Symmetric nonlinear functional differential equations at resonance

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Abstract. It is shown that a class of symmetric solutions of the scalar nonlinear functional differential equations at resonance with deviations from $\mathbb{R} \to \mathbb{R}$ can be investigated by using the theory of boundary-value problems. Conditions on a solvability and unique solvability are established. Examples are presented to illustrate given results.

Keywords: symmetric solution, solvability, Lyapunov–Schmidt reduction method.

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1   Introduction

The periodic solutions for differential equations or symmetric periodic equations are disseminated widely and could be found in the numerous publications (see, for example, [1,3,6,9–12] and [2,4,5,8]). The main goals of this paper are to show that solvability of a problem concerning a class of symmetric solutions to scalar nonlinear functional differential equations at resonance with perturbations from $\mathbb{R} \to \mathbb{R}$ can be investigated by using the theory of boundary-value problems. Furthermore, we establish conditions on (unique) solvability of scalar nonlinear functional differential equations with symmetries in general form. Several examples illustrate our theory.

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2   Problem formulation

A class of symmetric solutions of the scalar nonlinear functional differential equations is considered here:

$$x'(t) = \varepsilon \left(\sum_{i=1}^{m} \Big(p_i(t)x(\mu_i(t)) - g_i(t)x(\nu_i(t)) \Big) \right.$$

$$\left. + f(x(\tau_1(t)), x(\tau_2(t)), \ldots, x(\tau_m(t)), x(t), t) \right), \quad t \in \mathbb{R}, \quad (2.1)$$

where $t \in \mathbb{R}$, $\varepsilon \neq 0$, $f : \mathbb{R}^{m+2} \to \mathbb{R}$ is continuous, $m \geq 0$, $\mu_i, \nu_i, \tau_i : \mathbb{R} \to \mathbb{R}$, are measurable functions, $p_i, g_i \in L(\mathbb{R}, \mathbb{R})$, $i = 1, 2, \ldots, m$.

Definition 2.1. By a solution of the equation (2.1) we understand an absolutely continuous function $x : \mathbb{R} \to \mathbb{R}$ on every compact intervals which satisfies (2.1) almost everywhere.

The goal of this investigation is to find solutions $x : \mathbb{R} \to \mathbb{R}$ of the equation (2.1) with a symmetric property

$$x(t) = x(\psi(t)), \quad t \in (-\infty, +\infty), \quad (2.2)$$

where ψ is a monotonously increasing C^1-function. The condition (2.2) can describe not only periodic type of solutions, but rather more properties of solutions.

Example 2.2. Property (2.2) holds for the following choices of x and ψ.

$$
\begin{aligned}
x(t) &= (t + \tau)^{2m}, & \psi(t) &= -t - 2\tau, & \tau &\in \mathbb{R}, m \in \mathbb{N}; \\
x(t) &= (t + a)^{2m}(t + b)^{2m}, & \psi(t) &= -t - a - b, & \{a, b\} &\in \mathbb{R}, m \in \mathbb{N}; \\
x(t) &= \sum_{i=1}^{m} \Big((t + a)^{2i} + (t + b)^{2i} \Big), & \psi(t) &= -t - a - b, & \{a, b\} &\in \mathbb{R}, m \in \mathbb{N}; \\
x(t) &= (t + a)^{2m}(t - a)^{2m}, & \psi(t) &= -t, & m &\in \mathbb{N}; \\
x(t) &= (t + a)^{2m} + (t - a)^{2m}, & \psi(t) &= -t, & m &\in \mathbb{N}; \\
x(t) &= \cos t, & \psi(t) &= t + 2\pi; \\
x(t) &= \exp(t + a)^{2m}, & \psi(t) &= -t - 2a, & a &\in \mathbb{R}, m \in \mathbb{N}; \\
x(t) &= \ln(t + a)^{2m}, & \psi(t) &= -t - 2a, & a &\in \mathbb{R}, m \in \mathbb{N}.
\end{aligned}
$$

3   Symmetric properties

We consider a special case, where deviations of the arguments μ_i, ν_i, τ_i, $i = 1, 2, \ldots, m$, and function $f : \mathbb{R}^{m+2} \to \mathbb{R}$ in equation (2.1) are described in the next lemma.

Lemma 3.1. *If there exist such integers j_i, r_i, k_i, $i = 1, 2, \ldots, m$, $m \in \mathbb{N}$, that deviations of the argument μ_i, ν_i and τ_i, $i = 1, 2, \ldots, m$ have the next properties*

$$\mu_i \circ \psi = \psi^{j_i} \circ \mu_i, \quad i = 1, 2, \ldots, m, \quad (3.1)$$

$$\nu_i \circ \psi = \psi^{r_i} \circ \nu_i, \quad i = 1, 2, \ldots, m, \quad (3.2)$$

$$\tau_i \circ \psi = \psi^{k_i} \circ \tau_i, \quad i = 1, 2, \ldots, m, \quad (3.3)$$

then

$$\psi'(t)\left[\sum_{i=1}^{m}\Big(p_i(\psi(t))x(\mu_i(\psi(t))) - g_i(\psi(t))x(\nu_i(\psi(t)))\Big)\right.$$

$$\left. + f(x(\tau_1(\psi(t))), x(\tau_2(\psi(t))), \ldots, x(\tau_m(\psi(t))), x(\psi(t)), \psi(t))\right]$$

$$= \sum_{i=1}^{m}\Big(p_i(t)x(\psi(\mu_i(t))) - g_i(t)x(\psi(\nu_i(t)))\Big)$$

$$+ f(x(\psi(\tau_1(t))), x(\psi(\tau_2(t))), \ldots, x(\psi(\tau_m(t))), x(\psi(t)), t) \quad (3.4)$$

for all $x : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$, *with property* (2.2) *and every* $t \in \mathbb{R}$.

Proof. The property (3.4) is a symmetric property on operators p, g, f appearing in (2.1).

