = 0, 1, 2, \dots, k-1).$$

By this and (2.14) we get

$$\begin{aligned}
\beta_k' = {} & 2P_1 \sum_{i+j=k-1} \sum_{l=0}^{j} (\beta_j^l \bar{P}_1^{m+k-1-l} \overline{\bar{P}_1^l P_{2m+1}} + \beta_j^{m+l} \bar{P}_1^{k-1-l} \overline{\bar{P}_1^{m+l} P_{2m+1}})\\
& + 2P_1 \sum_{i+j=k-1} \hat{h}_i \sum_{l=0}^{j} \delta_j^l \bar{P}_1^{m+k-1-l} \overline{\bar{P}_1^l P_{2m+1}} + m\lambda_m C_{m+1}^1 \sum_{i+j=k} C_{m+i-1}^{m-1} \sum_{l=0}^{j} \delta_j^l \bar{P}_1^{k-l} \overline{\bar{P}_1^l P_{2m+1}}\\
& + P_{2m+1} \bar{P}_1^{k+m} C_{2m+2}^1 \sum_{i+j=k} C_{2m+i}^{2m} \hat{h}_j + \cdots,
\end{aligned}$$

solving this equation we get

$$\beta_k = \sum_{j=0}^{k} (\beta_k^j \bar{P}_1^{m+k-j} \overline{\bar{P}_1^j P_{2m+1}} + \beta_k^{m+j} \bar{P}_1^{k-j} \overline{\bar{P}_1^{m+j} P_{2m+1}}) + \sum_{j=0}^{m+k-1} \sum_{i=0}^{j} (\beta_{kji} \bar{P}_1^{j-i} \overline{\bar{P}_1^i P_{2m+1}}),$$

where β_{kji}, $(i = 0, 1, 2, \dots, j)$ are real numbers,

$$\beta_k^j = \frac{1}{m+k-j} \left(2 \sum_{i=0}^{k-j-1} \hat{h}_i \delta_{k-1-i}^j + 2 \sum_{i=j}^{k-1} \beta_i^j + m\lambda_m C_{m+1}^1 C_{2m+j-1}^j C_{m+k-j+1}^{k-j} \right), \quad (j = 0, 1, 2, \dots, k-1) \tag{2.19}$$

$$\beta_k^k = \lambda_m C_{m+1}^1 C_{2m+k-1}^k, \tag{2.20}$$

$$\beta_k^{m+j} = \frac{2}{k-j} \sum_{i=j}^{k-1} \beta_i^{m+j}, \qquad (j = 0, 1, 2, \dots, k-1), \tag{2.21}$$

$$\beta_k^{m+k} = C_{2m+2}^1 \sum_{i+j=k} C_{2m+i}^{2m} \hat{h}_j - \beta_k^k - \sum_{j=0}^{k-1} (\beta_k^j + \beta_k^{m+j}). \tag{2.22}$$

In order to prove the relations (2.16)–(2.18) are valid, first of all, we use mathematical induction to show that

$$2\sum_{j=0}^{k}\hat{h}_j+\lambda_m m C_{m+k+1}^{m}=(m+k+1)\hat{h}_{k+1}\qquad (k=0,1,2,\dots). \tag{2.23}$$

Indeed, by (2.6) we get $\hat{h}_0=\lambda_m$, $\hat{h}_1=\lambda_m\big(2+\frac{m}{m+1}C_{m-1}^{1}\big)$, thus

$$2\hat{h}_0+\lambda_m m C_{m+1}^{m}=\lambda_m(2+mC_{m+1}^{1})=(m+1)\hat{h}_1,$$

this means that the relation (2.23) is valid for $k=0$. Suppose that the identity (2.23) is valid for positive integer k, next we will show that

$$2\sum_{j=0}^{k+1}\hat{h}_j+\lambda_m m C_{m+k+2}^{m}=(m+k+2)\hat{h}_{k+2}. \tag{2.24}$$

Indeed, from the inductive hypothesis and (2.6) we have

$$\begin{aligned}2\sum_{j=0}^{k+1}\hat{h}_j&=(m+k+1)\hat{h}_{k+1}-\lambda_m m C_{m+k+1}^{m}+2\hat{h}_{k+1}\\&=\lambda_m(m+k+3)\sum_{j=0}^{k+1}\frac{m}{m+j}(k+2-j)C_{m+j-2}^{j}-\lambda_m m C_{m+k+1}^{m}.\end{aligned}$$

By this and $C_m^j=C_{m-1}^{j-1}+C_{m-1}^{j}$, we get

$$\begin{aligned}(m+k+2)\hat{h}_{k+2}-2\sum_{j=0}^{k+1}\hat{h}_j&=(m+k+2)\lambda_m\sum_{j=0}^{k+2}\frac{m}{m+j}(k+3-j)C_{m+j-2}^{j}\\&\quad-\lambda_m(m+k+3)\sum_{j=0}^{k+1}\frac{m}{m+j}(k+2-j)C_{m+j-2}^{j}+\lambda_m m C_{m+k+1}^{m}\\&=\lambda_m m\sum_{j=0}^{k+1}C_{m+j-2}^{m}+\lambda_m m C_{m+k}^{k+2}+\lambda_m m C_{m+k+1}^{m}\\&=\lambda_m m\sum_{j=0}^{k+2}C_{m+j-2}^{j}+\lambda_m m C_{m+k+1}^{k+1}=m\lambda_m C_{m+k+2}^{k+2}.\end{aligned}$$

Thus the identity (2.24) is correct. By mathematical induction, the identity (2.23) is valid.

Now, we prove that the relation (2.16) is correct. Indeed, by (2.19) and (2.20) we get

$$\beta_0^0=\lambda_m C_{m+1}^{1}=C_{m+1}^{1}\hat{h}_0,$$

and

$$\begin{aligned}\beta_1^0&=\frac{1}{m+1}(2\hat{h}_0+2\beta_0+\lambda_m m C_{m+1}C_{m+2}^{1})\\&=\frac{1}{m+1}C_{m+2}^{1}(2+mC_{m+1})=C_{m+2}^{1}\hat{h}_1,\end{aligned}$$

and

$$\beta_1^1=\lambda_m C_{2m}^{1}C_{m+1}^{1}=C_{2m}^{1}C_{m+1}^{1}\hat{h}_0.$$

Thus, the identity (2.16) is correct for $k = 0, 1$. Suppose that (2.16) is valid for natural numbers less than or equal to k, next we will prove that (2.16) holds for $k + 1$.

In fact, by (2.19) and (2.23) and Lemma 2.4 we get

$$\beta^j_{k+1} = \frac{1}{m+k+1-j}\left(2\sum_{i=0}^{k-j}\hat{h}_i\delta^j_{k-i} + 2\sum_{i=j}^{k}\beta^j_i + m\lambda_m C^1_{m+1}C^j_{2m+j-1}C^{k-j+1}_{m+k-j+2}\right)$$
$$= \frac{1}{m+k+1-j}C^j_{2m+j-1}C^1_{k+m-j+2}\left(2\sum_{i=0}^{k-j}\hat{h}_i + \lambda_m m C^m_{m+k+1-j}\right) = C^j_{2m+j-1}C^1_{k+m-j+2}\hat{h}_{k-j+1}.$$

So, by mathematical induction the identity (2.16) is correct.

Now we prove that the identity (2.17) is correct.

As $\beta^m_0 = C^1_{m+1}\lambda_m$, by (2.21) we get $\beta^m_1 = 2\beta^m_0$, thus, the relation (2.17) is correct for $k = 1$. Suppose that

$$\beta^{m+j}_{k-1} = (k-j)\beta^{m+j}_j, \qquad (j = 0, 1, 2, \ldots, k-2),$$

by this and (2.21) we get

$$\beta^{m+j}_k = \frac{2}{k-j}\sum_{i=j}^{k-1}\beta^{m+j}_i = \frac{2}{k-j}\sum_{i=j}^{k-1}(i-j+1)\beta^{m+j}_j = (k-j+1)\beta^{m+j}_j.$$

Therefore, the identity (2.17) is valid.

Substituting (2.16) and (2.17) into (2.22) implies that the identity (2.18) holds.

