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Infinitely many solutions for $2k$-th order BVP with parameters

Mariusz Jurkiewicz ✉

Military University of Technology, S. Kaliski 2, Warsaw, Poland

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Abstract. In this paper we consider a special case of BVP for higher-order ODE, where, the linear part consists of only even-order derivatives and depends on a set of real parameters. Among many questions related to this problem we are especially interested in the specific one, namely to work out assumptions which provide existence of infinitely many solutions. This task is dealt with by applying a combination of both topological and variational methods, including Chang's version of the Morse theory in particular.

Keywords: multidimensional spectrum, Lidstone boundary value problem, infinite dimensional Morse theory.

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1 Introduction

In general, a problem which is composed of a differential equation of even order whose nonlinear part depends on even derivatives and boundary value conditions of Dirichlet-type is commonly known as Lidstone BVP. Due to some physical and mechanical applications many authors have been studying diverse aspects of Lidstone BVP since the early 80s (see [1, 2, 9–13] for instance). Most of these papers are connected with the existence, uniqueness and multiplicity of solutions.

We begin by fixing a size parameter $k \geq 2$ being an integer number and choosing $i = 1, \dots, k-1$. The boundary value problem (BVP) which is considered here is composed of a nonlinear equation

$$(-1)^k x^{(2k)} + \sum_{j=1}^{k} \lambda_j x^{(2k-2j)} = (-1)^{i-1} f\left(t, x^{(2i-2)}\right), \tag{1.1}$$

together with Dirichlet-type boundary conditions

$$x^{(2j)}(0) = x^{(2j)}(1) = 0, \quad j = 0, \dots, k-1. \tag{1.2}$$

✉ Email: mariusz.jurkiewicz@wat.edu.pl

It is seen that the left-hand side of the equation (1.1) depends on the parameters $\lambda^1, \ldots, \lambda^k$. This means that for different unions of these parameters the respective differential operators can have various properties. For example, some of them may be invertible and some may not have an inverse. It turns out that the latter case is especially troublesome, because then the dimensions of the kernels varies from one to k. To be more specific, there exist some classes of parameters for which the corresponding operators have respectively one-, two-, and so to k- dimensional kernels. Therefore, (1.1)–(1.2) can be viewed as a generalization of Strum–Liouville problem. This in turn, lead us to the notion of *multidimensional-spectrum* which describes a set containing all these classes of parameters which cause invertibility of the linear part of (1.1)–(1.2). The multidimensional-spectrum is put across in [10] and [11]. It is worth mentioning that the former paper focuses on the case where lambdas belong to the spectrum and the nonlinear part is even more general than in (1.1), namely is a Carathéodory's function of k variables. The answer to a question regarding existence of solutions which is raised there is not obvious at all. It is because the non-invertibility of nonlinear part imposes a need to add some sort of integral conditions which are of Landesman–Lazer type. This means in turn that some nonstandard techniques have to be exploited to get desired results.

In this paper we point out assumptions which must be satisfied to provide existence of infinitely many solutions to (1.1)–(1.2), if the linear part is invertible. For this purpose, some methods of both infinite dimensional Morse theory (see [4,6,8,15]) and Leray–Schauder degree are used (see [5]). Furthermore, since the underlying idea is based on variational methods, the left-hand side of (1.1) must be in particular of class C^1. It is because we have to provide the respective differential operator with double-differentiability. In fact, consideration herein should be viewed as a continuation of the research published in [13] and [12]. Both of them solve multi-solutions problems but with this difference, that the former focuses on the existence of at least three solutions, whereas the latter on the existence of infinitely many solutions. Since the main result in [12] is based on the well known Rabinowitz's theorem (see [14, Theorem 6.5]), it was necessary to strictly control both a growth and behavior around zero of a function being on the right-hand side of the equation. It is a crucial difference between [12] and this paper. Namely, in spite of the fact that we also prove the existence of infinitely many solutions here, the combination of variational and topological methods let us to essentially weaken the assumptions and obtain more interesting result.

2 Preliminaries

Before starting the main part of our discussion, we pause to remind the reader of some important facts and ideas which will be used to justify the crucial result of this paper.

Let us define an auxiliary real function Λ, by the formula

$$\Lambda(n) = \sum_{j=1}^{k} (-1)^{k-j} \lambda_j \left(n^2\pi^2\right)^{k-j}, \quad \text{for } n \in \mathbb{N}. \tag{2.1}$$

Here, recall, the number $k \geq 2$ has been fixed in the previous section.

Definition 2.1 (see [10, 11])**.** A point $\lambda=(\lambda_1, \ldots \lambda_k) \in \mathbb{R}^k$ is called a k-dimensional eigenvalue iff the homogeneous problem

$$\begin{cases} (-1)^k x^{(2k)} + \lambda_1 x^{(2k-2)} + \lambda_2 x^{(2k-4)} + \cdots + \lambda_{k-1} x'' + \lambda_k x = 0, \\ x^{(2j)}(0) = x^{(2j)}(1) = 0, \quad \text{for } j = 0 \ldots k-1, \end{cases} \tag{2.2}$$

has a nonzero solution. The set of all such n-tuples is denoted by σ^k.