Assume that $x(t) = x(\psi(t))$ is the solution of the equation (2.1). Let us consider the deviation of arguments τ_i, $i = 1, 2, \ldots, m$, then from (2.2)

$$x(\tau_i(t)) = x(\psi(\tau_i(t)))$$

and

$$x(\psi^{k_i}(\tau_i(t))) = x(\psi(\tau_i(t))).$$

If (2.2) is a solution of the equation (2.1) then

$$x'(\psi(t))\psi'(t) = \varepsilon\left(\sum_{i=1}^{m}\Big(p_i(t)x(\psi(\mu_i(t))) - g_i(t)x(\psi(\nu_i(t)))\Big)\right.$$

$$\left. + f\big(x(\psi(\tau_1(t))), x(\psi(\tau_2(t))), \ldots, x(\psi(\tau_m(t))), x(\psi(t)), t\big)\right). \quad (3.5)$$

From the other hand

$$x'(\psi(t))\psi'(t) = \varepsilon\left(\psi'(t)\left[\sum_{i=1}^{m}\Big(p_i(\psi(t))x(\mu_i(\psi(t))) - g_i(\psi(t))x(\nu_i(\psi(t)))\Big)\right.\right.$$

$$\left.\left. + f\big(x(\tau_1(\psi(t))), \ldots, x(\tau_m(\psi(t))), x(\psi(t)), \psi(t)\big)\right]\right). \quad (3.6)$$

Obviously, (3.5) and (3.6) ensure the validity of the property (3.4).

Obtained results show that (3.4) is the natural symmetric property for equation (2.1) with symmetric deviation of the arguments (3.1)–(3.3) and symmetric solution (2.2). □

Remark 3.2. The proposition means that right side of the differential equation (2.1) has a property of symmetry which is in a sense natural by seeing on character of problem. For example, if μ_i, ν_i, τ_i, $i = 1, \ldots, m$, are linear delays $\mu_i(t) := \mu_i t$, where μ_i are constants, $i = 1, \ldots, m$, then conditions (3.1), (3.2), (3.3) are carried out obviously with $j_1 = \cdots = j_m = k_1 = \cdots = k_m = r_1 = \cdots = r_m = 1$. Equations with properties similar to (3.4) was considered in [5,9,10].

Let us fix some value $t_0 \in \mathbb{R}$. From the formulation of problem it is clear that a restriction $y = x|_{I_\psi}$ of every solution x on interval $I_\psi := [t_0, \psi(t_0)]$ satisfies a two-point boundary-value condition

$$y(t_0) = y(\psi(t_0)). \quad (3.7)$$

For further investigations we need the following notations and propositions:

a) The increasing function ψ generates increasing numerical sequence

$$\cdots < \psi^{-2}(t_0) < \psi^{-1}(t_0) < t_0 \leq \psi(t_0) < \psi^2(t_0) < \cdots \qquad (3.8)$$

b) Every points of the sequence from (3.8) divides $\mathbb{R}$ on a counted quantity of intervals

$$[\psi^j(t_0), \psi^{j+1}(t_0)], \qquad j \in \mathbb{Z}. \qquad (3.9)$$

c) Assume that the number j is a number of the interval $[\psi^j(t_0), \psi^{j+1}(t_0)]$.

Definition 3.3. For every $t \in \mathbb{R}$ we define number $l(t)$ by a number of such interval (3.9), which contains the point t.

Taking into account definition of the function $l : \mathbb{R} \to \mathbb{Z}$, we get that the next lemma is true.

Lemma 3.4. *If function* $y : I_\psi \to \mathbb{R}$ *satisfy two-point boundary-value condition* (3.7)*, then function*

$$x(t) := y\left(\psi^{-l(t)}(t)\right), \qquad t \in \mathbb{R} \qquad (3.10)$$

has the property (2.2).

Let us consider operators $\{\xi_i, \kappa_i, \sigma_i\} : C(I_\psi, \mathbb{R}) \to L_1(I_\psi, \mathbb{R})$ for $i = 1, 2, \ldots, m$,

$$(\xi_i x)(t) := \begin{cases} x(\mu_i(t)), & \text{if } \mu_i(t) \in I_\psi, \\ x\left(\psi^{-l(\mu_i(t))}(\mu_i(t))\right), & \text{if } \mu_i(t) \notin I_\psi, \end{cases} \qquad (3.11)$$

$$(\kappa_i x)(t) := \begin{cases} x(\nu_i(t)), & \text{if } \nu_i(t) \in I_\psi, \\ x\left(\psi^{-l(\nu_i(t))}(\nu_i(t))\right), & \text{if } \nu_i(t) \notin I_\psi, \end{cases} \qquad (3.12)$$

$$(\sigma_i x)(t) := \begin{cases} x(\tau_i(t)), & \text{if } \tau_i(t) \in I_\psi, \\ x\left(\psi^{-l(\tau_i(t))}(\tau_i(t))\right), & \text{if } \tau_i(t) \notin I_\psi, \end{cases} \qquad (3.13)$$

where $l(t)$ is the number of such interval which contains a point $t \in \mathbb{R}$ (see Definition 3.3).

Lemma 3.5. *Assume that function* $y : I_\psi \to \mathbb{R}$ *is a solution of the equation*

$$y'(t) = \varepsilon\left(\sum_{i=1}^{m}\Big(p_i(t)(\xi_i y)(t) - g_i(t)(\kappa_i y)(t)\Big)\right.$$

$$\left. + f\Big((\sigma_1 y)(t), (\sigma_2 y)(t), \ldots, (\sigma_m y)(t), y(t), t\Big)\right), \qquad t \in I_\psi, \quad (3.14)$$

and has the property (3.7).

Then the function $x : \mathbb{R} \to \mathbb{R}$ defined by (3.10) is a solution of the problem (2.1), (2.2).

Proof. Let us start by assuming that equation (3.14) is correct. Really, expression in right side is correctly defined for arbitrary absolutely continuous function $y : I_\psi \to \mathbb{R}$, such as, taking into account (3.11)–(3.13), in role of corresponding initial-value function on set $\Lambda_1 := \bigcup_{i=1}^{m} \mu_i(I_\psi) \setminus I_\psi$, $\Lambda_2 := \bigcup_{i=1}^{m} \nu_i(I_\psi) \setminus I_\psi$, and $\Lambda_3 := \bigcup_{i=1}^{m} \tau_i(I_\psi) \setminus I_\psi$ we can use values obtained

by "movement" or "broadcast" of the corresponding values on base interval I_ψ with saving of the symmetric property.

Suppose that function $y : I_\psi \to \mathbb{R}$ is a solution of the problem (3.7), (3.14) and function x is corresponding function (3.10). From lemma 3.4 it is evident that function (3.10) has a property (2.2). Note that on the set $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ the values of the function x coincide with values of the initial-value function using in construction of the equation (3.14). So it follows that x satisfy (2.1) on the interval I_ψ.