In summary, the conclusions of the present lemma is correct. □

Lemma 2.6. *If $\lambda_m \geq 0$ and $m \geq 2$, then*

$$\beta^{m+k}_k \geq 0, \qquad (k = 0, 1, 2, \ldots, m+1). \tag{2.25}$$

Proof. As $\lambda_m \geq 0$, by (2.6) we get

$$\hat{h}_k = \lambda_m \sum_{j=0}^{k}\frac{m}{m+j}(k-j+1)C^j_{m+j-2} \geq 0, \qquad (k = 0, 1, 2, \ldots, m+1).$$

Using (2.17) we get

$$\beta^m_0 = C^1_{m+1}\hat{h}_0 = \lambda_m C^1_{m+1} \geq 0.$$

By this and using (2.18) we have

$$\beta^{1+m}_1 = C^1_m\hat{h}_1 + C_{m+1}C^1_{2m+2}\hat{h}_0 - 2\beta^m_0 = C^1_m\hat{h}_1 + C^1_m C^1_{2m+2}\hat{h}_0 \geq 0.$$

Thus the inequality (2.25) is valid for $k = 0$ and $k = 1$. Now we suppose that inequality (2.25) holds for natural numbers less than k, i.e., $\beta^{i+m}_i \geq 0$, $(i = 0, 1, 2, \ldots, k-1)$. Next we will show that $\beta^{k+m}_k \geq 0$.

Indeed, by (2.18) we get

$$\beta^{k-1+m}_{k-1} = \sum_{j=0}^{k-1}(C^1_{2m+2}C^{j-1}_{2m+j-1} + C^j_{2m+j-1}C^1_{m+2+j-k})\hat{h}_{k-1-j} - \sum_{j=0}^{k-2}(k-j)\beta^{m+j}_j$$
$$= \sum_{j=1}^{k}(C^1_{2m+2}C^{j-2}_{2m+j-2} + C^{j-1}_{2m+j-2}C^1_{m+1+j-k})\hat{h}_{k-j} - \sum_{j=0}^{k-2}(k-j)\beta^{m+j}_j.$$

By this and (2.18) we get

$$\beta_k^{k+m} = C_{m+1-k}^1\hat{h}_k + \sum_{j=1}^{k}(C_{2m+2}^1 C_{2m+j-1}^{j-1} + C_{2m+j-1}^{j} C_{m+1+j-k}^1)\hat{h}_{k-j} - \sum_{j=0}^{k-2}(k-j)\beta_j^{m+j} - 2\beta_{k-1}^{k-1+m}$$
$$= \sum_{j=0}^{k-2}(k-1-j)\beta_j^{j+m} + C_{m+1-k}^1\hat{h}_k + (C_{2m+2}^1 + C_{m+2-k}^1 C_{2m-2}^1)\hat{h}_{k-1}$$
$$+ \sum_{j=2}^{k}\left(\frac{1}{j-1}C_{2m+2}^1 C_{2m+j-2}^{j-2} C_{2m-j+1}^1 + \frac{1}{j}C_{m+1+j-k}^1 C_{2m+j-2}^{j-1} C_{2m-1-j}^1\right)\hat{h}_{k-j}, \quad (2.26)$$

as $\beta_i^{i+m} \geq 0$ $(i = 0,1,2,\ldots,k-2)$ and $\hat{h}_i > 0$ $(i = 0,1,2,\ldots,m+1)$ and $m \geq 2$, then $\beta_k^{k+m} \geq 0$ holds. Thus, by mathematical induction, the present lemma is valid. The proof is complete. □

Lemma 2.7. *Suppose that h_k and δ_k are the solutions of equations (2.2) and (2.9) respectively, α_k and β_k are the solutions of equations (2.11) and (2.14) respectively and*

$$\tilde{\beta}_1' = C_{m+1}^1 P_m \beta_0 + P_m C_{m+1}^1 C_m^1 h_0\delta_0 + P_{2m+1}(C_{2m+2}^2 h_0^2 + C_{2m+2}^1\alpha_0), \quad (2.27)$$

$$\tilde{\beta}_2' = 2P_1(h_0\beta_0 + \alpha_0\delta_0 + \tilde{\beta}_1) + P_m C_{m+1}^1\left(\sum_{i+j=1} C_{m+i-1}^{m-1}\bar{P}_1^i\beta_j + C_m^1 \sum_{i+j+l=1} C_{m+i-2}^{m-2}\bar{P}_1^i h_j\delta_l\right)$$
$$+ P_{2m+1}\left(\sum_{i+j+l=1} C_{2m+2}^2 C_{2m+i-1}^{2m-1}\bar{P}_1^i h_j h_l + C_{2m+2}^1 \sum_{i+j=1} C_{2m+i}^{2m}\bar{P}_1^i\alpha_j + C_{2m+2}^1\delta_0\right). \quad (2.28)$$

Then

$$\tilde{\beta}_1 = \tilde{\beta}_1^0\bar{P}_1^{2m}\bar{P}_{2m+1} + \tilde{\beta}_1^m\bar{P}_1^m\overline{\bar{P}_1^m P_{2m+1}} + \tilde{\beta}_1^{2m}\overline{\bar{P}_1^{2m}P_{2m+1}} + \cdots, \quad (2.29)$$

$$\tilde{\beta}_2 = \tilde{\beta}_2^0\bar{P}_1^{2m+1}\bar{P}_{2m+1} + \tilde{\beta}_2^1\bar{P}_1^{2m}\overline{\bar{P}_1 P_{2m+1}} + \tilde{\beta}_2^m\bar{P}_1^{m+1}\overline{\bar{P}_1^m P_{2m+1}}$$
$$+ \tilde{\beta}_2^{m+1}\bar{P}_1^m\overline{\bar{P}_1^{m+1}P_{2m+1}} + \tilde{\beta}_2^{2m}\bar{P}_1\overline{\bar{P}_1^{2m}P_{2m+1}} + \tilde{\beta}_2^{2m+1}\overline{\bar{P}_1^{2m+1}P_{2m+1}} + C_{m+1}^1\bar{P}_{2m+1}^2 + \cdots, \quad (2.30)$$

where

$$\tilde{\beta}_1^0 = \frac{\lambda_m^2}{2}C_{m+1}^1 C_{2m+1}^1, \quad \tilde{\beta}_1^m = \lambda_m^2(C_{m+1}^1)^2, \quad \tilde{\beta}_1^{2m} = \frac{\lambda_m^2}{2}C_{m+1}^1 C_{2m+1}^1,$$
$$\tilde{\beta}_2^0 = \frac{1}{2m+1}\lambda_m^2 C_{m+1}^1(4m^3 + 4m^2 + 6m + 4), \quad \tilde{\beta}_2^1 = \lambda_m^2 m C_{m+1}^1 C_{2m+1}^1, \quad (2.31)$$
$$\tilde{\beta}_2^{m+1} = \lambda_m^2 m(3m^2 + 5m + 4), \quad \tilde{\beta}_2^m = \lambda_m^2(m^3 + 3m^2 + 4m + 4), \quad \tilde{\beta}_2^{2m} = 2\tilde{\beta}_1^{2m}$$
$$\tilde{\beta}_2^{2m+1} = 4m\lambda_m^2(m^2 + m + 1).$$

Proof. By Lemmas 2.2–2.5, and substituting (2.3) and (2.10) and (2.12) and (2.15) into the equations (2.27) and (2.28) and solving them we get (2.29) and (2.30) and in which

$$\tilde{\beta}_1^0 = \frac{\lambda_m}{2}C_{m+1}^1(\beta_0^0 + m\lambda_m), \quad \tilde{\beta}_1^m = \lambda_m C_{m+1}^1\beta_0^m,$$
$$\tilde{\beta}_1^{2m} = \lambda_m^2(m+1)(3m+2) - \frac{\lambda_m}{2}C_{m+1}^1(\beta_0^0 + m\lambda_m + 2\beta_0^m).$$
$$\tilde{\beta}_2^0 = \frac{1}{2m+1}(2\hat{h}_0\beta_0^0 + 2\hat{\alpha}_0 + 2\tilde{\beta}_1^0 + \lambda_m C_m^1 C_{m+1}^1(\beta_1^0 + m\beta_0^0) + \lambda_m m C_m^1 C_{m+1}^1(\hat{h}_1 + \hat{h}_0\delta_1^0 + C_{m-1}^1\hat{h}_0))$$

$$\begin{aligned}
\tilde{\beta}_2^1 &= \frac{\lambda_m}{2} C^1_{m+1}(\beta_1^1 + C^1_m \hat{h}_0 \delta_1^1),\\
\tilde{\beta}_2^{m+1} &= \lambda_m C^1_{m+1} \beta_1^{m+1},\\
\tilde{\beta}_2^m &= \frac{\lambda_m}{m+1}(2\beta_0^m + C^1_m C^1_{m+1}\beta_1^m + C^1_{m+1} m^2 \beta_0^m + 2\tilde{\beta}_1^m),\ \tilde{\beta}_2^{2m} = 2\tilde{\beta}_1^{2m},\\
\tilde{\beta}_2^{2m+1} &= C^2_{2m+2}(2\hat{h}_0\hat{h}_1 + C^1_{2m}\hat{h}_0^2) + C^1_{2m+2}(\hat{\alpha}_1 + C^1_{2m+1}\hat{\alpha}_0) - \tilde{\beta}_2^0 - \tilde{\beta}_2^1 - \tilde{\beta}_2^m - \tilde{\beta}_2^{1+m} - \tilde{\beta}_2^{2m}.
\end{aligned}$$