It is proven in [10] and [11] that σ^k has the following form

$$\sigma^k = \bigcup_{n \in \mathbb{N}} \Omega_n,$$

where a single $\Omega_n := \left\{\lambda \in \mathbb{R}^k \mid \left(n^2\pi^2\right)^k + \Lambda\left(n\right) = 0\right\}$ is a hyperplane in $\mathbb{R}^k$.

Note that if we define the following set

$$\Delta_+ := \bigcap_{n \in \mathbb{N}} \left\{\lambda \mid \Lambda\left(n\right) \geq 0\right\}, \tag{2.3}$$

then it is easy to see that $\Delta_+ \neq \emptyset$. Indeed, putting $D_+ := \left\{\lambda \in \mathbb{R}^k \mid (-1)^{k+s}\lambda_s \geq 0\right\}$, we can see that $D_+ \subset \Delta_+$. Furthermore, due to the fact that the inequality $\left(n^2\pi^2\right)^k + \Lambda\left(n\right) > 0$ holds for all positive integers, we have $\Delta_+ \cap \sigma^k = \emptyset$.

In [10,11], it is explained that if x is a solution to (1.1)–(1.2), then there exists a continuous function $\mathcal{P} : [0,1] \times [0,1] \to \mathbb{R}$, such that

$$x^{(2i-2)}\left(t\right) = \int_0^1 \mathcal{P}\left(t,s\right) f\left(s, x^{(2i-2)}\left(s\right)\right) ds,$$

where

$$\mathcal{P}\left(t,s\right) = \sum_{n=1}^{\infty} \frac{\left(n\pi\right)^{2i-2}}{\left(n^2\pi^2\right)^k + \Lambda\left(n\right)} \cdot \sin\left(n\pi s\right) \sin\left(n\pi t\right).$$

Substituting $y = x^{(2i-2)}$, we obtain

$$y\left(t\right) = \int_0^1 \mathcal{P}\left(t,s\right) f\left(s, y\left(s\right)\right) ds.$$

Consider the operator $\mathbf{T}$, mapping $C\left[0,1\right]$ into itself and defined by

$$\left(\mathbf{T}y\right)\left(t\right) = \int_0^1 \mathcal{P}\left(t,s\right) f\left(s, y\left(s\right)\right) ds.$$

It is easy to notice that $\mathbf{T}$ is a composition of respectively linear and nonlinear operators. To be more specific $\mathbf{T} = \mathbf{P} \circ \mathbf{f}$, where $\left(\mathbf{P}z\right)\left(t\right) = \int_0^1 \mathcal{P}\left(t,s\right) z\left(s\right) ds$ and $\left(\mathbf{f}y\right)\left(s\right) = f\left(s, y\left(s\right)\right)$. The operator $\mathbf{P}$ is semicontinuos and self-adjoint, furthermore $\sigma\left(\mathbf{P}\right) = \sigma_p\left(\mathbf{P}\right) \cup \sigma_c\left(\mathbf{P}\right)$, where

$$\sigma_p\left(\mathbf{P}\right) = \left\{\left(p\pi\right)^{2i-2} \cdot \left[\left(p^2\pi^2\right)^k + \Lambda\left(p\right)\right]^{-1} \,\middle|\, p = 1, 2, \ldots\right\}$$

and $\sigma_c\left(\mathbf{P}\right) = \{0\}$ (see [12] for details). In addition to that, if $\lambda \in \Delta_+$ then $\sigma\left(\mathbf{P}\right) \subset [0, +\infty)$ and this follows that $\mathbf{P}$ is a positive operator. A consequence is that there exists a unique positive and self-adjoint $\mathbf{S}$ such that $\mathbf{S}^2 = \mathbf{P}$ (see [3, Theorem 2.2.10]). From spectral theory in Hilbert spaces we know that $\mathbf{S}$ is an endomorphism of $L^2(0,1)$ which is defined by

$$\left(\mathbf{S}z\right)\left(t\right) = \int_0^1 \mathcal{S}\left(t,s\right) z\left(s\right) ds.$$

Here $\mathcal{S}$ is the kernel of $\mathbf{S}$ and it is of the form

$$\mathcal{S}(t,s) = \sum_{n=1}^{\infty} \frac{(n\pi)^{i-1}}{\sqrt{(n\pi)^{2k} + \Lambda(n)}} \sin(n\pi s)\sin(n\pi t),$$

(comp. [10–12]).

As we noticed earlier, the challenge here is to prove existence of infinitely many solutions to the problem (1.1)–(1.2). In [12], it is explained that to show that the problem has a solution it is enough to substantiate that $\mathbf{T} = \mathbf{P} \circ \mathbf{f}$ has a fixed point. Furthermore, whenever $\lambda \in \Delta_+$ and the operator $\mathbf{S} \circ \mathbf{f} \circ \mathbf{S}$ has infinitely many fixed points in $L^2(0,1)$, $\mathbf{T}$ has infinitely many fixed points in $C[0,1]$.