It is necessary to be convinced that for almost every $t \notin I_\psi$ for all x the equation (2.1) is true. It is proved directly by using (3.1)–(3.4). $\qquad\square$

Remark 3.6. If $\mu_i(I_\psi) \subset I_\psi$, $\nu_i(I_\psi) \subset I_\psi$ and $\tau_i(I_\psi) \subset I_\psi$, $i = 1, 2, \ldots, n$, then equation (3.14) does not need the definition of the initial-value function and can be recorded by (2.1) for $t \in I_\psi$.

It follows from Lemma 3.5 that problem of investigation of the solutions of the equation (2.1) with symmetric property (2.2) defined on $(-\infty, +\infty)$ can be changed by the investigations of solvability of two-point boundary value problem (3.7), (3.14) on the interval $[t_0, \psi(t_0)]$. However, introducing the properties (3.1)–(3.4) are necessary.

The possibility of study the scalar nonlinear functional differential equations with symmetric property only on the interval without any loss of general properties of solutions are illustrated by the following examples.

Example 3.7. Periodic type of solutions.

Let us consider at linear scalar functional-differential equation

$$x'(t) = \varepsilon \sum_{i=1}^{m} \alpha_i \cos(\theta_i t) x(\sin(t - b_i)), \quad t \in \mathbb{R}, \quad m \in \mathbb{N} \tag{3.15}$$

and find solution $x : \mathbb{R} \to \mathbb{R}$ with symmetric property

$$x(t) = x(t + 2\pi). \tag{3.16}$$

Then the equation (3.15) is the equation (2.1) with

$$p_i(t) := \alpha_i \cos(\theta_i t), \quad \mu_i = \sin(t - b_i), \quad g_i := 0, \quad \{\alpha_i, \theta_i, b_i\} \in \mathbb{R}, \quad i = 1, 2, \ldots, n, \quad f \equiv 0,$$

and equation (3.16) is the equation (2.2) with $\psi(t) := t + 2\pi$.

Here obviously that (3.1) and (3.4) are true with $j_i = 1$, $i = 1, 2, \ldots, m$. Let us consider interval $I_\psi = [t_0, t_0 + 2\pi]$ and study the two-point boundary value problem

$$y(t_0) = y(t_0 + 2\pi) \tag{3.17}$$

for scalar functional-differential equation

$$y'(t) = \varepsilon \sum_{i=1}^{m} \xi_i \cos(\theta_i t) \sin y(\sin(t - b_i)), \quad t \in I_\psi, \quad m \in \mathbb{N}. \tag{3.18}$$

Taking into account Lemma 3.5, equation (3.15) with symmetric property (3.16) on the interval $t \in \mathbb{R}$ is equivalent to the two-point boundary value problem (3.18), (3.17), $t \in [t_0, t_0 + 2\pi]$, without any loss of general properties of solutions.

Example 3.8. Let us review linear scalar functional-differential equation

$$x'(t) = \varepsilon \sum_{i=1}^{m} \alpha_i \sin(\gamma_i t) x(t + 2\pi)^2, \quad t \in \mathbb{R}, \quad m \in \mathbb{N} \tag{3.19}$$

and find solution $x : \mathbb{R} \to \mathbb{R}$ with symmetric property

$$x(t) = x(-t - 4\pi). \tag{3.20}$$

Then the equation (3.19) is the equation (2.1) with

$$p_i(t) := \alpha_i \sin(\alpha_i t), \quad \mu_i = (t + 2\pi)^2, \quad g_i := 0, \quad \{\alpha_i, \gamma_i\} \in \mathbb{R}, \quad i = 1, 2, \dots, m, \quad f \equiv 0$$

and equation (3.20) is the equation (2.2) with $\psi(t) := -t - 4\pi$.

Here obviously that (3.1) and (3.4) are true with $j_i = 1$, $i = 1, 2, \dots, m$. Let us consider interval $I_\psi = [t_0, -t_0]$ and study the two-point boundary value problem

$$y(t_0) = y(-t_0 - 4\pi) \tag{3.21}$$

for scalar functional-differential equation

$$y'(t) = \varepsilon \sum_{i=1}^{m} \alpha_i \sin(\gamma_i t) x(t + 2\pi)^2, \quad t \in I_\psi, \quad m \in \mathbb{N}. \tag{3.22}$$

Taking into account Lemma 3.5, equation (3.19) with symmetric property (3.20) on the interval $t \in \mathbb{R}$ is equivalent to the two-point boundary value problem (3.22), (3.21), $t \in [t_0, -t_0 - 4\pi]$, without any loss of general properties of solutions.

Example 3.9. Let us consider linear scalar functional-differential equation

$$x'(t) = \varepsilon \sum_{i=1}^{n} \beta_i t^{2i+1} x(t^{2i}), \quad t \in \mathbb{R} \tag{3.23}$$

and find solution $x : \mathbb{R} \to \mathbb{R}$ with symmetric property

$$x(t) = x(-t). \tag{3.24}$$

Then the equation (3.23) is the equation (2.1) with

$$p_i(t) := \beta_i t^{2i+1}, \quad \mu_i = t^{2i}, \quad g_i := 0, \quad \beta_i \in \mathbb{R}, \quad i = 1, 2, \dots, n, \quad f \equiv 0$$

and equation (3.24) is the equation (2.2) with $\psi(t) := -t$.

Here obviously that (3.1) and (3.4) are true with $j_i = 1$, $i = 1, 2, \dots, n$. Let us assume interval $I_\psi = [t_0, -t_0]$ and study the two-point boundary value problem

$$y(t_0) = y(-t_0) \tag{3.25}$$

for scalar functional-differential equation

$$y'(t) = \varepsilon \sum_{i=1}^{m} \beta_i t^{2i+1} y(t^{2i}), \quad t \in I_\psi, \quad m \in \mathbb{N}. \tag{3.26}$$

Taking into account Lemma 3.5, equation (3.23) with symmetric property (3.24) on the interval $t \in \mathbb{R}$ is equivalent to the two-point boundary value problem (3.26), (3.25), $t \in [t_0, -t_0]$, without any loss of general properties of solutions.

4 Two-point boundary-value problem on interval I_ψ

The study of a nonlinear scalar functional-differential equation (3.14) with two point boundary value problem (3.7) without any loss of general properties of solutions is presented follow. Here $p_i, g_i, f, i = 1, 2, \ldots, m$ have the properties (3.1)–(3.4).