By Lemma 2.2–2.5 and simple calculation we can obtain the present conclusions. □

3 Main theorem

Consider 2π-periodic equation

$$\frac{dr}{d\theta} = r(P_1(\cos\theta,\sin\theta)r + P_m(\cos\theta,\sin\theta)r^m + P_{2m+1}(\cos\theta,\sin\theta)r^{2m+1}), \tag{3.1}$$

where $P_k(\cos\theta,\sin\theta)$ is homogeneous polynomial in $\cos\theta,\sin\theta$ of degree k and $P_1 \neq 0$. In the case of $m = 1$ [21] I have studied its center-focus problem and proved that for this equation the composition conjecture is correct. In the following we only consider integer $m > 1$.

Theorem 3.1. *Suppose that* $\delta^{m+k}_{m+k} + \beta^{m+k}_k \neq 0\,(0 \leq k \leq m-1)$ *and* $m+k$ *is an odd number,* $\delta^{2m+1}_{2m+1} + \beta^{2m+1}_{m+1} + \tilde{\beta}^{2m+1}_2 \neq 0$. *Then* $r = 0$ *is a center of* (3.1)*, if and only if*

$$\int_0^{2\pi} \bar{P}_1^i P_m d\theta = 0, \quad (i = 0,1,2,\ldots,m), \qquad \int_0^{2\pi} \bar{P}_1^{2j+1} P_{2m+1} d\theta = 0, \quad (j = 0,1,2,\ldots,m). \tag{3.2}$$

Moreover, this center is a CC-center. Where the expressions of δ^j_k *and* β^j_k *and* $\tilde{\beta}^j_k$ *are given in Lemma 2.3, Lemma 2.5 and Lemma 2.7, respectively.*

Proof.

Necessity:

Let $r(\theta,c)$ be the solution of (3.1) such that $r(0,c) = c\,(0 < c \ll 1)$. We write

$$r(\theta,c) = c\sum_{n=0}^{\infty} a_n(\theta)c^n,$$

where $a_0(0) = 1$ and $a_n(0) = 0$ for $n > 1$. The origin of (3.1) is a center if and only if $r(\theta+2\pi,c) = r(\theta,c)$, i.e., $a_0(2\pi) = 1$, $a_n(2\pi) = 0\,(n = 1,2,3,\ldots)$ [5,6].

Substituting $r(\theta,c)$ into (3.1) we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n'(\theta)c^n = cP_1(\theta)\left(\sum_{n=0}^{\infty} a_n(\theta)c^n\right)^2 + c^m P_m\left(\sum_{n=0}^{\infty} a_n(\theta)c^n\right)^{m+1}\\
+ c^{2m+1}P_{2m+1}(\theta)\left(\sum_{n=0}^{\infty} a_n(\theta)c^n\right)^{2m+2}.
\end{aligned} \tag{3.3}$$

Equating the corresponding coefficients of c^n of (3.3) yields

$$a_0'(\theta) = 0, \qquad a_0(0) = 1,$$

$$a_k' = P_1 \sum_{i+j=k-1} a_i a_j, \qquad a_k(0) = 0, \qquad (k = 1, 2, \ldots, m-1),$$

solving these equations we get

$$a_0 = 1, \qquad a_k = \bar{P}_1^k, \qquad (k = 1, 2, \ldots, m-1).$$

$$a_{m+k}' = P_1 \sum_{i+j=m+k-1} a_i a_j + P_m \sum_{i_1+i_2+\cdots+i_{m+1}=k} a_{i_1} a_{i_2} \ldots a_{i_{m+1}}, a_{m+k}(0) = 0, \quad (k = 0, 1, 2, \ldots, m-1),$$

by these equations we have

$$a_{m+k} = \bar{P}_1^{m+k} + h_k, \qquad (k = 0, 1, 2, \ldots, m-1), \tag{3.4}$$

where

$$h_k' = 2P_1 \sum_{i+j=k-1} \bar{P}_1^i h_j + P_m C_{m+k}^m \bar{P}_1^k,$$

by Lemma 2.2, we get

$$h_k = \sum_{j=0}^{k} h_k^j \bar{P}_1^{k-j} \overline{\bar{P}_1^j P_m}, \qquad (k = 0, 1, 2, \ldots, m-1),$$

in which

$$h_k^j = (k - j + 1) C_{m+j-2}^j, \qquad (j = 0, 1, 2, \ldots, k-1), \qquad h_k^k = C_{m+k-2}^k.$$

Using (3.3) we have

$$a_{2m}' = P_1 \left(2m \bar{P}_1^{2m-1} + 2 \sum_{i+j=m-1} \bar{P}_1^i h_j \right) + P_m (C_{2m}^m \bar{P}_1^m + C_{m+1}^1 h_0),$$

solving this equation we get

$$a_{2m} = \bar{P}_1^{2m} + h_m + \alpha_0, \tag{3.5}$$

where

$$h_m = \sum_{j=0}^{m} h_m^j \bar{P}_1^{m-j} \overline{\bar{P}_1^j P_m}, \qquad \alpha_0 = \frac{m+1}{2} \bar{P}_m^2,$$

$$h_m^j = (m - j + 1) C_{m+j-2}^j, \qquad (j = 0, 1, 2, \ldots, m-1), \qquad h_m^m = C_{2m-2}^m.$$

As $C_{m+k-1}^k \neq 0 \quad (k = 0, 1, 2, \ldots, m)$, thus, by (3.4) and (3.5) from $a_{m+k}(2\pi) = 0$, $(k = 0, 1, 2, \ldots, m)$ follow that

$$\int_0^{2\pi} \bar{P}_1^k P_m d\theta = 0, \qquad (k = 0, 1, 2, \ldots, m). \tag{3.6}$$

By Lemma 2.1 we get

$$P_m = P_1 \sum_{i=1}^{m} i \lambda_i \bar{P}_1^{i-1}, \tag{3.7}$$

therefore,

$$\bar{P}_m = \lambda_m \bar{P}_1^m + \sum_{i=1}^{m-1} \lambda_i \bar{P}_1^i, \; \overline{\bar{P}_1^j P_m} = \frac{m \lambda_m}{m+j} + \sum_{i=1}^{m-1} \frac{i}{i+j} \bar{P}_1^{m+j}, \tag{3.8}$$

where λ_i $(i = 1, 2, \ldots, m)$ are real numbers. By this we obtain

$$h_k = \hat{h}_k \bar{P}_1^{m+k} + \sum_{j=1}^{k-1} \hat{h}_{kj} \bar{P}_1^{m+j}, \qquad \hat{h}_k = \lambda_m \sum_{j=0}^{k} \frac{m}{m+j} h_k^j, \qquad (k = 0, 1, 2, \ldots, m), \tag{3.9}$$

where $\hat{h}_{kj}$ $(j = 1, 2, \ldots, k-1)$ are real numbers.

$$\alpha_0 = \hat{\alpha}_0 \bar{P}_1^{2m} + \sum_{j=2}^{2m-1} \hat{\alpha}_{0j} \bar{P}_1^j, \qquad \hat{\alpha}_0 = \frac{m+1}{2} \lambda_m^2, \tag{3.10}$$

where $\hat{\alpha}_{0j}$ $(j = 2, 3, \ldots, 2m-1)$ are real numbers.