Let $\varphi : L^2(0,1) \to \mathbb{R}$, be a functional given by the formula

$$\varphi(y) = \frac{1}{2}\|y\|^2 - \int_0^1 F(t, (\mathbf{S}y)(t))\,dt.$$

Then φ is twice continuously differentiable and its first derivative is given by the formula (comp. [12])

$$\varphi'(y) = y - (\mathbf{S} \circ \mathbf{f} \circ \mathbf{S})\,y.$$

It turns out that there is an equivalence between critical points of φ and fixed points of $\mathbf{T}$. This in turn means that there is a relation between existence of solutions to the considered problem and critical points of the above functional. This relation is described by the following lemma.

Lemma 2.2 (see [12])**.** *To show that* (1.1)–(1.2) *has infinitely many solutions it is enough to prove that the functional φ has infinitely many critical points.*

Now we outline some preliminary knowledge about the infinite dimensional Morse theory, which will be used in the proofs of the main theorem.

Let X be a real separable Hilbert space, $\varphi \in C^1(X, \mathbb{R})$ be a functional and H_q be the q-th singular relative homology group. A point $p \in X$ is called a *critical point* of φ if $\varphi'(p) = 0$. The set $K(\varphi) = \{p \in X \mid \varphi'(p) = 0\}$ is called the *critical set*. A real number c is called a *critical value* if $\varphi^{-1}(c) \cap K(\varphi) \neq \emptyset$. Furthermore, define $K(\varphi)_c = \{p \in X \mid \varphi'(p) = 0 \text{ and } \varphi(p) = c\}$ and let $\varphi_a := \varphi^{-1}((-\infty, a])$. A real number is called a *regular value* iff it is not a critical value of φ.

Definition 2.3 (see [4])**.** Let p be an isolated critical point of φ, and let $c = \varphi(p)$, We call

$$C_q(\varphi, p) = H_q\left(\varphi_c \cap U_p, (\varphi_c \setminus \{p\}) \cap U_p\right)$$

the q-th critical group of φ at p, where U_p is a neighborhood of p such that $K(\varphi) \cap (\varphi_c \cap U_p) = \{p\}$.

Below, we will introduce the definition of Morse type numbers.

Definition 2.4 (see [4])**.** Assume that $a < b$ are two regular values of φ, φ has at most finitely many critical points on $\varphi^{-1}([a,b])$ and the rank of the critical group for every critical point is finite. Let $c_1 < c_2 < \cdots < c_m$ be all critical values of φ in $[a,b]$ and

$$K(\varphi)_{c_i} = \left\{p_1^i, p_2^i, \ldots, p_{n_i}^i\right\}, \quad i = 1, 2, \ldots, m.$$

Choose $0 < \varepsilon < \min\{c_1 - a, c_2 - c_1, \ldots, c_m - c_{m-1}, b - c_m\}$. We call

$$M_q = M_q(a,b) = \sum_{i=1}^{m} \operatorname{rank} H_q(\varphi_{c_i+\varepsilon}, \varphi_{c_i-\varepsilon}), \quad q = 0,1,2,\ldots$$

the q-th Morse type number of the functional φ with respect to (a,b).

Definition 2.5 (see [14])**.** A sequence $\{y_n\} \subset X$ is a Palais–Smale sequence for $\varphi \in C^1(X,\mathbb{R})$, if $\{\varphi(y_n)\}$ is bounded while $\varphi'(y_n) \to 0$ as $n \to \infty$.

Definition 2.6 (see [14])**.** We say that $\varphi \in C^1(X,\mathbb{R})$ satisfies (P.S.) condition if any Palais–Smale sequence has a (strongly) convergent subsequence.

For those functionals which satisfy the (P.S.) condition, Morse type numbers are well-defined, i.e., they are independent of the special choice of ε.

Corollary 2.7 (see [4])**.**

$$M_q(a,b) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \operatorname{rank} C_q\left(\varphi, p_j^i\right),$$

for $q = 0,1,2,\ldots$

Definition 2.8 (see [4])**.** Let $a < b$ be regular values of φ. We call

$$\beta_q = \beta_q(a,b) = \operatorname{rank} H_q(\varphi_b, \varphi_a), \quad q = 0,1,2,\ldots,$$

the q-dimensional Betti number.

The following two theorems indicate relation between the Morse-type numbers and the Betti numbers and relation between critical points and the Leray–Schauder degree, respectively.