Let us put

$$
\begin{aligned}
h(t) := \varepsilon \Big(\sum_{i=1}^{m} \big(p_i(t)(\xi_i y)(t) - g_i(t)(\kappa_i y)(t) \big) \\
+ f\big((\sigma_1 y)(t), (\sigma_2 y)(t), \ldots, (\sigma_m y)(t), y(t), t \big) \Big), \qquad t \in I_\psi,
\end{aligned}
$$

then from (3.14) we have that

$$
y'(t) = h(t), \quad t \in I_\psi
$$

and, taking into account (3.7), we get that

$$
\int_{t_0}^{\psi(t_0)} h(s)ds = 0. \tag{4.1}
$$

Let us solve boundary-value problem (3.7) for equation

$$
y'(t) = h(t) - a, \quad a = \text{const.} \tag{4.2}
$$

Then

$$
y(t) = y(t_0) + \int_{t_0}^{t} h(s)ds - a(t - t_0).
$$

and

$$
y(\psi(t_0)) = y(t_0) + \int_{t_0}^{\psi(t_0)} h(s)ds - a(\psi(t_0) - t_0)
$$

then

$$
a = \frac{1}{\psi(t_0) - t_0} \int_{t_0}^{\psi(t_0)} h(s)ds. \tag{4.3}
$$

So, one can write the solution of the equation (4.2), (3.7) by the next way

$$
y(t) = y(t_0) + \int_{t_0}^{t} h(s)ds - \frac{t - t_0}{\psi(t_0) - t_0} \int_{t_0}^{\psi(t_0)} h(s)ds. \tag{4.4}
$$

The obtained result is true.

Lemma 4.1. *The equation* (4.2) *has a solution with property* (3.7) *if and only if* (4.3) *is true and all solutions are given by* (4.4).

5 About the solvability of the problem (2.1) on $\mathbb{R}$

Let us consider the space $C([t_0, \psi(t_0)], \mathbb{R})$ of all functions with property (3.7). For further investigation we apply Lyapunov–Schmidt reduction method (see, for example, [7, 12]). Put

$$
y(t) := c + z(t), \quad t \in I_\psi, \tag{5.1}
$$

where c is constant.

Obviously, if

$$y(t_0) = y(\psi(t_0)) = c, \tag{5.2}$$

then

$$z(t_0) = z(\psi(t_0)) = 0. \tag{5.3}$$

Taking into account (3.11)–(3.13), we get that

$$(\xi_i y)(t) := c + (\xi_i z)(t), \quad i = 1, 2, \ldots, m,$$

$$(\kappa_i y)(t) := c + (\kappa_i z)(t), \quad i = 1, 2, \ldots, m,$$

and

$$(\sigma_i y)(t) := c + (\sigma_i z)(t), \quad i = 1, 2, \ldots, m.$$

Then one can write equation (4.4) by

$$z(t) = \varepsilon \Bigg(\sum_{i=1}^{m} \int_{t_0}^{t} \Big(p_i(s)(c + (\xi_i z)(s)) - g_i(s)(c + (\kappa_i z)(s)) \Big) ds$$

$$+ \int_{t_0}^{t} f\Big(c + (\sigma_1 z)(s), c + (\sigma_2 z)(s), \ldots, c + (\sigma_m z)(s), c + z(s), s \Big) ds$$

$$- \frac{t - t_0}{\psi(t_0) - t_0} \int_{t_0}^{\psi(t_0)} \Bigg(\sum_{i=1}^{m} \Big(p_i(s)(c + (\xi_i z)(s)) - g_i(s)(c + (\kappa_i z)(s)) \Big)$$

$$+ f\Big(c + (\sigma_1 z)(s), c + (\sigma_2 z)(s), \ldots, c + (\sigma_m z)(s), c + z(s), s \Big) \Bigg) ds \Bigg) \tag{5.4}$$

and, what is very important, we study the functions with property (4.1). This means that

$$\int_{t_0}^{\psi(t_0)} \Bigg(\sum_{i=1}^{m} \Big(p_i(s)(c + (\xi_i z)(s)) - g_i(s)(c + (\kappa_i z)(s)) \Big)$$

$$+ f\Big(c + (\sigma_1 z)(s), c + (\sigma_2 z)(s), \ldots, c + (\sigma_m z)(s), c + z(s), s \Big) \Bigg) ds = 0. \tag{5.5}$$

The next theorem about the unique solvability of the problem (5.4), (5.3) is true.

Theorem 5.1. *Assume that there exist such constants $K_i > 0$, $L_i > 0$, $i = 1, 2$, that for all $z_1, z_2 \in C(I_\psi, \mathbb{R})$, and $t \in \mathbb{R}$ the inequalities*

$$\Bigg| \sum_{i=1}^{m} \Big(p_i(t)(c_1 + (\xi_i z_1)(t)) + g_i(t)(c_1 + (\kappa_i z_1)(t)) \Big)$$

$$- \sum_{i=1}^{m} \Big(p_i(t)(c_2 + (\xi_i z_2)(t)) + g_i(t)(c_2 + (\kappa_i z_2)(t)) \Big) \Bigg|$$

$$\leq K_1 |c_1 - c_2| + K_2 |z_1(t) - z_2(t)| \tag{5.6}$$

and

$$\Bigg| f\Big(c_1 + (\sigma_1 z_1)(t), c_1 + (\sigma_2 z_1)(t), \ldots, c_1 + (\sigma_m z_1)(t), c + z_1(t), t \Big)$$

$$- f\Big(c_2 + (\sigma_1 z_2)(t), c_2 + (\sigma_2 z_2)(t), \ldots, c_2 + (\sigma_m z_2)(t), c + z_2(t), t \Big) \Bigg|$$

$$\leq L_1 |c_1 - c_2| + L_2 |z_1(t) - z_2(t)| \tag{5.7}$$

are true, and

$$|\varepsilon|(\psi(t_0) - t_0)(K_2 + L_2) < \frac{1}{2}. \tag{5.8}$$

Then the auxiliary equation (5.4) *has a unique solution* $z = z(\varepsilon, c, \cdot) \in C(I_\psi, \mathbb{R})$ *for any* $c \in \mathbb{R}$ *and* $z(\varepsilon, c, t_0) = 0, z(\varepsilon, c, \psi(t_0)) = 0$.