Denote

$$f = \sum_{i=0}^{\infty} a_i(\theta) c^i = g + c^m h + \alpha c^{2m} + \delta c^{2m+1} + \beta c^{3m+1},$$

where

$$g = \sum_{i=0}^{\infty} \bar{P}_1^i c^i, \quad h = \sum_{i=0}^{\infty} h_i c^i, \quad \alpha = \sum_{i=0}^{\infty} \alpha_i c^i, \quad \delta = \sum_{i=0}^{\infty} \delta_i c^i, \quad \beta = \sum_{i=0}^{\infty} \beta_i c^i.$$

Thus,

$$\begin{aligned} f^2 = {} & g^2 + 2ghc^m + (h^2 + 2g\alpha) c^{2m} + 2g\delta c^{2m+1} \\ & + 2h\alpha c^{3m} + 2(g\beta + h\alpha) c^{3m+1} + \alpha^2 c^{4m} + 2(h\beta + \alpha\delta) c^{4m+1} + \cdots, \end{aligned} \tag{3.11}$$

$$\begin{aligned} f^{m+1} = {} & g^{m+1} + C_{m+1}^1 g^m h c^m + (C_{m+1}^2 g^{m-1} h^2 + C_{m+1}^1 g^m \alpha) c^{2m} + C_{m+1}^1 g^m \delta c^{2m+1} \\ & + (C_{m+1}^3 g^{m-2} h^3 + C_{m+1}^1 C_m^1 g^{m-1} h\alpha) c^{3m} + C_{m+1}^1 (g^m \beta + C_m^1 g^{m-1} h\delta) c^{3m+1} \\ & + (C_{m+1}^4 g^{m-3} h^4 + C_{m+1}^1 C_m^2 g^{m-2} h^2 \alpha + C_{m+1}^2 g^{m-1} \alpha^2) c^{4m} + \cdots, \end{aligned} \tag{3.12}$$

$$\begin{aligned} f^{2m+2} = {} & g^{2m+2} + C_{2m+2}^1 g^{2m+1} h c^m + (C_{2m+2}^2 g^{2m} h^2 + C_{2m+2}^1 g^{2m+1} \alpha) c^{2m} \\ & + C_{2m+2}^1 \delta g^{2m+1} c^{2m+1} + \cdots, \end{aligned} \tag{3.13}$$

where

$$g^m = \sum_{i=0}^{\infty} C_{m+i-1}^{m-1} \bar{P}_1^i c^i, \qquad (m = 1, 2, 3, \ldots), \tag{3.14}$$

$$g^2 = \sum_{i=0}^{\infty} (i+1) \bar{P}_1^i c^i, \qquad g^{2m+2} = \sum_{i=0}^{\infty} C_{2m+1+i}^{2m+1} \bar{P}_1^i c^i. \tag{3.15}$$

Using (3.3) and (3.11)–(3.15), for $0 \le k \le m-1$, we have

$$\begin{aligned} a'_{2m+1+k} = {} & P_1 \left((2m+1+k) \bar{P}_1^{2m+k} + 2 \sum_{i+j=m+k} \bar{P}_1^i h_j + \sum_{i+j=k} (h_i h_j + 2\bar{P}_1^i \alpha_j) + 2 \sum_{i+j=k-1} \bar{P}_1^i \delta_j \right) \\ & + P_m \left(C_{2m+k+1}^m \bar{P}_1^{m+k+1} + C_{m+1}^1 \sum_{i+j=k+1} C_{m+i-1}^{m-1} \bar{P}_1^i h_j \right) \\ & + P_m \sum_{i+j=k-m+1} \left(C_{m+1}^2 C_{m+i-2}^{m-2} \bar{P}_1^i \sum_{i_1+i_2=j} h_{i_1} h_{i_2} + C_{m+1}^1 C_{m+i-1}^{m-1} \bar{P}_1^i \alpha_j \right) + P_{2m+1} C_{2m+k+1}^{2m+1} \bar{P}_1^k, \end{aligned}$$

solving this equation we get

$$a_{2m+1+k}=\bar{P}_1^{2m+k+1}+h_{m+k+1}+\alpha_{k+1}+\delta_k,\qquad (k=0,1,2,\dots,m-1), \tag{3.16}$$

where h_{m+k+1} is the solution of equation (2.2), in which $m+k+1$ taking the place of k, δ_k is the solution of equation (2.9), α_{k+1} is the solution of equation (2.11), in which $k+1$ taking the place of k. Thus, by Lemma 2.2, Lemma 2.3 and Lemma 2.4 we get

$$h_{m+k+1}=\sum_{j=0}^{m+k+1}h^j_{m+k+1}\bar{P}_1^{m+k+1-j}\overline{\bar{P}_1^jP_m}=\hat{h}_{m+k+1}\bar{P}_1^{2m+k+1}+\cdots, \tag{3.17}$$

where

$$h^j_{m+k+1}=\frac{2}{m+k+1-j}\sum_{i=j}^{m+k}h^j_i=(m+k-j)C^j_{m+j-2},\qquad (i=0,1,2,\dots,m+k),$$

$$h^{m+k+1}_{m+k+1}=C^m_{2m+k+1}-\sum_{j=0}^{m+k}h^j_{m+k+1}=C^{m-2}_{2m+k-1},\ \hat{h}_{m+k+1}=\lambda_m\sum_{j=0}^{m+k+1}\frac{m}{m+j}h^j_{m+k+1}.$$

$$\alpha_{k+1}=\hat{\alpha}_{k+1}\bar{P}_1^{2m+k+1}+\cdots \tag{3.18}$$

$$\begin{aligned}\hat{\alpha}_{k+1}=\frac{\lambda_m^2}{2m+k+1}\Bigg(&\sum_{i+j=k}\hat{h}_i\hat{h}_j+2\sum_{j=0}^{k}\hat{\alpha}_j\\ &+C^1_mC^2_{m+1}\sum_{i+j+l=k-m+1}C^{m-2}_{m+i-2}\hat{h}_j\hat{h}_l+C^1_mC^1_{m+1}\sum_{i+j=k-m+1}C^{m-1}_{m+i-1}\hat{\alpha}_j\Bigg).\end{aligned}$$

$$\delta_k=\sum_{j=0}^{k}\delta^j_k\bar{P}_1^{k-j}\overline{\bar{P}_1^jP_{2m+1}}, \tag{3.19}$$

$$\delta^j_k=\frac{2}{k-j}\sum_{i=j}^{k-1}\delta^j_i=(k-j+1)C^j_{2m+j-1},\qquad (j=0,1,2,\dots,k-1),$$

$$\delta^k_k=C^{2m+1}_{2m+k+1}-\sum_{j=0}^{k-1}\delta^j_k=C^k_{2m+k-1}.$$

By (3.17) and (3.18) we see that h_{m+k+1} and α_{k+1} are polynomials on $\bar{P}_1$ of degree $2m+k+1$, so they are 2π-periodic functions. As $\delta^k_k=C^k_{2m+k-1}\neq 0$, so from (3.16) we see that if $a_{2m+1+k}(2\pi)=0$ then

$$\int_0^{2\pi}\bar{P}_1^kP_{2m+1}d\theta=0,\qquad (k=0,1,2,\dots,m-1). \tag{3.20}$$