Theorem 2.9 (see [4])**.** *Assume that $a < b$ are two regular values of $\varphi \in C^1(X,\mathbb{R})$, φ satisfies the (P.S.) condition and it has at most finitely many critical points on $\varphi^{-1}([a,b])$ and the rank of the critical group for every critical point is finite. Then the following inequality holds*

$$\sum_{j=0}^{q} (-1)^{q-j} M_j \geq \sum_{j=0}^{q} (-1)^{q-j} \beta_j, \quad q = 0,1,2,\ldots,$$

and

$$\sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q,$$

if all M_q, β_q, $q = 0,1,2,\ldots$, are finite and the series converge.

Theorem 2.10 (see [4])**.** *Let $\varphi \in C^2(X,\mathbb{R})$ be a functional satisfying the (P.S.) condition. Assume that*

$$\varphi'(y) = y - Fy,$$

where $F : H \to H$ is completely continuous and that p_0 is an isolated critical point of φ. Then there exists a neighborhood U of p_0 such that p_0 is the unique critical point of φ in $\overline{U}$ and

$$\deg_{LS}(I - F, U, 0) = \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q(\varphi, p_0).$$

At the end of this section, we recall Borsuk's antipodal theorem (comp. [5,7]).

Theorem 2.11. *Let Ω be an open bounded and symmetric set in an infinite dimensional Hilbert space X, let $0 \in \Omega$, $F : \overline{\Omega} \to X$ be compact, $G = I - F$ and $0 \notin G(\partial\Omega)$. If for all $\lambda \geq 1$ and for all $x \in \partial\Omega$, $G(-x) \neq \lambda G(x)$ then $\deg_{LS}(I - F, \Omega, 0)$ is odd. In particular, this is true if $F|_{\partial\Omega}$ is odd.*

3 Main results

Before we formulate the main result and pass to the proof, we focus on the question of how the nonlinear part of (1.1) has to behave to provide φ with meeting the Palais–Smale condition.

Lemma 3.1. *Assume that there are $k \in (0,1/2)$ and $N > 0$, such that for $|w| \geq N$, we have*

$$\int_0^w f(t,u)\,du \leq kwf(t,w). \tag{3.1}$$

Then the functional φ satisfies the (P.S.) condition.

Proof. Let $\{y_n\}$ be a Palais–Smale sequence for φ. Firstly, we will show that $\{y_n\}$ is bounded. Suppose that it is unbounded. Then without loss of generality we can assume that $\|y_n\| \to \infty$, as $n \to \infty$. If this condition holds, then due to continuity of the function $(t,w) \mapsto \int_0^w f(t,u)\,du - kwf(t,w)$ in $[0,1] \times [-N.N]$, there exists $L > 0$ such that $\int_0^w f(t,u)\,du \leq kwf(t,w) + L$ in $[0,1] \times \mathbb{R}$. This implies that

$$\begin{aligned}
M > \varphi(y_n) &= \frac{1}{2}\|y_n\|^2 - \int_0^1 \int_0^{(Sy)(t)} f(t,u)\,dudt \\
&\geq \left(\frac{1}{2} - k\right)\|y_n\|^2 + k\left(\|y_n\|^2 - \int_0^1 f(t,(Sy)(t))\,S(y)(t)\,dt\right) - tL \\
&\geq \left(\frac{1}{2} - k\right)\|y_n\|^2 + k\left\langle \varphi'(y_n), y_n \right\rangle_{L^2} - L \\
&\geq \left(\frac{1}{2} - k\right)\|y_n\|^2 - k\left\|\varphi'(y_n)\right\|\|y_n\| - L.
\end{aligned}$$

We divide both sides of the above series of inequalities by $\|y_n\|$ to obtain

$$\frac{M}{\|y_n\|} > \left(\frac{1}{2} - k\right)\|y_n\| - k\left\|\varphi'(y_n)\right\| - \frac{L}{\|y_n\|}.$$

Now, we get contradiction after passing to the limit as $n \to \infty$. Clearly, $\{y_n\}$ is a bounded sequence. This fact together with complete continuity of $\mathbf{S}$ and the condition that $y_n - (\mathbf{S} \circ \mathbf{f} \circ \mathbf{S})\,y_n \to 0$ imply that $\{y_n\}$ has a convergent subsequence. □

Remark 3.2. If f is odd then functions $\mathbb{R} \ni w \mapsto \int_0^w f(t,u)\,du$ and $\mathbb{R} \ni w \mapsto w \cdot f(t,w)$ are even uniformly with respect to $t \in [0,1]$. Therefore, to verify assumption of Lemma 3.1 it suffices to check the inequality (3.1) for $w \geq N > 0$.

Corollary 3.3. *If $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is odd with respect to the second variable and there exists $p > 1$ such that*

$$\lim_{u \to +\infty} \frac{f(t,u)}{u^p} = \gamma, \quad \textit{uniformly for } t \in [0,1].$$

then f satisfies the condition (3.1).