Moreover, it satisfies

$$\|z(\varepsilon, c_1, \cdot) - z(\varepsilon, c_2, \cdot)\| \leq \frac{2|\varepsilon|(\psi(t_0) - t_0)(K_1 + L_1)|c_1 - c_2|}{1 - 2|\varepsilon|(\psi(t_0) - t_0)(K_2 + L_2)}, \tag{5.9}$$

$$\|z(\varepsilon, c_1, \cdot)\| \leq \frac{2|\varepsilon|(\psi(t_0) - t_0)\big((K_1 + L_1)|c_1| + \|f(0, 0, \dots, \cdot)\|\big)}{1 - 2|\varepsilon|(\psi(t_0) - t_0)(K_2 + L_2)}, \tag{5.10}$$

where $\|z\| = \max_{t \in I_\psi} |z(t)|$.

Proof. Let us put $F_\varepsilon : [t_0, \psi(t_0)] \to \mathbb{R}$ by

$$
\begin{aligned}
F_\varepsilon(c, z)(t) := \varepsilon\Bigg(& \sum_{i=1}^{m} \int_{t_0}^{t} \Big(p_i(s)\big(c + (\xi_i z)(s)\big) - g_i(s)\big(c + (\kappa_i z)(s)\big) \Big) ds \\
& + \int_{t_0}^{t} f\Big(c + (\sigma_1 z)(s), c + (\sigma_2 z)(s), \dots, c + (\sigma_m z)(s), c + z(s), s\Big) ds \\
& - \frac{t - t_0}{\psi(t_0) - t_0} \int_{t_0}^{\psi(t_0)} \Big(\sum_{i=1}^{m} \big(p_i(s)(c + (\xi_i z)(s)) - g_i(s)(c + (\kappa_i z)(s)) \big) \\
& + f\Big(c + (\sigma_1 z)(s), c + (\sigma_2 z)(s), \dots, c + (\sigma_m z)(s), c + z(s), s\Big) \Big) ds \Bigg).
\end{aligned}
$$

For any $c_1, c_2 \in \mathbb{R}$ and $z_1, z_2 \in C(I_\psi, \mathbb{R})$, using the conditions (5.6), (5.7), we get

$$
\begin{aligned}
& \Big| F_\varepsilon(c_1, z_1)(t) - F_\varepsilon(c_2, z_2)(t) \Big| \\
& \leq |\varepsilon| \Bigg| \int_{t_0}^{t} \Bigg(\sum_{i=1}^{m} \Big(p_i(s)\big(c_1 + (\xi_i z_1)(s) - c_2 - (\xi_i z_2)(s)\big) \\
& \quad + g_i(s)\big(c_1 + (\kappa_i z_1)(s) - c_2 - (\kappa_i z_2)(s)\big) \Big) \\
& \quad + f\Big(c_1 + (\sigma_1 z_1)(s), c_1 + (\sigma_2 z_1)(s), \dots, c_1 + (\sigma_m z_1)(s), c_1 + z_1(s), s\Big) \\
& \quad - f\Big(c_2 + (\sigma_1 z_2)(s), c_2 + (\sigma_2 z_2)(s), \dots, c_2 + (\sigma_m z_2)(s), c_2 + z_2(s), s\Big) \Bigg) ds \Bigg| \\
& \quad - |\varepsilon| \frac{t - t_0}{\psi(t_0) - t_0} \Bigg| \int_{t_0}^{\psi(t_0)} \Bigg(\sum_{i=1}^{m} \Big(p_i(s)\big(c_1 + (\xi_i z_1)(s) - c_2 - (\xi_i z_2)(s)\big) \\
& \quad + g_i(s)\big(c_1 + (\kappa_i z_1)(s) - c_2 - (\kappa_i z_2)(s)\big) \Big) \\
& \quad + f\Big(c_1 + (\sigma_1 z_1)(s), c_1 + (\sigma_2 z_1)(s), \dots, c_1 + (\sigma_m z_1)(s), c_1 + z_1(s), s\Big) \\
& \quad - f\Big(c_2 + (\sigma_1 z_2)(s), c_2 + (\sigma_2 z_2)(s), \dots, c_2 + (\sigma_m z_2)(s), c_2 + z_2(s), s\Big) \Bigg) ds \Bigg| \\
& \leq 2|\varepsilon|(\psi(t_0) - t_0)\Big(K_1|c_1 - c_2| + K_2\|z_1(t) - z_2(t)\| \Big) \\
& \quad + 2|\varepsilon|(\psi(t_0) - t_0)\Big(L_1|c_1 - c_2| + L_2\|z_1(t) - z_2(t)\| \Big) \\
& \leq 2|\varepsilon|(\psi(t_0) - t_0)\Big((K_1 + L_1)|c_1 - c_2| + (L_2 + K_2)\|z_1(t) - z_2(t)\| \Big)
\end{aligned}
$$

for any $t \in I_\psi$. Taking into account (5.8) and applying the Banach Fixed Point Theorem we get that problem (5.4) has a unique solution $z = z(\varepsilon, c, \cdot) \in C(I_\psi, \mathbb{R})$ satisfying (5.9).

Next, inequality (5.10) follows from

$$\|F_\varepsilon(c_1, z(\varepsilon, c_1, \cdot))\|$$
$$\leq 2|\varepsilon|(\psi(t_0) - t_0)\Big((K_1 + L_1)c_1 + (K_2 + L_2)\|z(\varepsilon, c_1, \cdot)\|\Big) + \|F_\varepsilon(0, 0)\|$$
$$\leq 2|\varepsilon|(\psi(t_0) - t_0)\Big((K_1 + L_1)c_1 + (K_2 + L_2)\|z(\varepsilon, c_1, \cdot)\| + \|f(0, 0, \ldots, \cdot)\|\Big).$$

The proof is finished. $\qquad\qquad\square$

Now one can return to the variable (5.1) with properties (5.3) and (5.2). Plugging $z(\varepsilon, c, \cdot)$ into (5.4), one can obtain the bifurcation equation

$$B(\varepsilon, c) := \int_{t_0}^{\psi(t_0)} \left(\sum_{i=1}^{m} \Big(p_i(s)(c + (\xi_i z)(\varepsilon, c, s)) - g_i(s)(c + (\kappa_i z)(\varepsilon, c, s)) \Big) \right.$$
$$+ f\Big(c + (\sigma_1 z)(\varepsilon, c, s), c + (\sigma_2 z)(\varepsilon, c, s), \ldots,$$
$$\left. c + (\sigma_m z)(\varepsilon, c, s), c + z(\varepsilon, c, s), s\Big) \right) ds = 0. \tag{5.11}$$

Let us put

$$M(c) := B(0, c) = \int_{t_0}^{\psi(t_0)} \left(\sum_{i=1}^{m} \Big(p_i(s) - g_i(s) \Big) c + f\Big(c, c, \ldots, c, s\Big) \right) ds. \tag{5.12}$$

5.1 Conditions on the solvability of the problem (2.1), (2.2)

The following global result is obtained.