Using (3.3) and (3.11)–(3.15), for $0 \le k \le m-2$, we have

$$
\begin{aligned}
a'_{3m+k+1} = {} & P_1((3m+k+1)\bar{P}_1^{3m+k} + 2\sum_{i+j=2m+k} \bar{P}_1^i h_j + \sum_{i+j=m+k} (h_i h_j + 2\bar{P}_1^i \alpha_j) \\
& + 2\sum_{i+j=m+k-1} \bar{P}_1^i \delta_j + 2\sum_{i+j=k} h_i \alpha_j + 2\sum_{i+j=k-1} (\bar{P}_1^i \beta_j + h_i \delta_j)) \\
& + P_m \Bigg(C_{3m+k+1}^m \bar{P}_1^{2m+k+1} + C_{m+1}^1 \sum_{i+j=m+k+1} C_{m+i-1}^{m-1} \bar{P}_1^i h_j + C_{m+1}^2 \sum_{i+j+l=k+1} C_{m+i-2}^{m-2} \bar{P}_1^i h_j h_l \\
& \quad + C_{m+1}^1 \sum_{i+j=k+1} C_{m+i-1}^{m-1} \bar{P}_1^i \alpha_j + C_{m+1}^1 \sum_{i+j=k} C_{m+i-1}^{m-1} \bar{P}_1^i \delta_j \Bigg) \\
& + P_{2m+1}(C_{3m+k+1}^{2m+1} \bar{P}_1^{m+k} + C_{2m+2}^1 \sum_{i+j=k} C_{2m+i}^{2m} \bar{P}_1^i h_j),
\end{aligned}
$$

solving this equation we get

$$
a_{3m+k+1} = \bar{P}_1^{3m+k+1} + h_{2m+k+1} + \alpha_{m+k+1} + \delta_{m+k} + \beta_k, \tag{3.21}
$$

where h_{2m+k+1} is the solution of equation (2.2), in which $2m+k+1$ taking the place of k, δ_{m+k} is the solution of equation (2.9), in which $m+k$ taking place of k, α_{m+k+1} is the solution of equation (2.11), in which $m+k+1$ taking the place of k, β_k is the solution of the equation (2.14). Thus, by Lemma 2.2, Lemma 2.3 and Lemma 2.4 and Lemma 2.5 we get

$$
h_{2m+k+1} = \sum_{j=0}^{2m+k+1} h_{2m+k+1}^j \bar{P}_1^{2m+k+1-j} \overline{\bar{P}_1^j P_m} = \hat{h}_{2m+k+1} \bar{P}_1^{3m+k+1} + \cdots, \tag{3.22}
$$

where

$$
h_{2m+k+1}^j = \frac{2}{2m+k+1-j} \sum_{i=j}^{2m+k} h_i^j = (2m+k+2-j) C_{m+j-2}^j, \qquad (i = 0, 1, 2, \ldots, 2m+k),
$$

$$
h_{2m+k+1}^{2m+k+1} = C_{3m+k+1}^m - \sum_{j=0}^{2m+k} h_{2m+k+1}^j = C_{3m+k-1}^{2m+k+1},
$$

$$
\hat{h}_{2m+k+1} = \lambda_m \sum_{j=0}^{2m+k+1} \frac{m}{m+j} h_{2m+k+1}^j.
$$

$$
\alpha_{m+k+1} = \hat{\alpha}_{m+k+1} \bar{P}_1^{3m+k+1} + \cdots, \tag{3.23}
$$

$$
\begin{aligned}
\hat{\alpha}_{m+k+1} = \frac{\lambda_m^2}{3m+k+1} \Bigg(& \sum_{i+j=m+k} \hat{h}_i \hat{h}_j + 2\sum_{j=0}^{m+k} \hat{\alpha}_j + 2\sum_{i+j=k} \hat{h}_i \hat{\alpha}_j + C_m^1 C_{m+1}^2 \sum_{i+j+l=k+1} C_{m+i-2}^{m-2} \hat{h}_j \hat{h}_l \\
& + C_m^1 C_{m+1}^1 \left(\sum_{i+j=k+1} C_{m+i-1}^{m-1} \hat{\alpha}_j + \sum_{i+j=m+k+1} C_{m+i-1}^{m-1} \hat{h}_j \right) \Bigg).
\end{aligned}
$$

$$
\delta_{m+k} = \sum_{j=0}^{m+k} \delta_{m+k}^j \bar{P}_1^{m+k-j} \overline{\bar{P}_1^j P_{2m+1}}, \tag{3.24}
$$

$$\delta^j_{m+k} = \frac{2}{m+k-j}\sum_{i=j}^{m+k-1}\delta^j_i = (m+k-j+1)C^j_{2m+j-1}, \qquad (j=0,1,2,\ldots,m+k-1),$$

$$\delta^{m+k}_{m+k} = C^{2m+1}_{3m+k+1} - \sum_{j=0}^{m+k-1}\delta^j_k = C^{m+k}_{3m+k-1}, \qquad \delta^0_0 = 1.$$

$$\beta_k = \sum_{j=0}^{k}\beta^j_k \bar{P}_1^{m+k-j}\overline{\bar{P}_1^j P_{2m+1}} + \beta^{m+j}_k \bar{P}_1^{k-j}\overline{\bar{P}_1^{m+j}P_{2m+1}} + \cdots, \tag{3.25}$$

where

$$\beta^j_k = \frac{1}{m+k-j}\left(2\sum_{j=0}^{k-j-1}\hat{h}_j\delta^j_{k-1-j} + 2\sum_{i=j}^{k-1}\beta^j_i + m\lambda_m C^1_{m+1}C^j_{2m+j-1}C^{k-j}_{m+k-j+1}\right),$$
$$j=(0,1,2,\ldots,k-2),$$

$$\begin{aligned}
\beta^{k-1}_k &= \frac{1}{m+1}(2\hat{h}_0\delta^{k-1}_{k-1} + 2\beta^{k-1}_{k-1} + m\lambda_m C^1_{m+1}C^1_{m+2}C^{k-1}_{2m+k-2}),\\
\beta^k_k &= \lambda_m C^1_{m+1}C^k_{2m+k-1},\\
\beta^{m+j}_k &= \frac{2}{k-j}\sum_{i=j}^{k-1}\beta^{m+j}_i, \qquad (j=0,1,2,\ldots,k-1),\\
\beta^{m+k}_k &= C^1_{2m+2}\sum_{i+j=k}C^{2m}_{2m+i}\hat{h}_j - \beta^k_k - \sum_{j=0}^{k-1}(\beta^j_k + \beta^{m+j}_k).
\end{aligned}$$

By (3.22) and (3.23) we see that h_{2m+k+1} and α_{m+k+1} are polynomials on $\bar{P}_1$ of degree $3m+k+1$, so they are $2\pi-$periodic functions, by (3.21) and (3.24) and (3.25) we see that if $a_{3m+1+k}(2\pi) = 0$, then

$$(\delta^{m+k}_{m+k} + \beta^{m+k}_k)\int_0^{2\pi}\bar{P}_1^{m+k}P_{2m+1}d\theta = 0, \qquad (k=0,1,2,\ldots,m-2).$$

If $m+k$ is an even integer, then $\bar{P}_1^{m+k}P_{2m+1}$ is an odd polynomial function in $\cos\theta, \sin\theta$, so $\int_0^{2\pi}\bar{P}_1^{m+k}P_{2m+1}d\theta = 0$. When $m+k$ is an odd integer, by the hypothesis, $\delta^{m+k}_{m+k} + \beta^{m+k}_k \neq 0$, then

$$\int_0^{2\pi}\bar{P}_1^{m+k}P_{2m+1}d\theta = 0, \qquad (k=0,1,2,\ldots,m-2). \tag{3.26}$$

For $0 \le k \le 2$, using (3.3) and (3.11)–(3.15) we get

$$\begin{aligned}
a'_{4m+k} = P_1\Bigg(&(4m+k)\bar{P}_1^{4m+k-1} + 2\sum_{i+j=3m+k-1}\bar{P}_1^i h_j + \sum_{i+j=2m+k-1}(h_ih_j + 2\bar{P}_1^i\alpha_j) + 2\sum_{i+j=2m+k-2}\bar{P}_1^i\delta_j\\
&+ 2\sum_{i+j=m+k-1}h_i\alpha_j + 2\sum_{i+j=m+k-2}(\bar{P}_1^i\beta_j + h_i\delta_j) + \sum_{i+j=k-1}\alpha_i\alpha_j + 2\sum_{i+j=k-2}(h_i\beta_j + \alpha_i\delta_j)\Bigg)\\
&+ P_m\Bigg(C^m_{4m+k}\bar{P}_1^{3m+k} + C^1_{m+1}\sum_{i+j=2m+k}C^{m-1}_{m+i-1}\bar{P}_1^i h_j + C^2_{m+1}\sum_{i+j+l=m+k}C^{m-2}_{m+i-2}\bar{P}_1^i h_j h_l
\end{aligned}$$