Indeed, let $k \in (1/(p+1), 1/2)$. By using L'Hôpital's rule, we have

$$\begin{aligned}
\lim_{w \to +\infty} \frac{\int_0^w f(t,u)\,du - kwf(t,w)}{w^{p+1}} &= \lim_{w \to +\infty} \left(\frac{f(t,w)}{(p+1)\,w^p} - k \cdot \frac{f(t,w)}{w^p}\right) \\
&= \lim_{w \to +\infty} \frac{f(t,w)}{w^p}\left(\frac{1}{p+1} - k\right) \\
&= \gamma\left(\frac{1}{p+1} - k\right) < 0.
\end{aligned}$$

So, there exists $N > 0$ such that $\int_0^w f(t,u)\,du \le kwf(t,w)$ for $w \ge N$ and $t \in [0,1]$.

Now, we are ready to formulate and prove the main result of this paper.

Theorem 3.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Delta_+$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be both of class C^1 and odd with respect to the second variable. If there exist $k \in (0, 1/2)$ and $M > 0$ such that*

$$0 < \int_0^w f(t,u)\,du \le kwf(t,w), \quad \text{for } w > M, \tag{3.2}$$

then the BVP (1.1)–(1.2) has infinitely many solutions.

Proof. By virtue of Lemma 3.1 and Remark 3.2, the functional φ satisfies the (P.S.) condition. Let us denote

$$F(t,w) = \int_0^w f(t,u)\,du.$$

It is easily seen that oddness of the function $u \mapsto f(t,u)$ implies evenness of $w \mapsto F(t,w)$ uniformly with respect to $t \in [0,1]$. Due to the assumptions, there exist $k \in (0,1/2)$ and $M > 0$, such that

$$qF(t,w) \le wf(t,w), \quad \text{for } |w| > M,$$

where $q := k^{-1}$. Moreover, continuity of f implies that the function $F(t,w) - kwf(t,w)$ is continuous as well. In particular, it is uniformly continuous on the compact set $[0,1] \times [-M, M]$. This, in turn, means that there exists $C_1 > 0$ which fulfills the following condition

$$F(t,w) - kwf(t,w) \le C_1, \quad \text{for } (t,w) \in [0,1] \times [-M,M].$$

Therefore, we have

$$F(t,w) \le kwf(t,w) + C_1, \quad \text{for } (t,w) \in [0,1] \times \mathbb{R}. \tag{3.3}$$

Next, there exist $C_2 > 0$ and $C_3 > 0$ such that

$$F(t,w) \ge C_2 |w|^q - C_3, \quad \text{for } (t,w) \in [0,1] \times \mathbb{R}. \tag{3.4}$$

Indeed, we have for $w > M$ and $t \in [0,1]$

$$\begin{aligned} \frac{\partial}{\partial w}\left(\frac{F(t,w)}{w^q}\right) &= \frac{w^q f(t,w) - qw^{q-1}F(t,w)}{w^{2q}} = \frac{w^q f(t,w) - qw^{q-1}F(t,w)}{w^{2q}} \\ &= w^{q-1}\frac{wf(t,w) - qF(t,w)}{w^{q-1}w^{q+1}} = \frac{wf(t,w) - qF(t,w)}{w^{q+1}} \ge 0. \end{aligned}$$

The above condition means that $w \mapsto F(t,w)/w^q$ is an increasing function for $w > M$. This implies that for $w > M$, we get

$$\frac{F(t,w)}{w^q} \ge \frac{F(t,M)}{M^q} \ge M^{-1} \cdot \min_{t \in [0,1]} F(t,M) =: C_2 > 0,$$

hence

$$F(t,w) \ge C_2 w^q, \quad \text{for } w > M,\ t \in [0,1].$$

Both $w \mapsto F(t,w)$ and $w \mapsto C_2 |w^q|$ are even functions, therefore the last inequality may be transformed to the following one

$$F(t,w) \ge C_2 |w|^q, \quad \text{for } |w| > M,\ t \in [0,1]. \tag{3.5}$$

Since $F(t,w) - C_2 |w^q|$ is continuous on the compact set $[0,1] \times [-M,M]$, there exists a constant $C_3 > 0$, such that

$$F(t,w) - C_2 |w|^q \geq -C_3 \quad \text{for } t \in [0,1], w \in [-M,M].$$

Finally, this condition together with (3.5) imply that

$$F(t,w) \geq C_2 |w|^q - C_3, \quad \text{for } w \in \mathbb{R},\ t \in [0,1].$$

Recall that the challenge here is to show that φ has infinitely many critical points. Assume conversely that it has finitely many critical points $\{y_1, -y_1, \ldots y_n, -y_n\}$ and choose numbers $\theta_1 < 0$, $\theta_2 > 0$ such that

$$\begin{aligned} &\theta_1 < \min\{\varphi(y_1), \ldots \varphi(y_n), -C_1\}, \\ &\theta_2 > \max\{\varphi(y_1), \ldots \varphi(y_n)\}. \end{aligned} \tag{3.6}$$

Let $y \in L^2(0,1)$, then using (3.4) we get the following estimation

$$\varphi(y) = \frac{1}{2}\|y\|^2 - \int_0^1 F(t, (\mathbf{S}y)(t))\,dt \leq \frac{1}{2}\|y\|^2 - C_2 \int_0^1 |(\mathbf{S}y)(t)|^q\,dt + C_3.$$