Theorem 5.2. *Assume, that inequalities (5.6) and (5.7) are fulfilled and there exist $a < b$ such that*

$$M(a)M(b) < 0. \tag{5.13}$$

Then for any $\varepsilon \neq 0$ small, there exists a symmetric and periodic solution $x_\varepsilon(t)$ of the equation (2.1) located in (a, b).

Proof. Taking into account (5.11), (5.12) and (5.13), one can conclude that there is an $\varepsilon_0 > 0$ small such that

$$B(\varepsilon, a)B(\varepsilon, b) < 0$$

for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. It is known from the Bolzano Theorem or Mean Value Theorem that there is an $c(\varepsilon) \in (a, b)$ solving

$$B(\varepsilon, c(\varepsilon)) = 0.$$

This means that problems (5.4) and (5.5) have a solution $z(\varepsilon, c(\varepsilon), t)$. Now, using (5.1) it is clearly seen that (3.14), (3.7) has a solution

$$y(\varepsilon, t) = c(\varepsilon) + z(\varepsilon, c(\varepsilon), t).$$

So, in view of (3.10) we get that the nonlinear symmetric functional differential equation (2.1) with argument's symmetric property (3.1)–(3.3) has at least one symmetric solution on $\mathbb{R}$ with property (2.2) located in (a, b). $\qquad\qquad\square$

5.2 Conditions on the unique solvability of the problem (2.1), (2.2)

The following local result is obtained.

Theorem 5.3. *Let $f \in C^1(\mathbb{R}, \mathbb{R})$. Assume that there exists such $c_0 \in \mathbb{R}$ that*

$$M(c_0) = 0 \quad and \quad M'(c_0) \neq 0.$$

Then for any $\varepsilon \neq 0$ small, there exists a unique symmetric and periodic solution $x_\varepsilon(t)$ of the equation (2.1) near c_0.

Proof. Obviously

$$B(\varepsilon, c) = M(c) + \tilde{B}(\varepsilon, c)$$

for $\tilde{B}(\varepsilon, c) = B(\varepsilon, c) - M(c)$. Then we derive

$$
\begin{aligned}
&\tilde{B}(\varepsilon, c_1) - \tilde{B}(\varepsilon, c_2) \\
&= \int_{t_0}^{\psi(t_0)} \Bigg(\sum_{i=1}^m p_i(s)(c_1 - c_2 + (\xi_i z)(\varepsilon, c_1, s) - (\xi_i z)(\varepsilon, c_2, s)) \\
&\qquad - \sum_{i=1}^m g_i(s)(c_1 - c_2 + (\kappa_i z)(\varepsilon, c_1, s) - (\kappa_i z)(\varepsilon, c_2, s)) \\
&\qquad + f(c_1 + (\sigma_1 z)(\varepsilon, c_1, s), \ldots, c_1 + z(\varepsilon, c_1, s), s) \\
&\qquad - f(c_2 + (\sigma_1 z)(\varepsilon, c_2, s), \ldots, c_2 + z(\varepsilon, c_2, s), s) \Bigg) ds \\
&\quad - \int_{t_0}^{\psi(t_0)} \Bigg(\sum_{i=1}^m p_i(s)(c_1 - c_2) - \sum_{i=1}^m g_i(s)(c_1 - c_2) + f(c_1, c_1, \ldots, s) - f(c_2, c_2, \ldots, s) \Bigg) ds \\
&= \int_{t_0}^{\psi(t_0)} \Bigg(\sum_{i=1}^m p_i(s) \big((\xi_i z)(\varepsilon, c_1, s) - (\xi_i z)(\varepsilon, c_2, s) \big) \\
&\qquad - \sum_{i=1}^m g_i(s) \big((\kappa_i z)(\varepsilon, c_1, s) - (\kappa_i z)(\varepsilon, c_2, s) \big) \\
&\qquad + \int_0^1 \Bigg(\sum_{i=1}^m f_{z_i} \big(\theta c_1 + (1-\theta)c_2 + \theta(\sigma_1 z)(\varepsilon, c_1, s) + (1-\theta)(\sigma_1 z)(\varepsilon, c_2, s), \\
&\qquad\qquad \theta c_1 + (1-\theta)c_2 + \theta(\sigma_2 z)(\varepsilon, c_1, s) + (1-\theta)(\sigma_2 z)(\varepsilon, c_2, s), \ldots, \\
&\qquad\qquad z(\varepsilon, c_1, s) + (1-\theta)z(\varepsilon, c_2, s), s \big) \big(c_1 - c_2 + \sigma_i(z)(\varepsilon, c_1, s) - \sigma_i(z)(\varepsilon, c_2, s) \big) \\
&\qquad\qquad + f_{z_{m+1}} \big(\theta c_1 + (1-\theta)c_2 + \theta(\sigma_1 z)(\varepsilon, c_1, s) + (1-\theta)(\sigma_1 z)(\varepsilon, c_2, s), \\
&\qquad\qquad \theta c_1 + (1-\theta)c_2 + \theta(\sigma_2 z)(\varepsilon, c_1, s) + (1-\theta)(\sigma_2 z)(\varepsilon, c_2, s), \ldots, \\
&\qquad\qquad z(\varepsilon, c_1, s) + (1-\theta)z(\varepsilon, c_2, s), s \big) \big(z(\varepsilon, c_1, s) - z(\varepsilon, c_2, s) \big) \Bigg) d\theta \Bigg) ds \\
&\quad - \int_{t_0}^{\psi(t_0)} \int_0^1 \Bigg(\sum_{i=1}^m f_{z_i} \big(\theta c_1 + (1-\theta)c_2, \theta c_1 + (1-\theta)c_2, \ldots, s \big) (c_1 - c_2) \Bigg) d\theta ds.
\end{aligned}
\tag{5.14}
$$