$$+ C^1_{m+1} \sum_{i+j=m+k} C^{m-1}_{m+i-1}\bar{P}^i_1\alpha_j + C^1_{m+1} \sum_{i+j=m+k-1} C^{m-1}_{m+i-1}\bar{P}^i_1\delta_j + C^3_{m+1} \sum_{i+j+r+s=k} C^{m-3}_{m+i-3}\bar{P}^i_1 h_j h_r h_s$$

$$+ C^1_{m+1}C^1_m \sum_{i+j+l=k} C^{m-2}_{m+i-2}\bar{P}^i_1 h_j\alpha_l + C^1_{m+1} \sum_{i+j=k-1} C^{m-1}_{m+i-1}\bar{P}^i_1\beta_j + C^1_{m+1}C^1_m \sum_{i+j+l=k-1} C^{m-2}_{m+i-2}\bar{P}^i_1 h_j\delta_l\Bigg)$$

$$+ P_{2m+1}\Bigg(C^{2m+1}_{4m+k}\bar{P}^{2m+k-1}_1 + C^1_{2m+2} \sum_{i+j=m+k-1} C^{2m}_{2m+k}\bar{P}^i_1 h_j + \sum_{i+j+l=k-1} C^2_{2m+2}C^{2m-1}_{2m+i-1}\bar{P}^i_1 h_j h_l$$

$$+ C^1_{2m+2} \sum_{i+j=k-1} C^{2m}_{2m+i}\bar{P}^i_1\alpha_j + C^1_{2m+2} \sum_{i+j=k-2} C^{2m}_{2m+i}\bar{P}^i_1\delta_j\Bigg),$$

solving this equation we get

$$a_{4m+k} = \bar{P}^{4m+k}_1 + h_{3m+k} + \alpha_{2m+k} + \tilde{\alpha}_k + \delta_{2m+k-1} + \beta_{m+k-1} + \tilde{\beta}_k, \qquad (k=0,1,2), \tag{3.27}$$

where

$$h'_{3m+k} = 2P_1 \sum_{i+j=3m+k-1} \bar{P}^i_1 h_j + P_m C^m_{4m+k}\bar{P}^{3m+k}_1,$$

by Lemma 2.2 we obtain

$$h_{3m+k} = \sum_{j=0}^{3m+k} h^j_{3m+k}\bar{P}^{3m+k-j}_1\overline{\bar{P}^j_1 P_m} = \hat{h}_{3m+k}\bar{P}^{4m+k}_1 + \sum_{j=1}^{4m+k-1} \hat{h}_{3m+k\,j}\bar{P}^j_1, \tag{3.28}$$

in which $\hat{h}_{3m+k\,j}$ $(j=1,2,\dots,4m+k-1)$ are real numbers,

$$h^j_{3m+k} = \frac{2}{3m+k-j}\sum_{i=j}^{3m+k-1} h^j_i = (3m+k-j+1)C^j_{m+j-2}, \qquad (j=0,1,2,\dots,3m+k-1),$$

$$h^{3m+k}_{3m+k} = C^{m-1}_{4m+k-1} - \sum_{j=0}^{3m+k-1} h^j_{3m+k} = C^{3m+k}_{4m+k-2}, \qquad \hat{h}_{3m+k} = \lambda_m \sum_{j=0}^{3m+k} \frac{m}{m+j} h^j_{3m+k}.$$

$$\alpha'_{2m+k} = P_1 \sum_{i+j=2m+k-1} (h_i h_j + 2\bar{P}^i_1\alpha_j) + 2P_1 \sum_{i+j=m+k-1} h_i\alpha_j + P_m C^1_{m+1} \sum_{i+j=2m+k} C^{m-1}_{m+i-1}\bar{P}^i_1 h_j$$
$$+ P_m C^2_{m+1} \sum_{i+j+l=m+k} C^{m-2}_{m+i-2}\bar{P}^i_1 h_j h_l + P_m C^1_{m+1} \sum_{i+j=m+k} C^{m-1}_{m+i-1}\bar{P}^i_1\alpha_j,$$

by Lemma 2.4 we have

$$\alpha_{2m+k} = \hat{\alpha}_{2m+k}\bar{P}^{4m+k}_1 + \sum_{j=0}^{4m+k-1} \hat{\alpha}_{2m+k\,j}\bar{P}^j_1,$$

where $\hat{\alpha}_{2m+k\,j}$ $(j=0,1,2,\dots,2m+k-1)$ are real numbers, $\hat{\alpha}_{2m+k}$ is expressed by (2.13) with taking the place of k by $2m+k$.

$$\tilde{\alpha}'_k = P_1 \sum_{i+j=k-1} \alpha_i\alpha_j + P_m C^3_{m+1} \sum_{i+j+r+s=k} C^{m-3}_{m+i-3}\bar{P}^i_1 h_j h_r h_s + P_m C^1_{m+1}C^1_m \sum_{i+j+l=k} C^{m-2}_{m+i-2}\bar{P}^i_1 h_j\alpha_l,$$

substituting (2.3) and (2.12) into it and solving this equation we get $\tilde{\alpha}_k = \hat{\tilde{\alpha}}_k \bar{P}_1^{4m+k} + \cdots$,

$$\hat{\tilde{\alpha}}_k = \frac{1}{4m+k} \sum_{i+j=k-1} \hat{\alpha}_i \hat{\alpha}_j + m\lambda_m C_{m+1}^3 \sum_{i+j+r+s=k} C_{m+i-3}^{m-3} \hat{h}_j \hat{h}_r \hat{h}_s + m\lambda_m C_{m+1}^1 C_m^1 \sum_{i+j+l=k} C_{m+i-2}^{m-2} \hat{h}_j \hat{\alpha}_l.$$

Thus,

$$\alpha_{2m+k} + \tilde{\alpha}_k = (\hat{\alpha}_{2m+k} + \hat{\tilde{\alpha}}_k)\bar{P}_1^{4m+k} + \cdots, \qquad (k = 0,1,2). \tag{3.29}$$
$$\delta'_{2m+k-1} = 2P_1 \sum_{i+j=2m+k-2} \bar{P}_1^i \delta_j + P_{2m+1} C_{4m+k}^{2m+1} \bar{P}_1^{2m+k-1},$$

by Lemma 2.3 we obtain

$$\delta_{2m+k-1} = \sum_{j=0}^{2m+k-1} \delta_{2m+k-1}^j \bar{P}_1^{2m+k-1-j} \overline{\bar{P}_1^j P_{2m+1}}, \tag{3.30}$$
$$\delta_{2m+k-1}^j = (2m+k-j) C_{2m+j-1}^j, \; \delta_{2m+k-1}^{2m+k-1} = C_{4m+k-2}^{2m+k-1}.$$
$$\beta'_{m+k-1} = 2P_1 \sum_{i+j=m+k-2} (\bar{P}_1^i \beta_j + h_i \delta_j)$$
$$+ P_m C_{m+1}^1 \sum_{i+j=m+k-1} C_{m+i-1}^{m-1} \bar{P}_1^i \delta_j + P_{2m+1} C_{2m+2}^1 \sum_{i+j=m+k-1} C_{2m+i}^{2m} \bar{P}_1^i h_j,$$

by Lemma 2.5, we get

$$\beta_{m+k-1} = \sum_{j=0}^{m+k-1} \beta_{m+k-1}^j \bar{P}_1^{2m+k-1-j} \overline{\bar{P}_1^j P_{2m+1}} + \beta_{m+k-1}^{m+j} \bar{P}_1^{m+k-j-1} \overline{\bar{P}_1^{m+j} P_{2m+1}} + \cdots, \tag{3.31}$$
$$\beta_m = \sum_{j=0}^{m} \beta_m^j \bar{P}_1^{2m-j} \overline{\bar{P}_1^j P_{2m+1}} + \beta_m^{m+j} \bar{P}_1^{m-j} \overline{\bar{P}_1^{m+j} P_{2m+1}} + \cdots,$$
$$\beta_{m+1} = \sum_{j=0}^{m+1} \beta_{m+1}^j \bar{P}_1^{2m+1-j} \overline{\bar{P}_1^j P_{2m+1}} + \beta_{m+1}^{m+j} \bar{P}_1^{m+1-j} \overline{\bar{P}_1^{m+j} P_{2m+1}} + \cdots,$$

where β_{m+k-1}^j $(j = 0,1,2,\ldots,m+k-1)$ are expressed by (2.16)–(2.18).