If, in addition, $y \in S^\infty$ and $\alpha \geq 0$, then applying the above estimation, we obtain that

$$\varphi(\alpha y) \leq \frac{1}{2}\|\alpha y\|^2 - C_2 \int_0^1 |[\mathbf{S}(\alpha y)](t)|^q\,dt + C_3 = \frac{1}{2}\alpha^2 - C_2\alpha^q \int_0^1 |(\mathbf{S}y)(t)|^q\,dt + C_3.$$

Since the operator $\mathbf{S}$ maps $L^2(0,1)$ onto $C[0,1]$, the last integral is finite. This fact together with the assumption that $q > 2$, imply the following condition

$$\lim_{\alpha\to\infty} \varphi(\alpha y) = -\infty, \quad \text{for } y \in S^\infty. \tag{3.7}$$

Note that since φ is continuous, the real function $\varphi_y : \mathbb{R}_+ \ni \alpha \mapsto \varphi(\alpha y)$, is also continuous for every $y \in S^\infty$. This implies that for each $y \in S^\infty$ there exists $\alpha_y > 0$, such that

$$\varphi(\alpha_y y) = \theta_1. \tag{3.8}$$

Further, we show that there exists $\delta > 0$, such that for every $y \in S^\infty$ we have $\alpha_y \geq \delta$. To do this, suppose that there exists a sequence $(y_n) \subset S^\infty$, such that $\alpha_{y_n} \to 0$, as $n \to \infty$. Then the condition $\|y_n\| = 1$ implies that $\alpha_{y_n} y_n \to 0$, as $n \to \infty$. Putting everything together, we have

$$0 > \theta_1 = \lim_{n\to\infty} \varphi(\alpha_{y_n} y_n) = \varphi\left(\lim_{n\to\infty} \alpha_{y_n} y_n\right) = \varphi(0) = 0.$$

This is a contradiction.

We next verify that for every $y \in S^\infty$ there exists exactly one $\alpha_y \geq \delta$, satisfying (3.8). Let $y \in S^\infty$ and $\alpha > 0$, then we obtain

$$\begin{aligned} \frac{d}{d\alpha}\varphi_y(\alpha) &= \langle \varphi'(\alpha y), y\rangle = \langle \alpha y - (\mathbf{S}\circ\mathbf{f}\circ\mathbf{S})(\alpha y), y\rangle \\ &= \langle \alpha y, y\rangle - \langle(\mathbf{S}\circ\mathbf{f}\circ\mathbf{S})(\alpha y), y\rangle = \alpha - \langle(\mathbf{f}\circ\mathbf{S})(\alpha y), \mathbf{S}y\rangle \\ &= \alpha - \int_0^1 f(t, [\mathbf{S}(\alpha y)](t)) \cdot (\mathbf{S}y)(t)\,dt \\ &= \alpha - \frac{1}{\alpha}\int_0^1 f(t, \alpha(\mathbf{S}y)(t)) \cdot \alpha(\mathbf{S}y)(t)\,dt. \end{aligned}$$

Applying estimation (3.3), we have

$$\begin{aligned}
&\alpha - \frac{1}{\alpha}\int_0^1 f\left(t, \alpha\left(\mathbf{S}y\right)(t)\right)\cdot\alpha\left(\mathbf{S}y\right)(t)\,dt \\
&\quad\leq \alpha - \frac{1}{\alpha}q\int_0^1 F\left(t, \alpha\left(\mathbf{S}y\right)(t)\right)dt + \frac{qC_1}{\alpha} \\
&\quad= \frac{q}{\alpha}\left(\alpha\cdot\frac{\alpha}{q} - \int_0^1 F\left(t, \left[\mathbf{S}\left(\alpha y\right)\right](t)\right)dt\right) + \frac{qC_1}{\alpha} \\
&\quad= \frac{q}{\alpha}\left(\frac{1}{q}\left\|\alpha y\right\|^2 - \int_0^1 F\left(t, \left[\mathbf{S}\left(\alpha y\right)\right](t)\right)dt\right) + \frac{qC_1}{\alpha} \\
&\quad< \frac{q}{\alpha}\left(\frac{1}{2}\left\|\alpha y\right\|^2 - \int_0^1 F\left(t, \left[\mathbf{S}\left(\alpha y\right)\right](t)\right)dt\right) + \frac{qC_1}{\alpha} \\
&\quad= \frac{q}{\alpha}\left(\varphi\left(\alpha y\right) + C_1\right).
\end{aligned}$$

Now the formula $(3.6)_1$ leads to the following conclusion

$$\left.\frac{d}{d\alpha}\varphi_y\left(\alpha\right)\right|_{\alpha=\alpha_y} < \frac{q}{\alpha}\left(\varphi\left(\alpha_y y\right) + C_1\right) = \frac{q}{\alpha}\left(\theta_1 + C_1\right) < 0. \tag{3.9}$$