Now, using property $\|\xi_i x\| \leq \|x\|$ and inequality (5.9), there is a constant $\tilde{K} \geq 0$ such that

$$\|(\xi_i z)(\varepsilon, c_1, s) - (\xi_i z)(\varepsilon, c_2, s)\| \leq \|\xi_i\big(z(\varepsilon, c_1, s) - z(\varepsilon, c_2, s)\big)\|$$
$$\leq \|z(\varepsilon, c_1, s) - z(\varepsilon, c_2, s)\| \leq \tilde{K}|\varepsilon||c_1 - c_2|.$$

Without loss of generality, we consider $M'(c_0) > 0$. Then taking $\delta > 0$ small, we obtain

$$M'(c) \geq \frac{M'(c_0)}{2} > 0$$

for $c \in (c_0 - \delta, c_0 + \delta)$. Then (5.14) implies that there exists an $\varepsilon_0 > 0$ small such that

$$|\tilde{B}(\varepsilon, c_1) - \tilde{B}(\varepsilon, c_2)| \leq \frac{M'(c_0)}{4}|c_1 - c_2|$$

for any $c_1, c_2 \in (c_0 - \delta, c_0 + \delta)$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. If $c_1 > c_2$, and using the Mean Value Theorem, we get

$$B(\varepsilon, c_1) - B(\varepsilon, c_2) = M(c_1) - M(c_2) + \tilde{B}(\varepsilon, c_1) - \tilde{B}(\varepsilon, c_2)$$
$$\geq M'(c)(c_1 - c_2) - \frac{M'(c_0)}{4}(c_1 - c_2) \geq \frac{M'(c_0)}{4}(c_1 - c_2) > 0.$$

Now we apply the Bolzano Theorem to get a unique $c(\varepsilon) \in (c_0 - \delta, c_0 + \delta)$ solving

$$B(\varepsilon, c(\varepsilon)) = 0.$$

We already know from the end of the proof of Theorem 5.2 that then (2.1) with the symmetric property (3.1)–(3.3) has a unique symmetric solution on $\mathbb{R}$ with property (2.2).   □

Remark 5.4. It should be noted, that solutions of the equation (5.11) are anticipated by zeroes of (5.12).

Remark 5.5. An alternative way in the proof of Theorem 5.3 is applying the Implicit Function Theorem, but our approach is constructive by allowing to estimate the magnitude of ε_0 for concrete functions p, g and f.

6   Application

Example 6.1. Let us find conditions necessaries for the unique solvability of the equation

$$x'(t) = \varepsilon \left(p(t)x(t - 2\pi) + f(t) \right), \tag{6.1}$$

and find solution $x : \mathbb{R} \to \mathbb{R}$ with symmetric property

$$x(t) = x\left(t + \frac{1}{2}\sin t + 1 \right), \tag{6.2}$$

where

$$\psi(t) = t + \frac{1}{2}\sin t + 1. \tag{6.3}$$

Here the equation (6.1) is the equation (2.1) with

$$m = 1, \quad p_1(t) := p(t), \quad \mu_1(t) := \mu(t) = t - 2\pi, \quad g_1 := 0, \quad f(\cdot, t) = f(t).$$

The symmetric property (6.2) is the property (2.2) with $\psi(t)$ defined by (6.3).
Easy to see, that

$$\mu(\psi(t)) = \psi(\mu(t)) = t - 2\pi + \frac{1}{2}\sin t + 1.$$

So, (3.1) and (3.4) are fulfilled with $j_i = 1$, $i = 1$. Let us consider the interval $I_\psi = [t_0, t_0 + \frac{1}{2}\sin t_0 + 1]$ with $t_0 = 0$. Then $\psi(t_0) = 1$ and $I_\psi = [0, 1]$. Note, it is necessary to introduce such functions $p(t)$ and $f(t)$, that

$$\psi'(t)p(\psi(t)) = p(t), \quad \psi'(t)f(\psi(t)) = f(t), \quad t \in \mathbb{R}.$$

If $t \in [\psi(0), \psi(\psi(0))]$ then

$$p(t) = \frac{p(\psi^{-1}(t))}{\psi'(\psi^{-1}(t))}$$

and if $t \in [0, \psi(0)]$ then

$$p(\psi^2(t)) = \frac{p(\psi(t))}{\psi'(\psi(t))} = \frac{p(t)}{\psi'(t)\psi'(\psi(t))}, \quad \text{for } \psi^2(t) \in [\psi^2(0), \psi^3(0)]$$

and, generaly, if $t \in [0, \psi(0)]$, then

$$p(\psi^n(t)) = \frac{p(t)}{\psi'(t)\psi'(\psi(t))\ldots\psi'(\psi^{n-1}(t))} \quad \text{for } \psi^n(t) \in [\psi^n(0), \psi^{n+1}(0)].$$

Function $p(t)$ in general case can be represented by the graph on Figure 6.1.
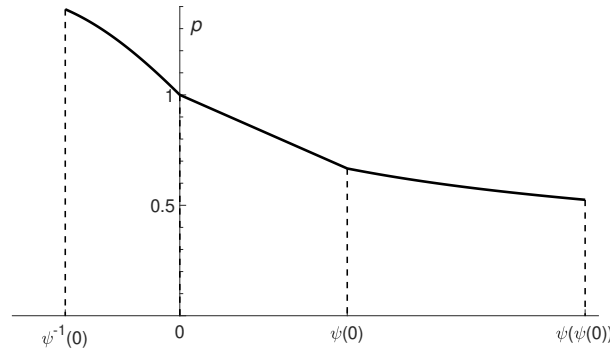

Figure 6.1: Function $p(t)$ on the interval $[\psi^{-1}(0), 0] \cup [0, \psi(0)] \cup [\psi(0), \psi(\psi(0))]$.

Here

$$p(t) = \left(1 - \frac{t + \frac{1}{2}\sin t + 1}{3}\right)\left(1 + \frac{\cos t}{2}\right) \quad \text{on } [\psi^{-1}(0), 0],$$

and

$$p(t) = 1 - \frac{t}{3} \quad \text{on } [0, \psi(0)],$$

and

$$p(t) = \frac{2\left(1 - \frac{1}{3}\alpha(t)\right)}{2 + \cos(\alpha(t))} \quad \text{on } [\psi(0), \psi(\psi(0))],$$

$$\alpha(t) := \psi^{-1}(t) = \frac{2(t-1)}{3} + \frac{4(t-1)^3}{243} + \frac{28(t-1)^5}{32805} + \frac{968(t-1)^7}{18600435}$$
$$+ \frac{14908(t-1)^9}{4519905705} + \frac{195704(t-1)^{11}}{958865710275} + \frac{81505976(t-1)^{13}}{7067799150437025} + O(t-1)^{15} \quad (6.4)$$

is the inverse Taylor series for function $\psi(t) = t + \frac{1}{2}\sin t + 1$.