$$\tilde{\beta}'_k = 2P_1 \sum_{i+j=k-2} (h_i \beta_j + \alpha_i \delta_j) + C_{m+1}^1 P_m \sum_{i+j=k-1} C_{m+i-1}^{m-1} \bar{P}_1^i \beta_j$$
$$+ P_m C_{m+1}^1 C_m^1 \sum_{i+j+l=k-1} C_{m+i-2}^{m-2} \bar{P}_1^i h_j \delta_l + P_{2m+1} \left(\sum_{i+j+l=k-1} C_{2m+2}^2 C_{2m+i-1}^{2m-1} \bar{P}_1^i h_j h_l \right.$$
$$\left. + C_{2m+2}^1 \sum_{i+j=k-1} C_{2m+i}^{2m} \bar{P}_1^i \alpha_j + C_{2m+2}^1 \sum_{i+j=k-2} C_{2m+i}^{2m} \bar{P}_1^i \delta_j \right), \qquad (k = 0,1,2),$$

from this and Lemma 2.7 we get

$$\tilde{\beta}_0 = 0,$$
$$\tilde{\beta}_1 = \tilde{\beta}_1^0 \bar{P}_1^{2m} \bar{P}_{2m+1} + \tilde{\beta}_1^m \bar{P}_1^m \overline{\bar{P}_1^m P_{2m+1}} + \tilde{\beta}_1^{2m} \overline{\bar{P}_1^{2m} P_{2m+1}} + \cdots, \tag{3.32}$$
$$\tilde{\beta}_2 = \tilde{\beta}_2^0 \bar{P}_1^{2m+1} \bar{P}_{2m+1} + \tilde{\beta}_2^1 \bar{P}_1^{2m} \overline{\bar{P}_1 P_{2m+1}} + \tilde{\beta}_2^m \bar{P}_1^{m+1} \overline{\bar{P}_1^m P_{2m+1}}$$
$$+ \tilde{\beta}_2^{m+1} \bar{P}_1^m \overline{\bar{P}_1^{m+1} P_{2m+1}} + \tilde{\beta}_2^{2m+1} \overline{\bar{P}_1^{2m+1} P_{2m+1}} + C_{m+1}^1 \bar{P}_{2m+1}^2 + \cdots, \tag{3.33}$$

By (3.28) and (3.29) we see that h_{3m+k} and $\alpha_{2m+k} + \tilde{\alpha}_k$ are polynomials on $\bar{P}_1$ of degree $4m + k$, so they are 2π-periodic functions. By (2.31) and (3.28) and (3.31) and (3.32) and (3.33) and (3.20) and (3.26) we see that from $a_{4m}(2\pi) = 0$ implies

$$(\delta_{2m-1}^{2m-1} + \beta_{m-1}^{2m-1}) \int_0^{2\pi} \bar{P}_1^{2m-1} P_{2m+1} d\theta = 0, \tag{3.34}$$

by the hypothesis, $\delta_{2m-1}^{2m-1} + \beta_{m-1}^{2m-1} \neq 0$, thus, from (3.34) we get

$$\int_0^{2\pi} \bar{P}_1^{2m-1} P_{2m+1} d\theta = 0. \tag{3.35}$$

From $a_{4m+2}(2\pi) = 0$ follows

$$(\delta_{2m+1}^{2m+1} + \beta_{m+1}^{2m+1} + \tilde{\beta}_2^{m+1}) \int_0^{2\pi} \bar{P}_1^{2m+1} P_{2m+1} d\theta = 0, \tag{3.36}$$

as $\delta_{2m+1}^{2m+1} + \beta_{m+1}^{2m+1} + \tilde{\beta}_2^{m+1} \neq 0$, from (3.36) we get

$$\int_0^{2\pi} \bar{P}_1^{2m+1} P_{2m+1} = 0. \tag{3.37}$$

In summary, by (3.6) and (3.20) and (3.26) and (3.35) and (3.37), the condition (3.2) is necessary for the origin to be a center of equation (3.1).

Sufficiency.

Now, we show that the condition (3.2) is also sufficient center condition.

By Lemma 2.1 and (3.2) we have

$$P_m = P_1 \sum_{j=1}^{m} j\lambda_j \bar{P}_1^{j-1}, \qquad P_{2m+1} = P_1 \sum_{j=0}^{2m} \mu_j \bar{P}_1^{j}.$$

where λ_j, μ_j are real numbers. Thus by Theorem 1.2, the $r = 0$ is a center of (3.1), moreover this center is CC-center, i.e, under condition of present theorem the composition conjecture is valid for equation (3.1).

In summary, Theorem 3.1 has been proved. □

Remark 3.2. By Theorem 3.1, we can derive all the focal quantity formulas of system (1.5) with arbitrary m and they contain exactly $\frac{[m]}{2} + m + 2$ relations (3.2) and they are more concise and beautiful than the results calculated by computer.

Corollary 3.3. *If $m > 1$ and $\lambda_m \geq 0$, then the $r = 0$ is a center of (3.1), if and only if the Moments Conditions:*

$$\int_0^{2\pi} \bar{P}_1^i P_m d\theta = 0, \quad (i = 0, 1, 2, \ldots, m), \qquad \int_0^{2\pi} \bar{P}_1^{2j+1} P_{2m+1} d\theta = 0, \quad (j = 0, 1, 2, \ldots, m).$$

are satisfied.

Proof. By Lemma 2.3 we see $\delta_k^k > 0$, from Lemma 2.6 follows that $\beta_k^{m+k} \geq 0$, using Lemma 2.7 we have $\tilde{\beta}_2^{2m+1} \geq 0$, so, $\delta_{m+k}^{m+k} + \beta_k^{m+k} \neq 0\,(0 \leq k \leq m-1)$ and $\delta_{2m+1}^{2m+1} + \beta_{m+1}^{2m+1} + \tilde{\beta}_2^{2m+1} \neq 0$, by Theorem3.1 the conclusion of the present corollary is true. □

Consider equation

$$r' = r(P_1 r + P_2 r^2 + P_5 r^5), \tag{3.38}$$

where $P_k = \sum_{i+j=k} p_{ij} \cos^i \theta \sin^j \theta$, $(k = 1, 2, 5)$ and $P_1 \neq 0$ and p_{ij} are real numbers.

In (2.6) taking $m = 2$, we get

$$\hat{h}_0 = \lambda_2, \qquad \hat{h}_1 = \frac{8}{3}\lambda_2, \qquad \hat{h}_2 = \frac{29}{6}\lambda_2,$$

by [21], $\lambda_2 = \frac{-p_{20}}{2p_{10}p_{01}}$ (if $p_{10}p_{01} \neq 0$) or $\lambda_2 = \frac{p_{11}}{2(p_{10}^2 - p_{01}^2)}$ (if $p_{10}^2 - p_{01}^2 \neq 0$). In Lemma 2.3, taking $m = 2$, $k = 3$, $k = 5$ we obtain

$$\delta_3^3 = 20, \qquad \delta_5^5 = 56.$$

In (2.26) taking $m = 2$, $k = 1, 2, 3$, we get

$$\beta_0^2 = 3\lambda_2, \quad \beta_1^3 = \frac{52}{3}\lambda_2, \quad \beta_2^4 = \frac{117}{2}\lambda_2, \quad \beta_3^5 = \frac{452}{3}\lambda_2, \quad \tilde{\beta}_2^5 = 56\lambda_2^2.$$

Theorem 3.1 implies the following corollary.

Corollary 3.4 ([23]). *If $(5 + \frac{13}{3}\lambda_2)(14 + \frac{113}{3}\lambda_2 + 14\lambda_2^2) \neq 0$, then the $r = 0$ is a center of (3.38) if and only if,*

$$\int_0^{2\pi} \bar{P}_1^{2i} P_2 d\theta = 0, \quad (i = 0, 1), \qquad \int_0^{2\pi} \bar{P}_1^{2j+1} P_5 d\theta = 0, \quad (j = 0, 1, 2), \tag{3.39}$$

where $\lambda_2 = \frac{-p_{20}}{2p_{10}p_{01}}$ (if $p_{10}p_{01} \neq 0$) or $\lambda_2 = \frac{p_{11}}{2(p_{10}^2 - p_{01}^2)}$ (if $p_{10}^2 - p_{01}^2 \neq 0$).