This means that the set $\Gamma_y := \left\{\alpha_y \mid \varphi_y\left(\alpha_y\right) = \theta_1\right\}$ contains only isolated points. Therefore, we can choose $\alpha_y, \beta_y \in \Gamma_y$, $\alpha_y < \beta_y$ such that $\left(\alpha_y, \beta_y\right) \cap \Gamma_y = \emptyset$. Applying (3.9), we see that there exist $\alpha_0 > \alpha_y$ and $\beta_0 < \beta_y$, $\alpha_0 < \beta_0$, such that $\varphi_y\left(\alpha_0\right) < \theta_1$ and $\varphi_y\left(\beta_0\right) > \theta_1$. Since φ_y is continuous it follows that there exists $\gamma_y \in \left(\alpha_0, \beta_0\right)$ such that $\varphi_y\left(\gamma_y\right) = \theta_1$ and finally $\gamma_y \in \Gamma_y$. This is impossible.

Note that a function $\alpha : S^\infty \to [\delta, +\infty)$, given by the formula $\alpha\left(y\right) = \alpha_y$ is continuous. Indeed, if we take a sequence $\left(y_n\right) \subset S^\infty$, such that $y_n \to y_0 \in S^\infty$ and apply the formula (3.4), we have

$$\begin{aligned}
\theta_1 = \varphi\left(\alpha_{y_n} y_n\right) = \varphi\left(\alpha\left(y_n\right)y_n\right) &= \frac{1}{2}\left\|\alpha\left(y_n\right)y_n\right\|^2 - \int_0^1 F\left(t, \left[\mathbf{S}\left(\alpha\left(y_n\right)y_n\right)\right](t)\right) \\
&\leq \frac{1}{2}\left(\alpha\left(y_n\right)\right)^2 - C_2\int_0^1 \left|\left[\mathbf{S}\left(\alpha\left(y_n\right)y_n\right)\right](t)\right|^q dt + C_3 \\
&= \frac{1}{2}\left(\alpha\left(y_n\right)\right)^2 - C_2\left(\alpha\left(y_n\right)\right)^q\int_0^1 \left|\left(\mathbf{S}y_n\right)(t)\right|^q dt + C_3.
\end{aligned} \tag{3.10}$$

Next, we show that there exist $N \in \mathbb{N}$ and $C_4 > 0$ such that

$$\int_0^1 \left|\left(\mathbf{S}y_n\right)(t)\right|^q dt > C_4. \tag{3.11}$$

In [12], it is explained that $\mathbf{H} = \mathbf{S}^2$ in $L^2(0,1)$ and that the kernel of $\mathbf{S}$ is continuous, therefore $\mathbf{S}$ is a continuous operator. This implies that $\mathbf{S}y_n \rightrightarrows \mathbf{S}y_0$ and that $\left|\mathbf{S}y_n\right|^q$ is uniformly convergent to $\left|\mathbf{S}y_0\right|^q$. Assume toward a contradiction that $\mathbf{S}y_0 = 0$. Then due to the fact that $0 \notin \sigma_p\left(\mathbf{S}\right)$ and that $\ker\mathbf{S}$ is trivial, we get $y_0 = 0$. It is impossible, because $y_0 \in S^\infty$, so $\mathbf{S}y_0 \neq 0$ and

$$\int_0^1 \left|\left(\mathbf{S}y_n\right)(t)\right|^q dt \to \int_0^1 \left|\left(\mathbf{S}y_0\right)(t)\right|^q dt > 0.$$

The above inequality means that (3.11) is satisfied. The conditions (3.10) and (3.11) lead us to the following conclusion

$$\theta_1 \leq \frac{1}{2}\left(\alpha\left(y_n\right)\right)^2 - C_2\cdot C_4\cdot\left(\alpha\left(y_n\right)\right)^q + C_3, \quad \text{for } n > N. \tag{3.12}$$

Further, since $\frac{1}{2}x^2 - C_2 \cdot C_4 \cdot x^q + C_3 \to -\infty$ as $x \to +\infty$, the sequence $(\alpha(y_n))$ is bounded, so there exists a subsequence $(\alpha(y_{n_k}))$ such that $\alpha(y_{n_k}) \to a$ and we have

$$\theta_1 = \lim_{k\to\infty} \varphi(\alpha(y_{n_k}) y_{n_k}) = \varphi(ay_0).$$

Then as an immediate consequence of uniqueness, we obtain that $a = \alpha(y_0)$. This means in turn that each convergent subsequence of the sequence is convergent to the number $\alpha(y_0)$ and proofs that α is a continuous function.