Remark 6.2. Note, that function $p(t)$ cannot to be a constant function because $p(t) = constant$ does not fulfill the symmetric property (3.4).

Similar arguments are applied to $f(t)$. So for instance we can take

$$f(t) = \frac{1}{2}\sqrt{1 - \frac{5(t + \frac{1}{2}\sin t + 1)}{9}}\left(1 + \frac{\cos t}{2}\right) \quad \text{on } [\psi^{-1}(0), 0],$$

and

$$f(t) = \frac{1}{2}\sqrt{1 - \frac{5t}{9}} \quad \text{on } [0, \psi(0)],$$

and

$$f(t) = \frac{\sqrt{1 - \frac{5}{9}\alpha(t)}}{2 + \cos(\alpha(t))} \quad \text{on } [\psi(0), \psi(\psi(0))],$$

where α is defined by (6.4). Function $f(t)$ in general case can be represented by the next graph (see Figure 6.2).
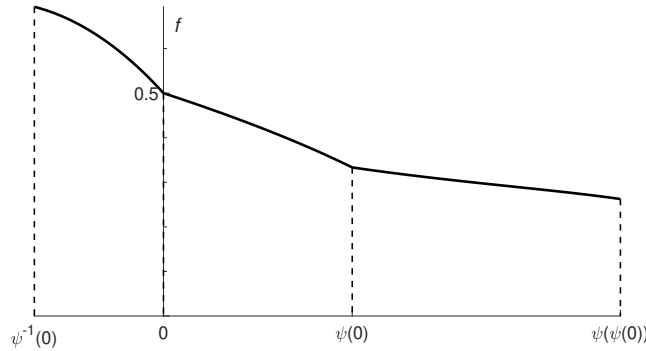

Figure 6.2: Function $f(t)$ on the interval $[\psi^{-1}(0), 0] \cup [0, \psi(0)] \cup [\psi(0), \psi(\psi(0))]$.

Now we are ready to study the existence of the symmetric solutions of the problem (6.1), (6.2).

Obviously, $M(c)$ defined by (5.12) is equal to

$$M(c) = \int_{t_0}^{t_0 + \frac{1}{2}\sin t_0 + 1} \left(p(s)c + f(s)\right) ds = c\int_0^1 p(s)\, ds + \int_0^1 f(s)\, ds. \quad (6.5)$$

Taking into account (6.5) the next corollary is obtained directly from Theorem 5.3.

Corollary 6.3. If $\int_0^1 p(s)\, ds \neq 0$, then the equation (6.1) with symmetric property (6.2) has a unique symmetric solution of order $-\frac{\int_0^1 f(s)\, ds}{\int_0^1 p(s)\, ds} + O(\varepsilon)$.

In our concrete case, we have $\frac{\int_0^1 f(s)\,ds}{\int_0^1 p(s)\,ds} = \frac{38}{75}$.

References

[1] A. A. Boichuk, V. F. Zhuravlev, A. M. Samoilenko, *Generalized inverse operators and Noether boundary-value problems* (in Russian), Proceedings of the Institute of Mathematics of the National Academy of Sciences of the Ukraine, Vol. 13, Kyiv, 1995. MR1367732; Zbl 0894.34001

[2] N. Dilna, M. Fečkan, Weakly non-linear and symmetric periodic systems at resonance, *Nonlinear Stud.* **16**(2009), No. 2, 149–170. MR2527178; Zbl 1188.34046

[3] A. Domoshnitsky, R. Hakl, J. Šremr, Component-wise positivity of solutions to periodic boundary problem for linear functional differential system, *J. Inequal. Appl.* **2012**(2012) Article No. 112, 23 pp. https://doi.org/10.1186/1029-242X-2012-112; MR2954534; Zbl 1278.34070

[4] N. Dilna, M. Fečkan, On the uniqueness, stability and hyperbolicity of symmetric and periodic solutions of weakly nonlinear ordinary differential equations, *Miskolc Math. Notes* **10**(2009), No. 1, 11–40. https://doi.org/10.18514/MMN.2009.200; MR2518222; Zbl 1199.34196

[5] M. Fečkan, A. Rontó, N. Dilna, On a kind of symmetric weakly non-linear ordinary differential systems, *Bull. Sci. Math.* **140**(2016), No. 2, 188–230. https://doi.org/10.1016/j.bulsci.2015.11.003; MR3456187; Zbl 1339.34048

[6] O. I. Kocherha, O. I. Nenya, V. I., Tkachenko, On positive periodic solutions of nonlinear impulsive functional differential equations, *Nonlinear Oscil. (N. Y.)* **11**(2008), No. 4, 527–538. https://doi.org/10.1007/s11072-009-0048-6; MR2515084; Zbl 1277.34100

[7] J. E. Marsden, Qualitative methods in bifurcation theory, *Bull. Amer. Math. Soc.* **84**(1978), No. 6, 1125–1148. https://doi.org/10.1090/S0002-9904-1978-14549-2; MR508450; Zbl 0404.35010

[8] A. Ronto, M. Rontó, On certain symmetry properties of periodic solutions, *Nonlinear Oscil. (N. Y.)* **6**(2003), No. 1, 83–108. https://doi.org/10.1023/A:1024827821289; MR2089762; Zbl 1083.34034

[9] A. N. Ronto, On periodic solutions of systems with "maxima" (in Russian), *Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Tekh. Nauki* **1999**, No. 12, 27–30. MR1751454; Zbl 0954.34059

[10] A. Rontó, M. Rontó, Successive approximation techniques in non-linear boundary value problems for ordinary differential equations, in: *Handbook of differential equations: ordinary differential equations*. Vol. IV, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 441–592. https://doi.org/10.1016/S1874-5725(08)80010-7; MR2440165; Zbl 1180.34013

[11] A. N. Ronto, M. Rontó, A. M. Samoilenko, S. I. Trofimchuk, On periodic solutions of autonomous difference equations, *Georgian Math. J.* **8**(2001), No. 1, 135–164. MR1828690; Zbl 0985.39010

[12] A. Vanderbauwhede, *Local bifurcation and symmetry*, Research Notes in Mathematics, Vol. 75, Pitman (Advanced Publishing Program), Boston, MA, 1982. MR697724; Zbl 0539.58022