Remark 3.5. By Corollary 3.4 follows that if $(5 + \frac{13}{3}\lambda_2)(14 + \frac{113}{3}\lambda_2 + 14\lambda_2^2) \neq 0$, then system

$$\begin{cases} x' = -y + x(\sum_{i+j=1} p_{ij}x^i y^j + \sum_{i+j=2} p_{ij}x^i y^j + \sum_{i+j=5} p_{ij}x^i y^j), \\ y' = x + y(\sum_{i+j=1} p_{ij}x^i y^j + \sum_{i+j=2} p_{ij}x^i y^j + \sum_{i+j=5} p_{ij}x^i y^j) \end{cases}$$

has a center at $(0, 0)$, if and only if its five focal values are equal to zero, i.e.,

$$\begin{aligned} p_{20} + p_{02} &= 0, \\ p_{20}(p_{01}^2 - p_{10}^2) - p_{11}p_{10}p_{01} &= 0, \\ p_{01}(5p_{50} + p_{32} + p_{14}) - p_{10}(5p_{05} + p_{23} + p_{41}) &= 0, \\ p_{10}^3(p_{23} + 10p_{05}) - 3p_{10}^2 p_{01}(2p_{14} + p_{32}) + 3p_{10}p_{01}^2(p_{23} + 2p_{41}) - p_{01}^3(p_{32} + 10p_{50}) &= 0, \\ p_{50}p_{01}^5 - p_{41}p_{01}^4 p_{10} + p_{32}p_{01}^3 p_{10}^2 - p_{23}p_{01}^2 p_{10}^3 + p_{14}p_{01}p_{10}^4 - p_{05}p_{10}^5 &= 0. \end{aligned}$$

Taking $p_{10} = p_{01} = 1$, $p_{20} = -2$, $p_{11} = 0$, $p_{02} = 2$, $p_{50} = p_{05} = a$, $p_{41} = p_{14} = b$, $p_{23} = p_{32} = c$, $\lambda_2 = 1$, the above conditions are satisfied, from Remark 3.5 we get the following example.

Example 3.6. Differential system

$$\begin{cases} x' = -y + x(x + y - 2x^2 + 2y^2 + ax^5 + bx^4y + cx^3y^2 + cx^2y^3 + bxy^4 + ay^5), \\ y' = x + y(x + y - 2x^2 + 2y^2 + ax^5 + bx^4y + cx^3y^2 + cx^2y^3 + bxy^4 + ay^5) \end{cases}$$

has a CC-center at $(0, 0)$. Here a, b, c are arbitrary numbers.

Acknowledgements

Funding: This study was funded by the National Natural Science Foundation of China (61773017, 11571301) and the National Natural Science Foundation of Province Jiangsu (BK20161327).

Conflict of interest: The authors declare that they have no conflict of interest.

References

[1] A. Algaba, M. Reyes, Computing center conditions for vector fields with constant angular speed, *J. Comput. Appl. Math.* **1154**(2003), 143–159. https://doi.org/10.1016/S0377-0427(02)00818-X; MR3091844; Zbl 032.34026.

[2] M. A. M. Alwash, Composition conditions for two-dimentionalpolynomial systems, *Differ. Equ. Appl.* **1**(2013), No. 5, 1–12. https://doi.org/10.7153/dea-05-01; MR3087263; Zbl 1287.34022.

[3] M. A. M. Alwash, Computing the Poincaré–Lyapunov constants, *Differ. Equ. Dyn. Syst.* **6**(1998), 349–361. MR1664026; Zbl 0991.34029.

[4] M. A. M. Alwash, On the center conditions of certain cubic systems, *Proc. Amer. Math. Soc.* **126**(1998), 3335–3336. https://doi.org/10.1090/S0002-9939-98-04715-7; MR1485453; Zbl 0903.34026.

[5] M. A. M. Alwash, The composition conjecture for Abel equation, *Expos. Math.* **27**(2009), 241–250. https://doi.org/10.1016/j.exmath.2009.02.0020; MR2555370; Zbl 0618.34026.

[6] M. A. M. Alwash, N. G. Lloyd, Non-autonomous equations related to polynomial two-dimensional systems, *Proc. Roy. Soc. Edinburgh Sect. A* **105**(1987), 129–152. https://doi.org/10.1017/S0308210500021971; MR0890049; Zbl 0618.34026.

[7] M. A. M. Alwash, On a condition for a center of cubic non-autonomous equations, *Proc. Roy. Soc. Edinburgh Sect. A* **113**(1989), 289–291. https://doi.org/10.1017/S030821050002415X; MR1037734.

[8] M. A. M. Alwash, Periodic solutions of Abel differential equations, *J. Math. Anal. Appl.* **329**(2007), 1161–1169. https://doi.org/10.1016/j.jmaa.2006.07.039; MR2296914.

[9] L. A. Cherkas, Number of limit cycles of an autonomous second-order system, *Differ. Uravn.* **5**(1976), 666–668. MR0422759; Zbl 0365.34039.

[10] A. Cima, A. Gasull, F. Mañosas, Centers for trigonometric Abel equations, *Qual. Theory Dyn. Syst.* **11**(2012), 19–37. MR2902723; Zbl 1264.34055.

[11] A. Cima, A. Gasull, F. Mañosas, A simple solution of some composition conjectures for Abel equations, *J. Math. Anal. Appl.* **398**(2013) 477–486. https://doi.org/10.1016/j.jmaa.2012.09.006; MR2990073; Zbl 1269.34035.

[12] A. Cima, A. Gasull, F. Mañosas, An explicit bound of the number of vanishing double moments forcing composition, *J. Differential Equations* **255**(2013) 339–350. https://doi.org/10.1016/j.jde.2013.04.009; MR3053469; Zbl 1290.34015.

[13] V. V. Ivanov, E. P. Volokitin, Uniformly isochronous polynomial centers, *Electron. J. Differ. Equ.* **2005**, No. 133, 1–10. MR2181277; Zbl 1094.14019.

[14] J. Devlin, N. G. Lloyd, J. M. Pearson, Cubic systems and Abel equations, *J. Differential Equations* **147**(1998), 435–454. https://doi.org/10.1006/jdeq.1998.3420; MR1633961; Zbl 0911.34020.

[15] N. G. Lloyd, J. M. Pearson, Computing centre conditions for certain cubic systems, *J. Comput. Appl. Math.* **40**(1992), 323–336. https://doi.org/10.1016/0377-0427(92)90188-4; MR1170911; Zbl 0754.65072.

[16] V. A. Lunkevich, S. K. Sibirskii, On the conditions for a center, *Differ. Uravn.* **10**(1965), 53–66.

[17] M. Muzychuk, F. Pakovich, Solution of the polynomial moment problem, *Proc. Lond. Math. Soc.* **99**(2009), No. 3, 633–657. MR2551466.

[18] F. Pakovich, A counterexample to the composition Conjecture, *Proc. Am. Math. Soc.* **130**(2002), 3747–3749. https://doi.org/10.1090/S0002-9939-02-06755-2; MR1920083; Zbl 1008.34024.

[19] J. Xu; Z. Lu, Sufficient and necessary center conditions for the Poincaré systems $P(2, 2n)$ $(n \leq 5)$, *J. Appl. Math.* **2011**, Art. ID 762762, 18 pp. MR2861924.

[20] L. Yang, Y. Tang, Some new results on Abel equations, *J. Math. Anal. Appl.* **261**(2001), 100–112. https://doi.org/10.1006/jmaa.2001.7478; MR1850959; Zbl 0995.34031.

[21] Z. Zhou, The Poincaré center-focus problem for a class of higher order polynomial differential systems, *Math. Meth. Appl. Sci.* **42**(2019), 1804–1818. https://doi.org/10.1002/mma.5475; MR3937634; Zbl 07063454.

[22] Z. Zhou, V. G. Romanovski, The center problem and the composition condition for a family of quartic differential systems, *Electron. J. Qual. Theory. Differ. Equ.* **2018**, No. 15, 1–17. https://doi.org/10.14232/ejqtde.2018.1.15; MR3789406; Zbl 1413.34135.

[23] Z. Zhou, Y. Yan, On the composition center of a class of rigid system, *Bull. Braz. Math. Soc. New Series*, published online, 2019, 1–17. https://doi.org/10.1007/s00574-019-00147-y.