Let us choose $0 < \varepsilon < \delta$ such that $\varphi_{\theta_1} \cap B(0,\varepsilon) = \emptyset$ and consider the map $R : [0,1] \times L^2 \backslash B(0,\varepsilon) \to L^2 \backslash B(0,\varepsilon)$, given by the formula

$$R(t,y) = \begin{cases} (1-t)\,y + t\alpha\left(\frac{y}{\|y\|}\right)\frac{y}{\|y\|} & \text{for } y \in \left(L^2\backslash B(0,\varepsilon)\right)\backslash \varphi_{\theta_1}, \\ y & \text{for } y \in \varphi_{\theta_1}. \end{cases}$$

It is easily seen that R is a homotopy and φ_{θ_1} is a strong deformation retract of $L^2\backslash B(0,\varepsilon)$. By both $(3.6)_2$ and the deformation lemma (see [14]), φ_{θ_2} is a strong deformation retract of $L^2(0,1)$, so we have

$$\beta_q = \beta_q(\theta_1,\theta_2) = \operatorname{rank} H_q(\varphi_{\theta_2},\varphi_{\theta_1}) = \operatorname{rank} H_q\left(L^2, L^2\backslash B(0,\varepsilon)\right).$$

It is well known that S^∞ is contractible (see [7]), thus $L^2\backslash B(0,\varepsilon)$ is contractible too. Furthermore, it is easily seen that $B(0,\varepsilon)$ is homotopy equivalent to S^∞. Therefore, we get

$$H_q\left(L^2, L^2\backslash B(0,\varepsilon)\right) \cong 0 \quad \text{and} \quad \beta_q = 0 \quad \text{for } q = 0,1,2,\dots$$

It follows that

$$\sum_{q=0}^{\infty} (-1)^q \beta_q = 0. \tag{3.13}$$

Let us choose $\rho > 0$ such that the balls $B(0,\rho)$, $B(y_i,\rho)$, $B(-y_i,\rho)$, $i = 1\dots n$ are mutually disjoint. According to Corollary 2.7, we obtain the following formula

$$M_q = M_q(\theta_1,\theta_2) = \operatorname{rank} C_q(\varphi,0) + \sum_{i=1}^{n}\left[\operatorname{rank} C_q(\varphi,y_i) + \sum_{i=1}^{n}\operatorname{rank} C_q(\varphi,-y_i)\right],$$

for $q = 0,1,2,\dots$ Further, Borsuk's Theorem (theorem 2.11) implies that

$$\begin{aligned} \text{an odd number} = {} & \deg_{LS}\left(I - \mathbf{S}\circ\mathbf{f}\circ\mathbf{S}, B(0,\rho) \cup \bigcup_{i=1}^{n}\left(B(y_i,\rho)\cup B(-y_i,\rho)\right), 0\right) \\ = {} & \deg_{LS}(I - \mathbf{S}\circ\mathbf{f}\circ\mathbf{S}, B(0,\rho), 0) \\ & + \sum_{i=1}^{n}\deg_{LS}(I - \mathbf{S}\circ\mathbf{f}\circ\mathbf{S}, B(y_i,\rho), 0) \\ & + \sum_{i=1}^{n}\deg_{LS}(I - \mathbf{S}\circ\mathbf{f}\circ\mathbf{S}, B(-y_i,\rho), 0) \end{aligned}$$

If we apply Theorem 2.10, we get

$$\begin{aligned}
\text{an odd number} &= \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q(\varphi, 0) \\
&\quad + \sum_{i=1}^{n} \left[\sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q(\varphi, y_i) + \sum_{q=0}^{\infty} (-1)^q \operatorname{rank} C_q(\varphi, -y_i) \right] \\
&= \sum_{q=0}^{\infty} (-1)^q \left[\sum_{i=1}^{n} \left(\operatorname{rank} C_q(\varphi, y_i) + \operatorname{rank} C_q(\varphi, -y_i) \right) + \operatorname{rank} C_q(\varphi, 0) \right] \\
&= \sum_{q=0}^{\infty} (-1)^q M_q.
\end{aligned} \tag{3.14}$$

To summarize, conditions (3.13) and (3.14) imply that the series $\sum_{q=0}^{\infty} (-1)^q \beta_q$ and $\sum_{q=0}^{\infty} (-1)^q M_q$ are summable. On the other hand, we have

$$\sum_{q=0}^{\infty} (-1)^q \beta_q \neq \sum_{q=0}^{\infty} (-1)^q M_q.$$

This contradicts Theorem 2.9 and means that the functional φ has infinitely many critical points in $L^2(0,1)$. According to the conclusion of Lemma 2.2 we obtain the existence of infinitely many solutions to the considered BVP (1.1)–(1.2). □

Example 3.5. *Let us consider the following problem*

$$\begin{cases} x^{(100)}(t) - x^{(9)}(t) + \pi^2 x''(t) = -\left(x^{(6)}(t)\right)^2 \cdot \arctan\left(x^{(6)}(t) + t\right), \\ x^{(2j)}(0) = x^{(2j)}(1) = 0, \quad j = 0, \dots, 49. \end{cases}$$

It easy to verify that the above BVP satisfies the assumptions of the theorem, thus it has infinitely many solutions.

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